State polynomials for nonlinear Bell inequalities

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Joint work with Igor Klep, Jurij Volčič & Jie Wang

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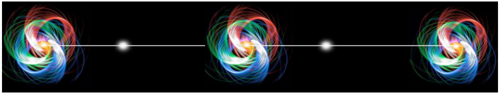

Motivation: Bell inequalities

State polynomials

NPA hierarchy for state polynomials

Back to Bell inequalities

Linear Bell inequalities

“Pillars” of quantum physics: violations imply that properties (e.g. entanglement) can’t be represented by classical physics

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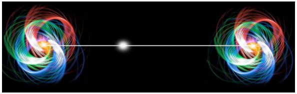

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Correlations = conditional joint probabilities:
$P(ab|st) = P$ (Alice, Bob answer a, b |Alice, Bob are asked s, t)

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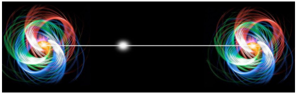

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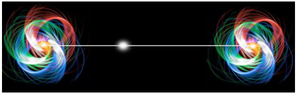

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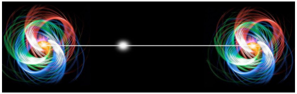

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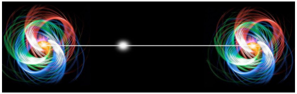

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Bell inequalities = linear in the correlations $P(ab|st)$ & marginals $P(a|s)$, $P(b|t)$, valid for all classical correlations

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Clauser-Horne-Shimony-Holt (CHSH) inequality is violated by quantum systems [Tsirelson 80]

$$-P_a(1|0) - P_b(1|0) + P(11|00) + P(11|01) + P(11|10) - P(11|11) \leqslant 0$$

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Inequality on eigenvalues of noncommutative polynomials

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- If $\mathcal{H}_A$ is finite-dim then a quantum state $\lambda \in \mathcal{S}(\mathcal{H}_A)$ = SDP matrix with unit trace

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- entangled states cannot be written as mixtures of product states

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$\hookrightarrow$ noncommutative analogue of [Lasserre 01] hierarchy for classical polynomial optimization

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Device-independent cryptography protocols: guaranteed by quantum mechanics laws without the need of trusted device

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Covariance in the quantum (commuting) model
$\operatorname{cov}(A, B) = \lambda(AB) - \lambda(A)\lambda(B)$ where λ is a state
$\hookrightarrow$ max over all states λ = **quantum violation**

Covariance Bell inequalities [Pozsgay et al. 17]

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The state λ is given by the integration w.r.t. a proba μ built from E [Schmüdgen 12]

$$\lambda(f) = \int_{\mathbb{R}^n} f \mathrm{d}\mu$$

Covariance Bell inequalities [Pozsgay et al. 17]

$$\begin{aligned}
\mathrm{cov}_{3322} = &\,\mathrm{cov}(A_1, B_1) + \mathrm{cov}(A_1, B_2) + \mathrm{cov}(A_1, B_3) \\
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$(A_1 A_2 A_3 / B_1 B_2 B_3)$: strategy where Alice and Bob deterministically output A_x and B_y for inputs x and y

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Concrete λ yields $\mathrm{cov}_{3322} = 5$

Covariance Bell inequalities [Pozsgay et al. 17]

$$\begin{aligned}\mathrm{cov}_{3322} = &\, \mathrm{cov}(A_1, B_1) + \mathrm{cov}(A_1, B_2) + \mathrm{cov}(A_1, B_3) \\ &+ \mathrm{cov}(A_2, B_1) + \mathrm{cov}(A_2, B_2) - \mathrm{cov}(A_2, B_3) \\ &+ \mathrm{cov}(A_3, B_1) - \mathrm{cov}(A_3, B_2)\end{aligned}$$

What is the max of cov_{3322}?

