Au sujet du problème de transport martingale

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Journée MAS - MODE 2e édition, mars 2022

Outline

1. The martingale transport problem
2. Definition of the shadow transports
3. Stability of the solutions

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The martingale problem

The martingale transport problem

Minimize

$$\pi \in \mathrm{Mart}(\mu,\nu) \mapsto \iint c(x,y)\mathrm{d}\pi(x,y)$$

For a given cost function c, one is looking for:

- Existence of minimizers? Uniqueness?
- Geometry/shape of the minimizers?
- Stability of the value function/minimizers?

Origin of the problem

- **Hobson, +Klimmek '15, +Neuberger '12:**

  "Model independent finance/robust price bounds"

  For a given risk-neutral martingale $(X_n)_{n\in\mathbb{N}}$, one observes on the market the prices (at time zero) of European options

$$t \in \mathbb{R} \longmapsto U_n(t) := \mathbb{E}(|X_n - t|_\pm)$$

  Differentiating twice in t one obtains the family $(\mu_n)_{n\in\mathbb{N}}$ where $\mu_n = \mathrm{Law}(X_n)$.

  However, $\mathrm{Law}((X_n)_n)$ remains unknown.

Origin of the problem

- **Hobson, +Klimmek '15, +Neuberger '12:**

  "Model independent finance/robust price bounds"

  The correct price for an exotic option c is $\mathbb{E}(c(X_1, \ldots, X_n))$ has to be computed

  with respect to the risk-neutral martingale law.

  Hobson and coauthors try to obtain upper/lower bounds for

  $$\mathbb{E}(|X_2 - X_1|)$$

  where (X_1, X_2) is a martingale with marginals μ_1 and μ_2.

Origin of the problem

- **Beiglböck – Henry-Labordère – Penkner (2013):**

  Primal ("pricing") and dual ("hedging") martingale optimal transport problems in discrete time.

- **Galichon – Henry-Labordère – Touzi (2014):**

  Hedging from a stochastic control perspective.

Martingale transport

Definition 1 (Martingale transport plan)

A *martingale transport* is the distribution of a two-times martingale $(X_i)_{i\in\{1,2\}}$.

Martingale transport

Definition 2 (Martingale transport plan)

$\pi = \text{Law}(X, Y)$ is a martingale transport IFF the **transport kernel**

$$k : x \mapsto \text{Law}(Y|X = x)$$

is a **dilation** a.s, that is

$$\int y\, k(x, dy) = \mathbb{E}(Y|X = x) = x.$$

Mart(μ, ν) can be empty

The convex order

Strassen Theorem (1965)

Let μ and ν be two positive measures

$$\mathrm{Mart}(\mu,\nu) \neq \emptyset \iff \underbrace{\forall \varphi \text{ convex}, \int \varphi \, d\mu \leq \int \varphi \, d\nu}_{\text{notation: } \mu \preceq_C \nu}.$$

The martingale problem

Thus we assume $\mu \preceq_C \nu$.

The martingale transport problem

Minimize

$$\pi \in \mathrm{Mart}(\mu,\nu) \mapsto \iint c(x,y)\mathrm{d}\pi(x,y)$$

For a given cost function c, one is looking for:

- Existence of minimizers? Uniqueness?
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The martingale problem

Théorème (Beiglböck–J. 2016)

Let $\mu, \nu \in \mathcal{P}_3(\mathbb{R})$ be probability measures and $\pi \in \mathrm{Mart}(\mu,\nu)$. The following statements are equivalent:

1. The martingale transport π is optimal for the cost $c(x,y) = (y-x)^3$
2. The martingale transport π is concentrated on a martingale-monotone (*see the figure*) set of routes R ,
3. The martingale transport π is **the left-curtain transport** (*defined later*)

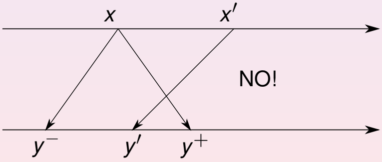
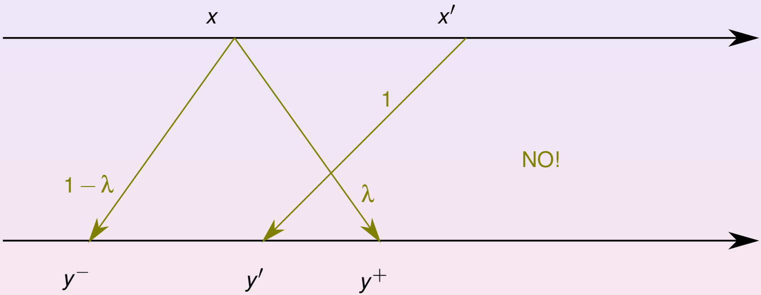


