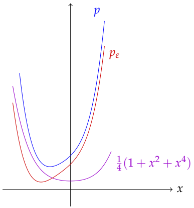
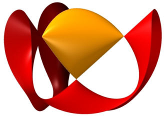
Exact Polynomial Optimization

Victor Magron, LAAS CNRS



TU Chemnitz
22 October 2019



Deciding Nonnegativity & Exact Optimization

$X = (X_1, \ldots, X_n)$

$f \in \mathbb{Q}[X]$

co-NP hard problem: check $f \geqslant 0$ **on** $\mathbf{K}$

NP hard problem: $\min\{f(\mathbf{x}) : \mathbf{x} \in \mathbf{K}\}$

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1. Unconstrained $\rightsquigarrow \mathbf{K} = \mathbb{R}^n$

2. Constrained
   $\rightsquigarrow \mathbf{K} = \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geqslant 0, \dots, g_m(\mathbf{x}) \geqslant 0\} \quad g_j \in \mathbb{Q}[X]$

$\deg f, \deg g_j \leqslant d$

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[Collins 75] CAD **doubly exp. in** n **poly. in** d

[Grigoriev-Vorobjov 88, Basu-Pollack-Roy 98]
Critical points **singly exponential time** $(m+1)\,\tau\, d^{\mathcal{O}(n)}$

Deciding Nonnegativity & Exact Optimization

Sums of squares (SOS)
$\sigma = {h_1}^2 + \cdots + {h_p}^2$

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HILBERT 17TH PROBLEM: f SOS of rational functions?

[Artin 27] **YES**!

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Semidefinite programming (SDP) $\rightsquigarrow$ **approximate** certificates

$$\boxed{\simeq \quad \to \quad =}$$

The Question of Exact Certification

How to go from **approximate** to **exact** certification?

Motivation

Positivity certificates

- Stability proofs of critical control systems (Lyapunov)
- Certified function evaluation [Chevillard et. al 11]
- Formal verification of real inequalities [Hales et. al 15]:

COQ HOL-LIGHT

Decomposing Nonnegative Polynomials

1. **Polya**'s representation $f = \frac{\sigma}{(X_1^2 + \cdots + X_n^2)^D}$
   positive definite form f
   [Reznick 95]

2. **Hilbert-Artin**'s representation $f = \frac{\sigma}{h^2}$
   $f \geqslant 0$
   [Artin 27]

3. **Putinar**'s representation
   $f = \sigma_0 + \sigma_1 \, g_1 + \cdots + \sigma_m \, g_m$ $f > 0$ on compact K
   $\deg \sigma_i \leqslant 2D$
   [Putinar 93]

Decomposing Nonnegative Polynomials

- Deciding **polynomial nonnegativity**

$$f(a,b) = a^2 - 2ab + b^2 \geqslant 0$$

- $f(a,b) = \begin{pmatrix} a & b \end{pmatrix} \underbrace{\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}}_{\succcurlyeq 0} \begin{pmatrix} a \\ b \end{pmatrix}$

- $a^2 - 2ab + b^2 = z_1 a^2 + 2 z_2 ab + z_3 b^2 \qquad (\mathbf{A}\,\mathbf{z} = \mathbf{d})$

- $\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\mathbf{F}_1} z_1 + \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\mathbf{F}_2} z_2 + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{F}_3} z_3 \succcurlyeq \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}_{\mathbf{F}_0}$

Decomposing Nonnegative Polynomials

- Choose a cost $\mathbf{c}$ e.g. $(1, 0, 1)$ and solve **SDP**

$$\begin{aligned} \min_{\mathbf{z}} \quad & \mathbf{c}^\top \mathbf{z} \\ \text{s.t.} \quad & \sum_i \mathbf{F}_i \, z_i \succcurlyeq \mathbf{F}_0 \ , \quad \mathbf{A}\,\mathbf{z} = \mathbf{d} \end{aligned}$$

- Solution $\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \succcurlyeq 0$ (eigenvalues 0 and 2)

- $a^2 - 2ab + b^2 = \begin{pmatrix} a & b \end{pmatrix} \underbrace{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}_{\succcurlyeq 0} \begin{pmatrix} a \\ b \end{pmatrix} = (a - b)^2$

- Solving **SDP** $\implies$ Finding **SUMS OF SQUARES** certificates

Decomposing Nonnegative Polynomials

4. **Circuit** polynomial

$$f = b_{\boldsymbol{\alpha}(1)} X^{\boldsymbol{\alpha}(1)} + \cdots + b_{\boldsymbol{\alpha}(r)} X^{\boldsymbol{\alpha}(r)} + b_{\boldsymbol{\beta}} X^{\boldsymbol{\beta}}$$

$b_{\boldsymbol{\alpha}(j)} > 0 \quad \boldsymbol{\alpha}(j) \in (2\mathbb{N})^n$

$\boldsymbol{\beta} = \lambda_1 \boldsymbol{\alpha}(1) + \cdots + \lambda_r \boldsymbol{\alpha}(r) \quad \lambda_j > 0$ and $\lambda_1 + \cdots + \lambda_r = 1$

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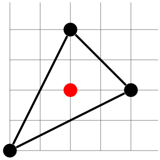
$$f = 1 + X_1^2 X_2^4 + X_1^4 X_2^2 - 3 X_1^2 X_2^2$$