Concrete μ yields $\mathrm{cov}_{3322} = 4.5$

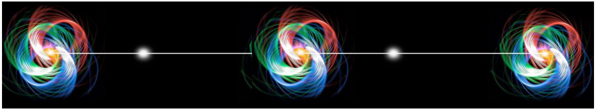
$$\mu = \frac{3}{8}(+++/+++) + \frac{3}{8}(--+/--+) + \frac{1}{4}(-+-/-+-)$$

$(A_1 A_2 A_3 / B_1 B_2 B_3)$: strategy where Alice and Bob deterministically output A_x and B_y for inputs x and y

Concrete λ yields $\mathrm{cov}_{3322} = 5$

What are the classical bound and the maximal quantum violation?

Bilocal Bell inequality [Tavakoli et al. 21-22]



Alice $\leftsquigarrow$ Source 1 $\rightsquigarrow$ Bob $\leftsquigarrow$ Source 2 $\rightsquigarrow$ Charlie

Observers hold particles from different sources and therefore a priori share no correlations

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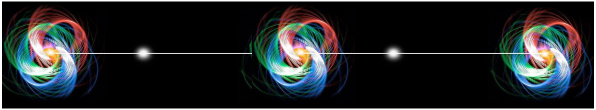

Alice $\leftsquigarrow$ Source 1 $\rightsquigarrow$ Bob $\leftsquigarrow$ Source 2 $\rightsquigarrow$ Charlie

Observers hold particles from different sources and therefore a priori share no correlations
A party that holds multiple shares originating from different sources can perform entangled measurements to a posteriori distribute entanglement between [$\cdots$] systems in the network

Bilocal Bell inequality [Tavakoli et al. 21-22]

Binary random variables A_i, B_j, C_k

$$\frac{1}{3} \sum_{i \in \{1,2,3\}} \Big(E(B_i C_i) - E(A_i B_i) \Big) - \sum_{\{i,j,k\} = \{1,2,3\}} E(A_i B_j C_k)$$

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+ similar factorization constraints & vanishing constraints

$$E(A_i) = E(B_i) = E(C_i) = 0 \quad \text{for } i \in \{1, 2, 3\}$$
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Concrete proba μ yields 3 — What is the classical max?

Concrete state λ yields 4 — What is the quantum violation?

Moment polynomials

Elements of $\mathscr{M}[x]$

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at a proba μ on $\mathbb{R}^2$ with fourth order moments and a pair $X = (X_1, X_2) \in \mathbb{R}^2$, f evaluates as

$$f(\mu, X) = X_1 X_2 \int x_1 x_2^3 \, \mathrm{d}\mu - X_2^2 \left(\int x_1^2 \, \mathrm{d}\mu \right)^3 + X_2 - 2$$

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at a state $\lambda : \mathcal{B}(\mathcal{H}) \to \mathbb{R}$ and a pair $X = (X_1, X_2) \in \mathcal{B}(\mathcal{H})^2$, f evaluates as

$$f(\lambda, X) = X_1 X_2 \lambda(X_1 X_2^3) - X_2^2 \left(\lambda(X_1^2)\right)^3 + X_2 - 2$$

Related business

Trace polynomials $\mathrm{tr}(x_1^2)\,\mathrm{tr}(x_2) + \mathrm{tr}(x_2)$ with $\mathrm{tr}(uv) = \mathrm{tr}(vu)$

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Proba is a state $\Rightarrow$ moment polynomials are state polynomials

Moment polynomial optimization

Objective function $f \in \mathscr{M}[x]$

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for Bell $f = \frac{1}{3} \sum_{i \in \{1,2,3\}} \left(\mathtt{m}(b_i c_i) - \mathtt{m}(a_i b_i) \right) - \sum_{\{i,j,k\} = \{1,2,3\}} \mathtt{m}(a_i b_j c_k) \in \mathscr{M}$

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Two types of constraints:

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NPA hierarchy to optimize over $\mathscr{M}[x]$

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State polynomial optimization

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for Bell $f = \frac{1}{3} \sum_{i \in \{1,2,3\}} \Big(\tau(b_i c_i) - \tau(a_i b_i) \Big) - \sum_{\{i,j,k\} = \{1,2,3\}} \tau(a_i b_j c_k) \in \mathscr{S}$

Two types of constraints:

- $s_1(X) \geqslant 0$ with $s_1 \in \mathbb{R}\langle x \rangle \qquad X \in \mathcal{B}(\mathcal{H})^n$

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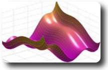
💡 NPA hierarchy to optimize over $\mathscr{S}\langle x \rangle$

Hierarchies for polynomial optimization

NP-hard NON CONVEX problem $f_{\min} = \inf f(X)$

Theory

(Primal)		(Dual)
$\inf \int f \, d\mu$		$\sup \lambda$
with μ proba $\Rightarrow$	**INFINITE LP**	$\Leftarrow$ with $f - \lambda \geqslant 0$

Hierarchies for polynomial optimization

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Practice

(Primal **Relaxation**)

(Dual **Strengthening**)

moments $\int X^\alpha \, d\mu$

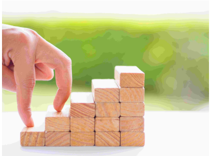
$f - \lambda =$ **sum of squares**

finite number $\Rightarrow$ **SDP** $\Leftarrow$ **fixed** degree

LASSERRE'S HIERARCHY of **CONVEX PROBLEMS** $\uparrow f_{\min}$

[Lasserre '01]

degree r & n vars $\Longrightarrow \binom{n+2r}{n}$ **SDP** VARIABLES



A simple example

$f_{\min} = \min f(X)$ over $K(S)$

Semialgebraic set $K(S) = \{X \in \mathbb{R}^n : s(X) \geqslant 0, \quad s \in S\}$

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$$\overbrace{X_1 X_2}^{f} =$$

$$-\frac{1}{8} + \overbrace{\frac{1}{2}\left(X_1 + X_2 - \frac{1}{2}\right)^2}^{\sigma_0} + \overbrace{\frac{1}{2}}^{\sigma_1} \overbrace{X_1(1 - X_1)}^{s_1} + \overbrace{\frac{1}{2}}^{\sigma_2} \overbrace{X_2(1 - X_2)}^{s_2}$$

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Quadratic module: $\mathrm{QM}(S)_r = \left\{ \sigma_0 + \sum_j \sigma_j s_j, \ \deg \sigma_j \, s_j \leqslant 2r \right\}$

Hierarchies for polynomial optimization

$$f_{\min} = \min_{X \in K(S)} f(X)$$

- $\mathcal{P}(K(S))$: proba on $K(S)$
- quadratic module $\mathrm{QM}(S) = \left\{ \sigma_0 + \sum_j \sigma_j s_j, \text{ with } \sigma_j \text{ SOS} \right\}$

Infinite-dimensional linear programs (LP)

(Primal)		(Dual)
$\inf \quad \int_{K(S)} f \, d\mu$	$=$	$\sup \quad \lambda$
s.t. $\quad \mu \in \mathcal{P}(K(S))$		s.t. $\quad \lambda \in \mathbb{R}$
		$f - \lambda \in \mathrm{QM}(S)$

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$$f_{\min} = \min_{X \in K(S)} f(X)$$

- Pseudo-moment sequences $\mathbf{y}$ up to order r
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Hierarchies for polynomial optimization

Moment matrices are indexed by monomials

$$\mathbf{M}_1(\mathbf{y}) = \begin{array}{cc} & \begin{array}{cccc} 1 & & x_1 & x_2 \end{array} \\ \begin{array}{c} 1 \\ \\ x_1 \\ x_2 \end{array} & \begin{pmatrix} 1 & | & y_{10} & y_{01} \\ & - & - & - \\ y_{10} & | & y_{20} & y_{11} \\ y_{01} & | & y_{11} & y_{02} \end{pmatrix} \end{array}$$

Hierarchies for polynomial optimization

Theorem [Putinar 93, Lasserre 01]: positive polynomials

For $f \in \mathbb{R}[x]$, $S \subseteq \mathbb{R}[x]$, if $\underbrace{N}_{>0} - \sum_i x_i^2 \in \mathrm{QM}(S)$ then

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✓ Can be computed with SDP solvers (`CSDP`, `SDPA`, `MOSEK`)

