Au sujet du problème de transport martingale

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The martingale transport problem

Definition of the shadow transports

Stability of the solutions
Outline

1. The martingale transport problem
2. Definition of the shadow transports
3. Stability of the solutions
The martingale problem

Minimize

$$\pi \in \text{Mart}(\mu, \nu) \mapsto \iint c(x, y) d\pi(x, y)$$

For a given cost function $c$, one is looking for:

- Existence of minimizers? Uniqueness?
- Geometry/shape of the minimizers?
- Stability of the value function/minimizers?
Origin of the problem

- Hobson, Klimmek ’15, Neuberger ’12:
  
  “Model independent finance/robust price bounds”

For a given risk-neutral martingale \((X_n)_{n \in \mathbb{N}}\), one observes on the market the prices (at time zero) of European options

\[ t \in \mathbb{R} \mapsto U_n(t) := \mathbb{E}(|X_n - t|_\pm) \]

Differentiating twice in \(t\) one obtains the family \((\mu_n)_{n \in \mathbb{N}}\) where \(\mu_n = \text{Law}(X_n)\).

However, \(\text{Law}((X_n)_n)\) remains unknown.
Origin of the problem

**Hobson, +Klimmek ’15, +Neuberger ’12:**

"Model independent finance/robust price bounds"

The correct price for an exotic option $c$ is $\mathbb{E}(c(X_1, \ldots, X_n))$ has to be computed with respect to the risk-neutral martingale law.

Hobson and coauthors try to obtain upper/lower bounds for

$$\mathbb{E}(|X_2 - X_1|)$$

where $(X_1, X_2)$ is a martingale with marginals $\mu_1$ and $\mu_2$. 
Origin of the problem

- **Beiglböck – Henry-Labordère – Penkner (2013):**
  
  Primal ("pricing") and dual ("hedging") martingale optimal transport problems in discrete time.

  
  Hedging from a stochastic control perspective.
Definition 1 (Martingale transport plan)

A *martingale transport* is the distribution of a two-times martingale $(X_i)_{i \in \{1,2\}}$. 
Martingale transport

Definition 2 (Martingale transport plan)

\( \pi = \text{Law}(X, Y) \) is a martingale transport \( \text{IFF} \) the \textbf{transport kernel} 

\[ k : x \mapsto \text{Law}(Y|X = x) \]

is a \textbf{dilation} a.s., that is

\[ \int y k(x, dy) = \mathbb{E}(Y|X = x) = x. \]
Mart(\mu, \nu) can be empty
The convex order

**Strassen Theorem (1965)**

Let $\mu$ and $\nu$ be two positive measures

$$\text{Mart}(\mu, \nu) \neq \emptyset \iff \forall \phi \text{ convex}, \int \phi \, d\mu \leq \int \phi \, d\nu.$$  

notation: $\mu \leq_C \nu$
Thus we assume $\mu \preceq_C \nu$.

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The martingale problem

Théorème (Beiglböck–J. 2016)

Let $\mu, \nu \in \mathcal{P}_3(\mathbb{R})$ be probability measures and $\pi \in \text{Mart}(\mu, \nu)$. The following statements are equivalent:

1. The martingale transport $\pi$ is optimal for the cost $c(x, y) = (y - x)^3$
2. The martingale transport $\pi$ is concentrated on a martingale-monotone (see the figure) set of routes $R$, 
3. The martingale transport $\pi$ is the left-curtain transport (defined later)

Figure: $(x, y^-) \in R$, $(x, y^+) \in R$ and $(x', y') \in R$ is forbidden for a martingale-monotone $R$. 

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The martingale transport problem

The initial pattern

\[ c(x', y') + [\lambda c(x, y^+) + (1 - \lambda)c(x, y^-)] \]
The improved pattern

\[ c(x, y') + [\lambda c(x', y^+) + (1 - \lambda) c(x', y^-)] \]
Comparaison of the patterns

\[ c(x', y') + [\lambda c(x, y^+) + (1 - \lambda)c(x, y^-)] \]

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The martingale problem

Théorème (Beiglböck–J. 2016)

Let $\mu, \nu \in P_3(\mathbb{R})$ be probability measures and $\pi \in \text{Mart}(\mu, \nu)$. The following statements are equivalent:

1. The martingale transport $\pi$ is optimal for the cost $\partial^3_{xyy} c < 0$,
2. The martingale transport $\pi$ is concentrated on a martingale-monotone (see the figure) set of routes $R$,
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**Figure:** $(x, y^-) \in R$, $(x, y^+) \in R$ and $(x', y') \in R$ is forbidden for a martingale-monotone $R$. 
Martingale transport that avoids the forbidden crossing
Martingale transport that avoids the forbidden crossing
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Notion of Shadow
Associativity of the shadows
Left-curtain transport for uniform measures

\[ \text{Law}(X|U \leq \alpha) \]

\[ \text{Law}(Y|U \leq \alpha) \]
Left-curtain transport for uniform measures
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Other parametrizations
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Other parametrizations
There is a problem

**Question**

How to transport $\mathbb{P}(X \in \cdot \mid U = \alpha)$ to $\mathbb{P}(Y \in \cdot \mid U = \alpha)$?

Beiglböck-J (2021): there is a unique martingale way to do that!
There is a problem

**Question**

How to transport $\mathbb{P}(X \in \cdot | U = \alpha)$ to $\mathbb{P}(Y \in \cdot | U = \alpha)$?

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Definition of the shadow transports

Figure: The Kellerer dilation used to transport shadow increments
Properties of the shadow transports

1. Also solutions for transport problems (Multi-OT, WOT)

2. Interpretation as Skorkhod embedding.
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Stability as usual

Theorem (J., 2014, two proofs)

The map

\[(\mu, \nu) \in \{\mu \preceq_C \nu\} \mapsto \pi := \text{LCur}t(\mu, \nu) \in \text{Mart}(\mu, \nu)\]

is continuous.

Let $W$ be the Kantorovich ($=L^1$-Wasserstein) distance.

Theorem (J., 2014)

If $S^\nu(\mu)$ and $S^{\nu'}(\mu')$ are defined and $\mu(\mathbb{R}) = \mu'(\mathbb{R})$, $\nu(\mathbb{R}) = \nu'(\mathbb{R})$, then

\[W(S^\nu(\mu), S^{\nu'}(\mu')) \leq W(\mu, \mu') + 2W(\nu, \nu').\]
More recent stability results

- **Guo – Obłój (19):** Other costs. Stability of the value.
- **Wiesel (’19+):** Stability of the value function for general continuous costs.
- **Pammer – Backhoff-Veraguas (’22):** Stability for continuous costs $c$ uniform limits of $(c_n)_n$.
- **Beiglböck – Pammer – Jourdain – Margheriti (’21+):** Every optimal $\pi$ can be approached by $(\pi_n)_n$ almost optimal where $(\mu_n, \nu_n)_n$ are given. Stability for l.s.c continuous costs.
Non stability for dimension $d \geq 2$

- **Ghoussoub - Kim - Lim (’19):** MOT in dimension $d \geq 2$

**Theorem (Brückerhoff – J., 2021+)**

There exists $\mu_n \to \mu \in \mathcal{P}(\mathbb{R}^2)$, $\nu_n \to \nu \in \mathcal{P}(\mathbb{R}^2)$ and $(\pi_n)_n \in \text{Mart}(\mu_n, \nu_n)$ a sequence of optimal transport plans such that the limit $\pi$ exists and is not optimal.
The picture-proof
The picture-proof
The picture-proof

\[ y \]

\[ \rightarrow x \]