Decomposing Nonnegative Polynomials

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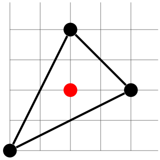

Circuit number $\Theta_f = \prod_{j=1}^r \left(\frac{b_{\alpha(j)}}{\lambda_j} \right)^{\lambda_j}$

Decomposing Nonnegative Polynomials

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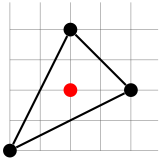

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Theorem (Illiman-de Wolff 16)

$f \geqslant 0 \Leftrightarrow |b_\beta| \leqslant \Theta_f$ or ($b_\beta \geqslant -\Theta_f$, β even)

SONC (SUMS OF NONNEGATIVE CIRCUITS)

Decomposing Nonnegative Polynomials

5 **arithmetic-geometric-mean-exponential (AGE)**

$f = c_1 \exp[X \cdot \boldsymbol{\alpha}(1)] + \cdots + c_t \exp[X \cdot \boldsymbol{\alpha}(t)] + \beta \exp[X \cdot \boldsymbol{\alpha}(0)]$

$c_j \in \mathbb{Q}_{>0} \quad \beta \in \mathbb{Q} \quad \boldsymbol{\alpha}(j) \in \mathbb{N}^n$

Decomposing Nonnegative Polynomials

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relative entropy $\qquad D(\boldsymbol{\nu}, \mathbf{c}) = \sum_j \nu_j \log \frac{\nu_j}{c_j}$

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Theorem (Chandrasekaran-Shah 16)

$f \geqslant 0 \Leftrightarrow \exists \boldsymbol{\nu} \mid D(\boldsymbol{\nu}, \mathbf{c}) \leqslant \beta$ and $\sum_j \boldsymbol{\alpha}(j) \nu_j = (\mathbf{1} \cdot \boldsymbol{\nu}) \boldsymbol{\alpha}(0)$

SAGE (SUMS OF AGE)

From Approximate to Exact Solutions

APPROXIMATE SOLUTIONS

sum of squares of $a^2 - 2ab + b^2$?

$(1.00001a - 0.99998b)^2$!

$$a^2 - 2ab + b^2 \simeq (1.00001a - 0.99998b)^2$$
$$a^2 - 2ab + b^2 \neq 1.0000200001a^2 - 1.9999799996ab + 0.9999600004b^2$$

$\boxed{\simeq \quad \rightarrow \quad =}$?

Rational SOS Decompositions

- Let $f \in \mathbb{R}[X]$ and $f \geqslant 0$ on $\mathbb{R}$ ($n = 1$)

Theorem

There exist $f_1, f_2 \in \mathbb{R}[X]$ s.t. $f = {f_1}^2 + {f_2}^2$.

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Proof.

$f = h^2(q + ir)(q - ir)$ □

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Examples

$$1 + X + X^2 = \left(X + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

$$1 + X + X^2 + X^3 + X^4 = \left(X^2 + \frac{1}{2}X + \frac{1+\sqrt{5}}{4}\right)^2 + \left(\frac{\sqrt{10+2\sqrt{5}} + \sqrt{10-2\sqrt{5}}}{4}X + \frac{\sqrt{10-2\sqrt{5}}}{4}\right)^2$$

Rational SOS Decompositions

- $f \in \mathbb{Q}[X] \cap \mathring{\Sigma}[X]$ (interior of the SOS cone)

Existence Question

Does there exist $f_i \in \mathbb{Q}[X], c_i \in \mathbb{Q}^{>0}$ s.t. $f = \sum_i c_i f_i^2$?

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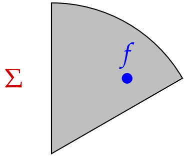
Examples

$$1 + X + X^2 = \left(X + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1\left(X + \frac{1}{2}\right)^2 + \frac{3}{4}(1)^2$$

$$1 + X + X^2 + X^3 + X^4 = \left(X^2 + \frac{1}{2}X + \frac{1+\sqrt{5}}{4}\right)^2 + \left(\frac{\sqrt{10+2\sqrt{5}} + \sqrt{10-2\sqrt{5}}}{4}X + \frac{\sqrt{10-2\sqrt{5}}}{4}\right)^2 = ???$$

Round & Project Algorithm [Peyrl-Parrilo 08]



$f \in \mathring{\Sigma}[X]$ with $\deg f = 2D$

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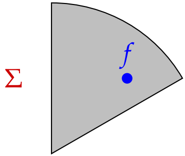

$f \in \mathring{\Sigma}[X]$ with $\deg f = 2D$

Find $\tilde{\mathbf{G}}$ with SDP at tolerance $\tilde{\delta}$ satisfying

$f(X) \simeq {\mathbf{v}_D}^T(X)\, \tilde{\mathbf{G}}\, \mathbf{v}_D(X) \quad \tilde{\mathbf{G}} \succ 0$

$\mathbf{v}_D(X)$: vector of monomials of $\deg \leqslant D$

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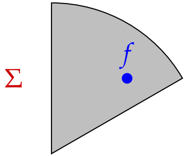

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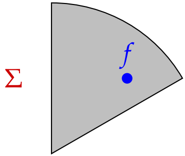

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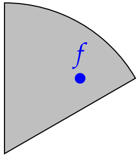
Round & Project Algorithm [Peyrl-Parrilo 08]