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💡 moment matrices & quadratic modules

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Moment matrices are (slightly) more complicated than in $\mathbb{R}[x]$

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 & \begin{array}{cccccc} 1 & & x_1 & x_2 & \mathtt{m}_{10} & \mathtt{m}_{01} \end{array} \\
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 & - & - & - & - & - \\
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y_{0100} & | & y_{1100} & y_{0200} & y_{0110} & y_{0101} \\
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Quadratic module $\mathrm{QM}(S_1, S_2)$ is also more complicated

$$\sum p^2 \mathtt{m}(q^2 s): \quad s \in \{1\} \cup S_1 \quad p, q \in \mathscr{M}[x]$$
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$\tau(x_1 x_2) = \tau(x_2 x_1)$ but $\tau(x_1^2 x_2) \neq \tau(x_1 x_2 x_1)$ in general

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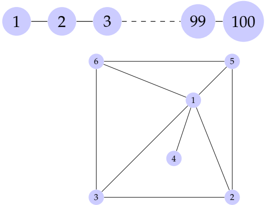

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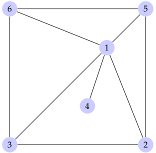

Term sparsity: few terms in f

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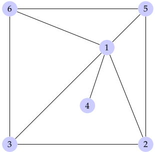
Correlative sparsity: few variable products in f

$\rightsquigarrow f = x_1 x_2 + x_2 x_3 + \cdots + x_{99} x_{100}$



Term sparsity: few terms in f

$\rightsquigarrow f = x_1^{99} x_2 + x_1 x_2^{100}$

More efficient NPA hierarchies

“SPARSE” cost f and constraints

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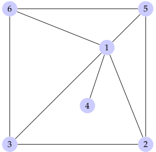

Term sparsity: few terms in f
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Universal algebras of binary observables:

More efficient NPA hierarchies

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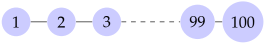

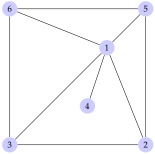

Term sparsity: few terms in f

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Universal algebras of binary observables:

$\rightsquigarrow$ group G of constraints $x_i^2 = 1 \quad x_i x_j = x_j x_i$

More efficient NPA hierarchies

"SPARSE" cost f and constraints

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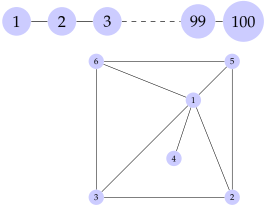

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Index SDP matrices by $H \subseteq G$ generated by the support of f

More efficient NPA hierarchies

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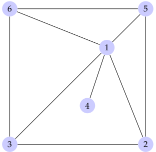

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PERFORMANCE  VS  ACCURACY

More efficient NPA hierarchies

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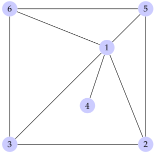

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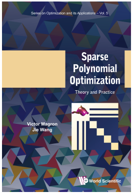
PERFORMANCE  VS  ACCURACY

Tons of applications: computer arithmetic, deep learning, entanglement, optimal power-flow, analysis of dynamical systems, matrix ranks



Back to Bell inequalities

Binary A_i, B_j

$$\begin{aligned}\mathrm{cov}_{3322} = {} & \mathrm{cov}(A_1, B_1) + \mathrm{cov}(A_1, B_2) + \mathrm{cov}(A_1, B_3) \\ & + \mathrm{cov}(A_2, B_1) + \mathrm{cov}(A_2, B_2) - \mathrm{cov}(A_2, B_3) \\ & + \mathrm{cov}(A_3, B_1) - \mathrm{cov}(A_3, B_2)\end{aligned}$$

Back to Bell inequalities

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NPA hierarchy for $\mathscr{M}$ and $r = 2$: SDP with $4\,146$ variables $\qquad f_2 = 4.5$

Back to Bell inequalities

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NPA hierarchy for $\mathscr{M}$ and $r = 2$: SDP with $4\,146$ variables $f_2 = 4.5$
same local bound as [Pozsgay et al. 17] classical bound = $f_{\max} = 4.5$