Σ



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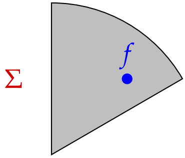

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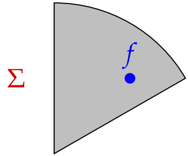
   $G_{\alpha,\beta} \leftarrow \hat{G}_{\alpha,\beta} - \frac{1}{\eta(\alpha+\beta)}\left(\sum_{\alpha'+\beta'=\alpha+\beta} \hat{G}_{\alpha',\beta'} - f_{\alpha+\beta}\right)$

Round & Project Algorithm [Peyrl-Parrilo 08]



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Small enough $\tilde{\delta}, \hat{\delta} \implies f(X) = \mathbf{v}_D{}^T(X)\, \mathbf{G}\, \mathbf{v}_D(X)$ and $\mathbf{G} \succcurlyeq 0$

One Answer when $\mathbf{K} = \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geqslant 0\}$

Hybrid **SYMBOLIC/NUMERIC** methods

Magron-Allamigeon-Gaubert-Werner 14

$$f \simeq \tilde{\sigma}_0 + \tilde{\sigma}_1 \, g_1 + \cdots + \tilde{\sigma}_m \, g_m$$

$$u = f - \tilde{\sigma}_0 + \tilde{\sigma}_1 \, g_1 + \cdots + \tilde{\sigma}_m \, g_m$$

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Magron-Allamigeon-Gaubert-Werner 14

$$f \simeq \tilde{\sigma}_0 + \tilde{\sigma}_1 \, g_1 + \cdots + \tilde{\sigma}_m \, g_m$$

Compact $\mathbf{K} \subseteq [0,1]^n$

$$u = f - \tilde{\sigma}_0 + \tilde{\sigma}_1 \, g_1 + \cdots + \tilde{\sigma}_m \, g_m$$

$$\boxed{\simeq \quad \to \quad =}$$

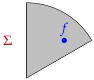
$\forall \mathbf{x} \in [0,1]^n, u(\mathbf{x}) \leqslant -\varepsilon$

$\min_{\mathbf{K}} f \geqslant \varepsilon$ when $\varepsilon \to 0$

COMPLEXITY?

From Approximate to Exact Solutions

Win TWO-PLAYER GAME

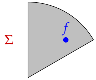


sum of squares of f?

$\simeq$ Output!

From Approximate to Exact Solutions

Win TWO-PLAYER GAME



Hybrid Symbolic/Numeric Algorithms

sum of squares of $f + \varepsilon$?

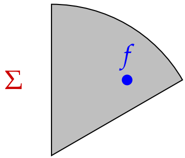
$\simeq$ Output!

Error Compensation

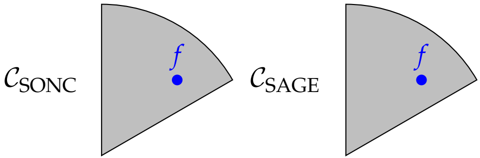
$\simeq \quad \rightarrow \quad =$

From Approximate to Exact Solutions

Exact SOS



Exact SONC/SAGE



Software: RealCertify and POEM

Exact optimization via SOS: RealCertify

Maple & arbitrary precision SDP solver SDPA-GMP [Nakata 10]

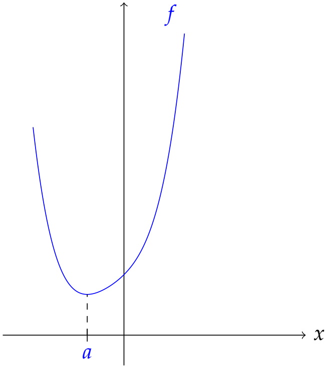
univsos	$n = 1$
multivsos	$n > 1$

Exact optimization via SONC/SAGE: POEM

Python (SymPy) & geometric programming/relative entropy ECOS [Domahidi-Chu-Boyd 13]

univsos: Outline [Schweighofer 99]

$f \in \mathbb{Q}[X]$ and $f > 0$
Minimizer a may not be in $\mathbb{Q} \ldots$



$f = 1 + X + X^2 + X^3 + X^4$

$a = \frac{5}{4(135+60\sqrt{6})^{1/3}} - \frac{4(135+60\sqrt{6})^{1/3}}{12} - \frac{1}{4}$

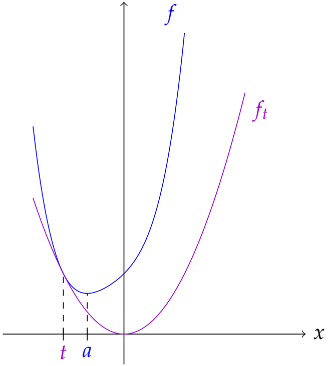
univsos: Outline [Schweighofer 99]

$f \in \mathbb{Q}[X]$ and $f > 0$

Minimizer a may not be in $\mathbb{Q}$ ...

Find $f_t \in \mathbb{Q}[X]$ s.t. :

- $\deg f_t \leqslant 2$
- $f_t \geqslant 0$
- $f \geqslant f_t$
- $f - f_t$ has a root $t \in \mathbb{Q}$



$f = 1 + X + X^2 + X^3 + X^4$

$a = \frac{5}{4(135+60\sqrt{6})^{1/3}} - \frac{4(135+60\sqrt{6})^{1/3}}{12} - \frac{1}{4}$

$f_t = X^2$

$t = -1$

univsos: Outline [Schweighofer 99]

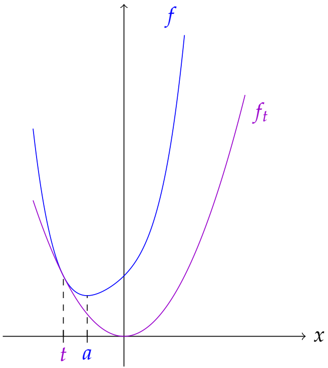
$f \in \mathbb{Q}[X]$ and $f > 0$

Minimizer a may not be in $\mathbb{Q}$ ...

Square-free decomposition:

$$f - f_t = g h^2$$

- $\deg g \leqslant \deg f - 2$
- $g > 0$
- Do it again on g



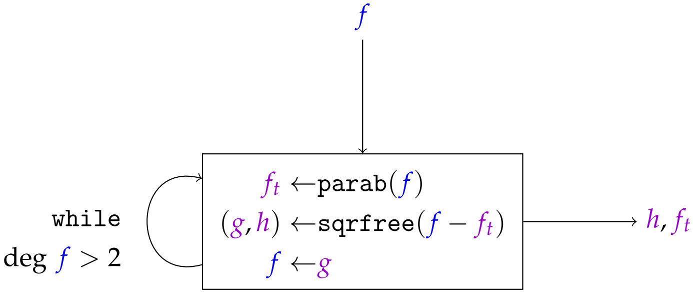
$f = 1 + X + X^2 + X^3 + X^4$

$f_t = X^2$

$f - f_t = (X^2 + 2X + 1)(X + 1)^2$

univsos: Algorithm [Schweighofer 99]

- **Input**: $f \geqslant 0 \in \mathbb{Q}[X]$ of degree $d \geqslant 2$
- **Output**: SOS decomposition with coefficients in $\mathbb{Q}$



univsos: Output Bitsize

Theorem

Let $0 < f \in \mathbb{Q}[X]$ with bitsize τ, $\deg f = d$.
The output bitsize τ' of `univsos` on f is $\mathcal{O}\left(\left(\frac{d}{2}\right)^{\frac{3d}{2}} \tau\right)$.

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Let $0 < f \in \mathbb{Q}[X]$ with bitsize τ, $\deg f = d$.
The output bitsize τ' of `univsos` on f is $\mathcal{O}\left(\left(\frac{d}{2}\right)^{\frac{3d}{2}}\tau\right)$.

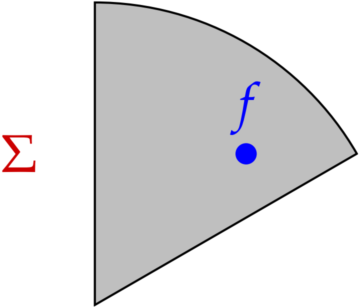
Proof.

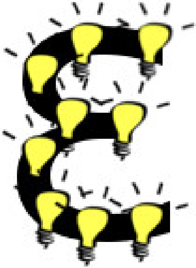
- Worst-case: $k = d/2$ induction steps

$$\implies \tau' = \mathcal{O}\left(\tau + k^3\tau + (k-1)^3 k^3 \tau + \cdots + (k!)^3 \tau\right)$$

□

intsos with $n \geqslant 1$: Perturbation





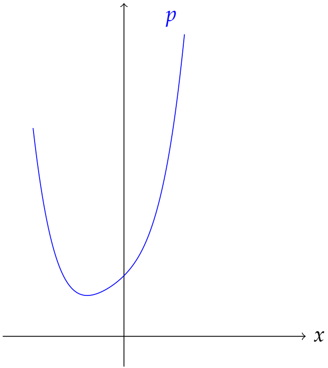
PERTURBATION idea

Approximate SOS Decomposition

$$f(X) - \varepsilon \sum_{\alpha \in \mathcal{P}/2} X^{2\alpha} = \tilde{\sigma} + u$$

intsos with $n = 1$ [Chevillard et. al 11]

$p \in \mathbb{Q}[X]$, $\deg p = d = 2k$, $p > 0$



$p = 1 + X + X^2 + X^3 + X^4$

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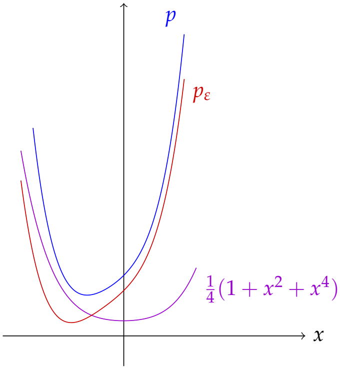
$p \in \mathbb{Q}[X]$, $\deg p = d = 2k$, $p > 0$

PERTURB: find $\varepsilon \in \mathbb{Q}$ s.t.

$$p_\varepsilon := p - \varepsilon \sum_{i=0}^{k} X^{2i} > 0$$



$$p = 1 + X + X^2 + X^3 + X^4$$

$$\varepsilon = \frac{1}{4}$$

$$p > \frac{1}{4}(1 + X^2 + X^4)$$

intsos with $n = 1$ [Chevillard et. al 11]