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NPA hierarchy for $\mathscr{S}$ and $r=2$: $f_2 = 5$

Back to Bell inequalities

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NPA hierarchy for $\mathscr{M}$ and $r = 2$: SDP with $4\,146$ variables $f_2 = 4.5$
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NPA hierarchy for $\mathscr{S}$ and $r = 2$: $f_2 = 5$
same local bound as [Pozsgay et al. 17] quantum bound = $f_{\max} = 5$

Back to Bell inequalities

Binary A_i, B_j, C_k

$$\frac{1}{3}\sum_{i\in\{1,2,3\}}\Big(E(B_iC_i) - E(A_iB_i)\Big) - \sum_{\{i,j,k\}=\{1,2,3\}} E(A_iB_jC_k)$$

Back to Bell inequalities

Binary A_i, B_j, C_k

$$\frac{1}{3} \sum_{i \in \{1,2,3\}} \Big(E(B_i C_i) - E(A_i B_i) \Big) - \sum_{\{i,j,k\} = \{1,2,3\}} E(A_i B_j C_k)$$

satisfying bilocality constraints

$$E(A_1 A_2 A_3 \; C_1 C_2 C_3) = E(A_1 A_2 A_3) \; E(C_1 C_2 C_3)$$

+ similar factorization constraints

Back to Bell inequalities

Binary A_i, B_j, C_k

$$\frac{1}{3} \sum_{i \in \{1,2,3\}} \Big(E(B_i C_i) - E(A_i B_i) \Big) - \sum_{\{i,j,k\} = \{1,2,3\}} E(A_i B_j C_k)$$

satisfying bilocality constraints

$$E(A_1 A_2 A_3 \; C_1 C_2 C_3) = E(A_1 A_2 A_3) \; E(C_1 C_2 C_3)$$

+ similar factorization constraints & vanishing constraints

$$E(A_i) = E(B_i) = E(C_i) = 0 \quad \text{for } i \in \{1, 2, 3\}$$
$$E(A_i B_j) = E(B_i C_j) = 0 \quad \text{for } i \neq j$$
$$E(A_i B_j C_k) = 0 \quad \text{for } |\{i, j, k\}| \leq 2$$

Back to Bell inequalities

[Tavakoli et al. 21-22] local classical bound of 3

Back to Bell inequalities

[Tavakoli et al. 21-22] local classical bound of 3

$$\sup \frac{1}{3} \sum_{i \in \{1,2,3\}} \Big(\mathtt{m}(b_i c_i) - \mathtt{m}(a_i b_i)\Big) - \sum_{\{i,j,k\}=\{1,2,3\}} \mathtt{m}(a_i b_j c_k)$$

s.t.

$$\mathtt{m}(a_1 a_2 a_3 \; c_1 c_2 c_3) = \mathtt{m}(a_1 a_2 a_3) \, \mathtt{m}(c_1 c_2 c_3)$$

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$\mathtt{m}(a_i b_j c_k) = 0 \quad$ for $\; |\{i, j, k\}| \leq 2$

Back to Bell inequalities

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$r = 3$: SDP with $31\,017$ variables $\qquad f_3 = 4$

Back to Bell inequalities

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$r = 3$: SDP with $31\,017$ variables $f_3 = 4$

We extracted a local classical bound of 4 💡 classical bound = $f_{\max} = 4$

Back to Bell inequalities

[Tavakoli et al. 21-22] local quantum bound of 4

Back to Bell inequalities

[Tavakoli et al. 21-22] local quantum bound of 4

$$\sup \ \frac{1}{3} \sum_{i \in \{1,2,3\}} \Big(\tau(b_i c_i) - \tau(a_i b_i) \Big) - \sum_{\{i,j,k\} = \{1,2,3\}} \tau(a_i b_j c_k)$$

s.t.