$p \in \mathbb{Q}[X]$, $\deg p = d = 2k$, $p > 0$

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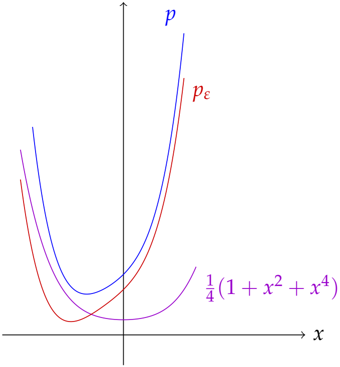
SDP Approximation:

$$p - \varepsilon \sum_{i=0}^{k} X^{2i} = \tilde{\sigma} + u$$

ABSORB: small enough u_i
$\implies \varepsilon \sum_{i=0}^{k} X^{2i} + u$ SOS



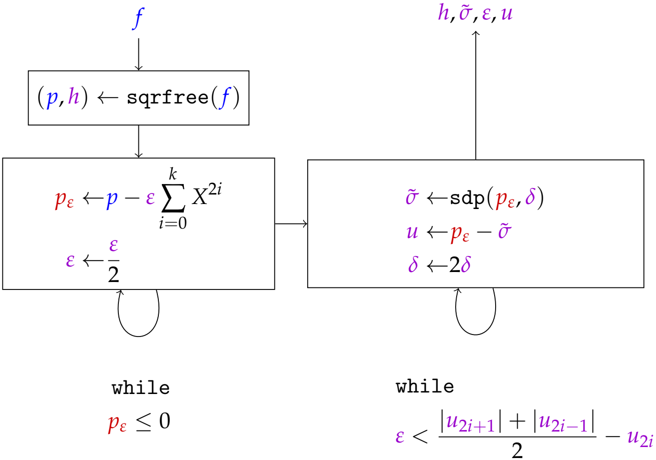
$$p = 1 + X + X^2 + X^3 + X^4$$

$$\varepsilon = \frac{1}{4}$$

$$p > \frac{1}{4}(1 + X^2 + X^4)$$

intsos with $n = 1$ and SDP Approximation

- **Input** $f \geqslant 0 \in \mathbb{Q}[X]$ of degree $d \geqslant 2$, $\varepsilon \in \mathbb{Q}^{>0}$, $\delta \in \mathbb{N}^{>0}$
- **Output**: SOS decomposition with coefficients in $\mathbb{Q}$



intsos with $n = 1$: Absorbtion

- $X = \frac{1}{2}\left[(X+1)^2 - 1 - X^2\right]$
- $-X = \frac{1}{2}\left[(X-1)^2 - 1 - X^2\right]$

intsos with $n = 1$: Absorbtion

💡 $X = \frac{1}{2}\left[(X+1)^2 - 1 - X^2\right]$

💡 $-X = \frac{1}{2}\left[(X-1)^2 - 1 - X^2\right]$

$$u_{2i+1} X^{2i+1} = \frac{|u_{2i+1}|}{2}\left[(X^{i+1} + \operatorname{sgn}(u_{2i+1}) X^i)^2 - X^{2i} - X^{2i+2}\right]$$

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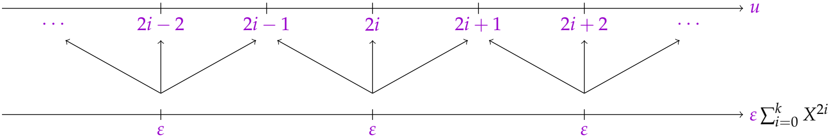

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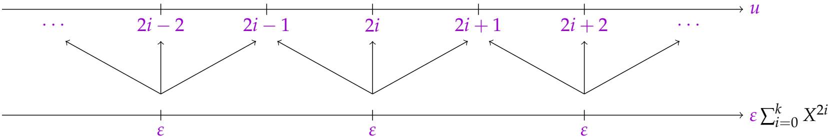
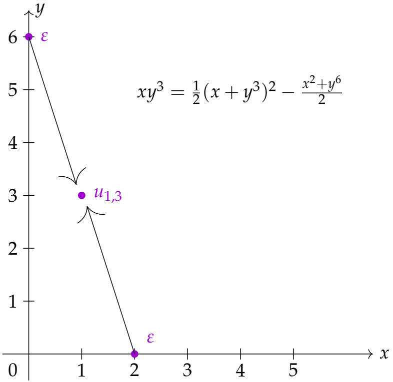


$$\varepsilon \geqslant \frac{|u_{2i+1}| + |u_{2i-1}|}{2} - u_{2i} \implies \varepsilon \sum_{i=0}^{k} X^{2i} + u \quad \text{SOS}$$

intsos with $n \geqslant 1$: Absorbtion

$$f(X) - \varepsilon \sum_{\alpha \in \mathcal{P}/2} X^{2\alpha} = \tilde{\sigma} + u$$

Choice of $\mathcal{P}$?



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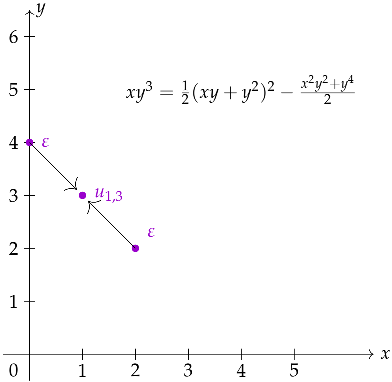

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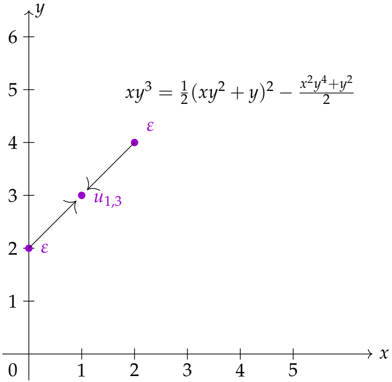
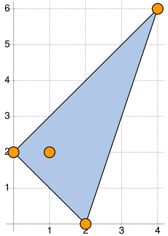


intsos with $n \geqslant 1$: Absorbtion

$$f(X) - \varepsilon \sum_{\alpha \in \mathcal{P}/2} X^{2\alpha} = \tilde{\sigma} + u$$

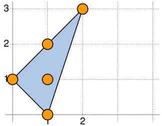
Choice of $\mathcal{P}$?

$f = 4x^4y^6 + x^2 - xy^2 + y^2$
spt(f) = $\{(4,6),(2,0),(1,2),(0,2)\}$



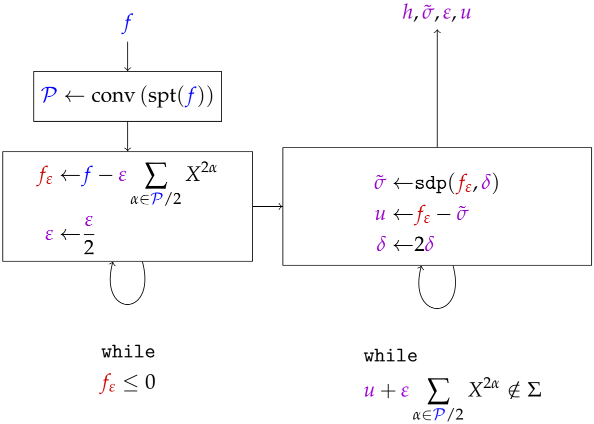
Newton Polytope $\mathcal{P} = \mathrm{conv}\,(\mathrm{spt}(f))$

Squares in SOS decomposition $\subseteq \frac{\mathcal{P}}{2} \cap \mathbb{N}^n$
[Reznick 78]



Algorithm `intsos`

- **Input** $f \in \mathbb{Q}[\mathbf{X}] \cap \mathring{\Sigma}[X]$ of degree d, $\varepsilon \in \mathbb{Q}^{>0}$, $\delta \in \mathbb{N}^{>0}$
- **Output**: SOS decomposition with coefficients in $\mathbb{Q}$



Algorithm `intsos`

Theorem (Exact Certification Cost in $\mathring{\Sigma}$)

$f \in \mathbb{Q}[X] \cap \mathring{\Sigma}[X]$ with $\deg f = d = 2k$ and bit size τ

$\Longrightarrow$ `intsos` terminates with SOS output of bit size $\boxed{\tau\, d^{\mathcal{O}(n)}}$

Algorithm intsos

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$f \in \mathbb{Q}[X] \cap \mathring{\Sigma}[X]$ with $\deg f = d = 2k$ and bit size τ

$\implies$ `intsos` terminates with SOS output of bit size $\boxed{\tau\, d^{\mathcal{O}(n)}}$

Proof.

- $\{\varepsilon \in \mathbb{R} : \forall \mathbf{x} \in \mathbb{R}^n, f(\mathbf{x}) - \varepsilon \sum_{\alpha \in \mathcal{P}/2} \mathbf{x}^{2\alpha} \geqslant 0\} \neq \emptyset$

Quantifier Elimination [Basu et. al 06] $\implies \tau(\varepsilon) = \tau\, d^{\mathcal{O}(n)}$

- # Coefficients in SOS output = size($\mathcal{P}/2$) = $\binom{n+k}{n} \leqslant d^n$

- Ellipsoid algorithm for SDP [Grötschel et. al 93] □

Algorithm `Polyasos`

f positive definite form has **Polya**'s representation:

$$f = \frac{\sigma}{(X_1 + \cdots + X_n)^{2D}} \quad \text{with } \sigma \in \Sigma[X]$$

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$$f\,(X_1 + \cdots + X_n)^{2D} \in \Sigma[X] \implies f\,(X_1 + \cdots + X_n)^{2D+2} \in \mathring{\Sigma}[X]$$

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Theorem (Exact Certification Cost of Polya's representations)

$f \in \mathbb{Q}[X]$ positive definite form with $\deg f = d$ and bit size τ

$$\implies D \leqslant 2^{\tau\, d^{\mathcal{O}(n)}} \quad \text{OUTPUT BIT SIZE} = \boxed{\tau\, D^{\mathcal{O}(n)}}$$

Algorithm Putinarsos

Assumption: $\exists i$ s.t. $g_i = 1 - \|X\|_2^2$

$f > 0$ on $\mathbf{K} := \{\mathbf{x} : g_j(\mathbf{x}) \geqslant 0\}$ has **Putinar**'s representation:

$$f = \sigma_0 + \sum_j \sigma_j \, g_j \quad \text{with } \sigma_j \in \Sigma[X] \,, \deg \sigma_j \leqslant 2D$$