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Back to Bell inequalities

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$r = 3$: SDP with $3\,018$ constraints (few seconds) $f_3 = 4.46$

Back to Bell inequalities

[Tavakoli et al. 21-22] local quantum bound of 4

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$r = 3$: SDP with $3\,018$ constraints (few seconds) $\quad f_3 = 4.46$

$r = 4$: SDP with $64\,878$ constraints (few hours) $\quad f_4 = 4.38$

Back to Bell inequalities

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$r = 3$: SDP with $3\,018$ constraints (few seconds) $f_3 = 4.46$
$r = 4$: SDP with $64\,878$ constraints (few hours) $f_4 = 4.38$
$r = 5$: SDP with $1\,352\,093$ constraints (one week) $f_5 = 4.37$

Back to Bell inequalities

[Tavakoli et al. 21-22] local quantum bound of 4

$$\sup \ \frac{1}{3} \sum_{i \in \{1,2,3\}} \Big(\tau(b_i c_i) - \tau(a_i b_i) \Big) - \sum_{\{i,j,k\} = \{1,2,3\}} \tau(a_i b_j c_k)$$

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$r = 3$: SDP with $3\,018$ constraints (few seconds) $f_3 = 4.46$

$r = 4$: SDP with $64\,878$ constraints (few hours) $f_4 = 4.38$

$r = 5$: SDP with $1\,352\,093$ constraints (one week) $f_5 = 4.37$

We still don't know the quantum bound $f_{\max}$!

Conclusion

Conclusion

Positivity certificates for moment and state polynomials under compact polynomial inequality constraints

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NPA hierarchies to certify classical and quantum bounds of nonlinear Bell inequalities

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NPA hierarchies to certify classical and quantum bounds of nonlinear Bell inequalities

Hilbert-Artin analogues (theoretical results not explained in this talk):

Conclusion

Positivity certificates for moment and state polynomials under compact polynomial inequality constraints

NPA hierarchies to certify classical and quantum bounds of nonlinear Bell inequalities

Hilbert-Artin analogues (theoretical results not explained in this talk):

State polynomials, positive over all matrices and matricial states, are sums of squares with denominators

Conclusion

Positivity certificates for moment and state polynomials under compact polynomial inequality constraints

NPA hierarchies to certify classical and quantum bounds of nonlinear Bell inequalities

Hilbert-Artin analogues (theoretical results not explained in this talk):

State polynomials, positive over all matrices and matricial states, are sums of squares with denominators

Moment polynomials positive on measures are sums of squares, up to arbitrarily small perturbation (generalization of [Lasserre 06])

Open EU PhD/Postdoc positions

TENORS
Tensor modEliNg, geOmetRy and optimiSation
Marie Skłodowska-Curie Doctoral Network
2024-2027

Tensors are nowadays ubiquitous in many domains of applied mathematics, computer science, signal processing, data processing, machine learning and in the emerging area of quantum computing. TENORS aims at fostering cutting-edge research in tensor sciences, stimulating interdisciplinary and intersectoriality knowledge developments between algebraists, geometers, computer scientists, numerical analysts, data analysts, physicists, quantum scientists, and industrial actors facing real-life tensor-based problems.

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(recruitment expected around Oct. 2024)

Scientific coord: B. Mourrain
Adm. manager: Linh Nguyen

COMPUTE

nonCommutative polynOMial oPtimisation for qUanTum nEtworks

Thank you for listening!

- Klep, M. & Volčič. Sums of squares certificates for polynomial moment inequalities. arXiv:2306.05761
- Klep, M., Volčič & Wang. State polynomials: positivity, optimization and nonlinear Bell inequalities. *Math. Programming*, arXiv:2301.12513
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- Huber, Klep, M. & Volčič. Dimension-free entanglement detection in multipartite Werner states. *Communications in Math. Physics*, arXiv:2108.08720
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- Huber. Positive maps and trace polynomials from the symmetric group. Journal of Mathematical Physics, 2021
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- Beckermann, Putinar, Saff & Stylianopoulos. Perturbations of Christoffel–Darboux Kernels: Detection of Outliers. Foundations of Computational Mathematics, 2021
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- Belinschi, Magron & Vinnikov. Noncommutative Christoffel-Darboux Kernels, Transactions of the AMS, 2022