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Theorem [M.-Safey El Din 18]

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💡 **ABSORBTION** as in Algorithm `intsos`: $u = f_\varepsilon - \tilde{\sigma}_0 - \sum_j \tilde{\sigma}_j \, g_j$

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ABSORBTION as in Algorithm `intsos`: $u = f_\varepsilon - \tilde{\sigma}_0 - \sum_j \tilde{\sigma}_j \, g_j$

OUTPUT BIT SIZE $= \boxed{\tau \, D^{\mathcal{O}(n)}}$

Algorithm `optsonc`: numerical steps

SONC (SUMS OF NONNEGATIVE CIRCUITS)

- **Input** $f = \sum_{\alpha} b_{\alpha} X^{\alpha}$ of degree d, $\hat{\delta} \in \mathbb{Q}^{>0}$, $\tilde{\delta} \in \mathbb{Q}^{>0}$
  Monomial squares = $\mathrm{MoSq}(f)$ Complement = $\mathrm{NoSq}(f)$

Algorithm optsonc: numerical steps

SONC (SUMS OF NONNEGATIVE CIRCUITS)

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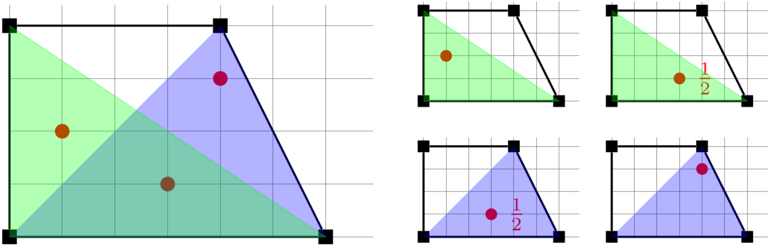
  Monomial squares = $\mathrm{MoSq}(f)$ Complement = $\mathrm{NoSq}(f)$

1. **Cover** each $\beta \in \mathrm{NoSq}(f)$ to get nonnegative circuit f_β
   $\Longrightarrow \lambda^\beta \geqslant 0$ with $\sum_{\alpha \in \mathrm{MoSq}(f)} \lambda_\alpha^\beta \cdot \alpha = \beta$



Algorithm `optsonc`: numerical steps

- **Input** $f = \sum_{\alpha} b_{\alpha} X^{\alpha}$, $\hat{\delta}$, $\tilde{\delta}$

2. **Numerical** resolution of GEOMETRIC PROGRAM

$$
\begin{aligned}
f_{\text{SONC}} = \min_{G > \mathbf{0}} \quad & \sum_{\beta \in \mathrm{NoSq}(f)} G_{\beta,\mathbf{0}} \\
\text{s.t.} \quad & \sum_{\beta \in \mathrm{NoSq}(f)} G_{\beta,\alpha} \leqslant b_{\alpha}\,, \quad \alpha \in \mathrm{MoSq}\,(f)\,, \alpha \neq \mathbf{0} \\
& \prod_{\alpha \in \mathrm{Cov}^{\beta}} \left(\frac{G_{\beta,\alpha}}{\lambda_{\alpha}^{\beta}} \right)^{\lambda_{\alpha}^{\beta}} = -b_{\beta}\,, \quad \beta \in \mathrm{NoSq}\,(f)
\end{aligned}
$$

Algorithm `optsonc`: symbolic steps

- **Input** $f = \sum_{\boldsymbol{\alpha}} b_{\boldsymbol{\alpha}} X^{\boldsymbol{\alpha}}$, $\hat{\delta}$, $\tilde{\delta}$

GEOMETRIC PROGRAM provides "IN THEORY"

$f_{\boldsymbol{\beta}} = \sum_{\boldsymbol{\alpha} \in \mathrm{Cov}^{\boldsymbol{\beta}}} G_{\boldsymbol{\beta},\boldsymbol{\alpha}} \cdot X^{\boldsymbol{\alpha}} + b_{\boldsymbol{\beta}} X^{\boldsymbol{\beta}}$, $f + \sum_{\boldsymbol{\beta}} G_{\boldsymbol{\beta},\mathbf{0}} - b_{\mathbf{0}} = \sum_{\boldsymbol{\beta}} f_{\boldsymbol{\beta}} \geqslant 0$

Algorithm `optsonc`: symbolic steps

- **Input** $f = \sum_{\alpha} b_{\alpha} X^{\alpha}$, $\hat{\delta}$, $\tilde{\delta}$

GEOMETRIC PROGRAM provides "IN THEORY"

$f_{\beta} = \sum_{\alpha \in \mathrm{Cov}^{\beta}} G_{\beta,\alpha} \cdot X^{\alpha} + b_{\beta} X^{\beta}$, $f + \sum_{\beta} G_{\beta,\mathbf{0}} - b_{\mathbf{0}} = \sum_{\beta} f_{\beta} \geqslant 0$

Tolerance $\tilde{\delta} \implies$ "IN PRACTICE" $\tilde{G}$ violates the constraints

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Tolerance $\tilde{\delta} \implies$ "IN PRACTICE" $\tilde{G}$ violates the constraints

3. **Rounding step** $\hat{G} \leftarrow \texttt{round}\left(\tilde{G}, \hat{\delta}\right)$

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Tolerance $\tilde{\delta} \implies$ "IN PRACTICE" $\tilde{G}$ violates the constraints

3. **Rounding step** $\hat{G} \leftarrow \texttt{round}\left(\tilde{G}, \hat{\delta}\right)$
4. **Projection step**

$$G_{\beta,\alpha} \leftarrow b_{\alpha} \cdot \hat{G}_{\beta,\alpha} / \sum_{\beta' \in \mathrm{NoSq}(f)} \hat{G}_{\beta',\alpha}$$

$$\tilde{G}_{\beta,\mathbf{0}} \leftarrow \lambda_{\mathbf{0}}^{\beta} \left(-b_{\beta} \cdot \prod_{\alpha \in \mathrm{Cov}^{\beta}} \left(\frac{\lambda_{\alpha}^{\beta}}{G_{\beta,\alpha}} \right)^{\lambda_{\alpha}^{\beta}} \right)^{\frac{1}{\lambda_{\mathbf{0}}^{\beta}}}$$

$$\hat{G}_{\beta,\mathbf{0}} \leftarrow \texttt{round}\uparrow \left(\tilde{G}_{\beta,\mathbf{0}}, \hat{\delta} \right)$$

Algorithm optsonc: symbolic steps

- **Input** $f = \sum_{\alpha} b_{\alpha} X^{\alpha}$, $\hat{\delta}$, $\tilde{\delta}$

GEOMETRIC PROGRAM provides "IN THEORY"

$f_{\beta} = \sum_{\alpha \in \mathrm{Cov}^{\beta}} G_{\beta,\alpha} \cdot X^{\alpha} + b_{\beta} X^{\beta}$, $f + \sum_{\beta} G_{\beta,0} - b_{0} = \sum_{\beta} f_{\beta} \geqslant 0$

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$$\hat{G}_{\beta,0} \leftarrow \texttt{round}\uparrow\left(\tilde{G}_{\beta,0}, \hat{\delta}\right)$$

$$\boxed{f \geqslant b_{0} - \sum_{\beta} \hat{G}_{\beta,0}}$$

SOS Benchmarks

- rounding-projection (SOS) [Peyrl-Parrilo]
- `RAGLib` (critical points) [Safey El Din]
- `SamplePoints` (CAD) [Moreno Maza-Alvandi et al.]

Id	n	d	`RealCertify` τ_1 (bits)	`RealCertify` t_1 (s)	`RoundProject` τ_2 (bits)	`RoundProject` t_2 (s)	`RAGLib` t_3 (s)	`CAD` t_4 (s)
f_{20}	2	20	745 419	110.	78 949 497	141.	0.16	0.03
M	3	8	17 232	0.35	18 831	0.29	0.15	0.03
f_2	2	4	1 866	0.03	1 031	0.04	0.09	0.01
f_6	6	4	56 890	0.34	475 359	0.54	598.	–
f_{10}	10	4	344 347	2.45	8 374 082	4.59	–	–

SONC vs SAGE

terms	bit size		time	
	`optsonc`	`optsage`	`optsonc`	`optsage`
6	432	1005	0.06	0.26
9	806	2696	0.19	0.66
12	1261	5568	0.37	1.29
20	2592	19203	0.64	4.00
24	3826	32543	0.97	6.66
30	5029	53160	1.34	10.58
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SONC vs SAGE

terms	bit size		time	
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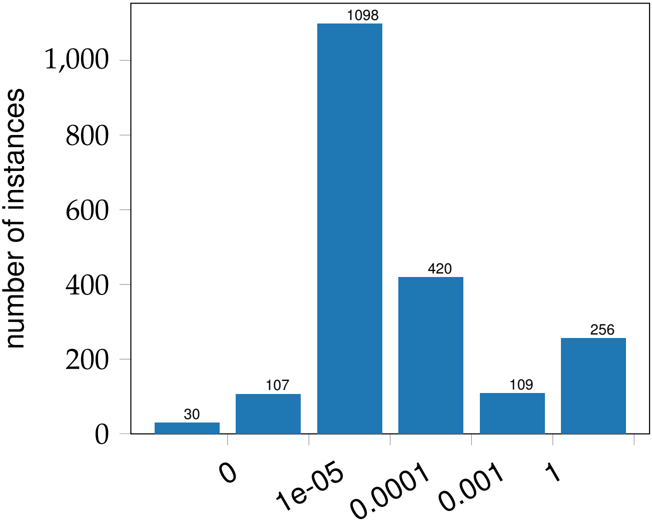
- "IN PRACTICE" `optsonc` faster and more concise than `optsage`
- "IN THEORY" `optsonc` less accurate than `optsage`

SONC: Gap between Numeric & Symbolic



Conclusion and Perspectives

Input f on $\mathbf{K}$ with $\deg f = d$ and bit size τ

Algo	Input	$\mathbf{K}$	OUTPUT BIT SIZE
`intsos`	$\mathring{\Sigma}$	$\mathbb{R}^n$	$\tau\, d^{\mathcal{O}(n)}$
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- How to handle degenerate situations?
- Arbitrary precision SDP/GP/REP solvers
- Extension to other relaxations

Crucial need for polynomial systems certification
Available PhD/Postdoc Positions

Thank you for your attention!

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https://homepages.laas.fr/vmagron

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