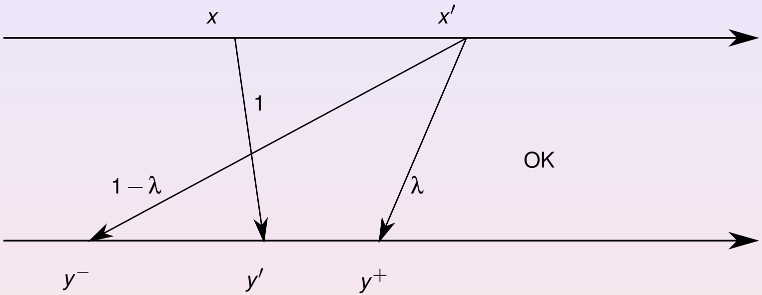
Figure: $(x,y^-) \in R$, $(x,y^+) \in R$ and $(x',y') \in R$ is forbidden for a martingale-monotone R.

The initial pattern



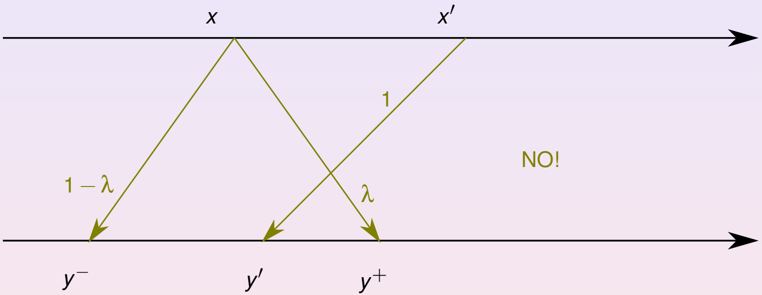
$$c({\color{red}x'},y')+[\lambda c({\color{blue}x},y^+)+(1-\lambda)c({\color{blue}x},y^-)]$$

The improved pattern



$$c(x, y') + [\lambda c(x', y^+) + (1-\lambda) c(x', y^-)]$$

Comparaison of the patterns



$$c({\color{red}x'},y')+[\lambda c({\color{blue}x},y^+)+(1-\lambda)c({\color{blue}x},y^-)]$$

$$c({\color{blue}x},y')+[\lambda c({\color{red}x'},y^+)+(1-\lambda)c({\color{red}x'},y^-)]$$

The martingale problem

Théorème (Beiglböck–J. 2016)

Let $\mu, \nu \in \mathcal{P}_3(\mathbb{R})$ be probability measures and $\pi \in \mathrm{Mart}(\mu, \nu)$. The following statements are equivalent:

1. The martingale transport π is optimal for the cost $\partial^3_{xyy} c < 0$,
2. The martingale transport π is concentrated on a martingale-monotone (*see the figure*) set of routes R ,
3. The martingale transport π is **the left-curtain transport** (*defined later*)

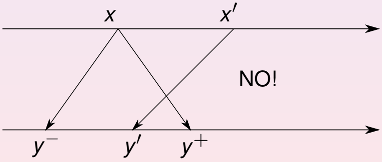
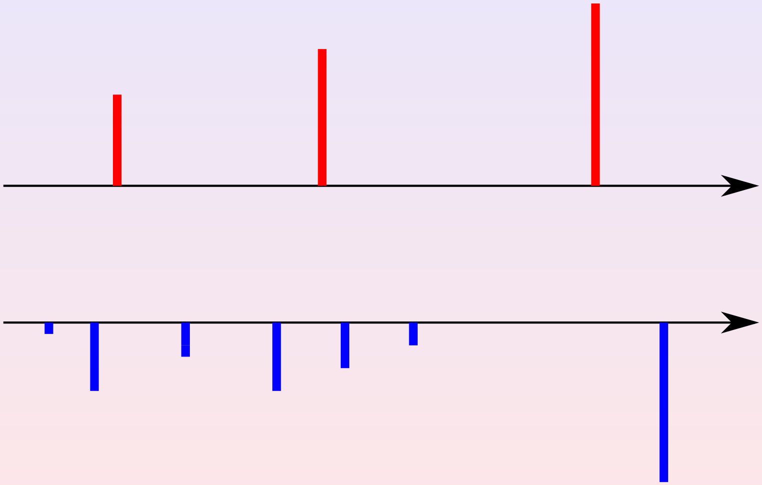


Figure: $(x, y^-) \in R$, $(x, y^+) \in R$ and $(x', y') \in R$ is forbidden for a martingale-monotone R.

Martingale transport that avoids the forbidden crossing



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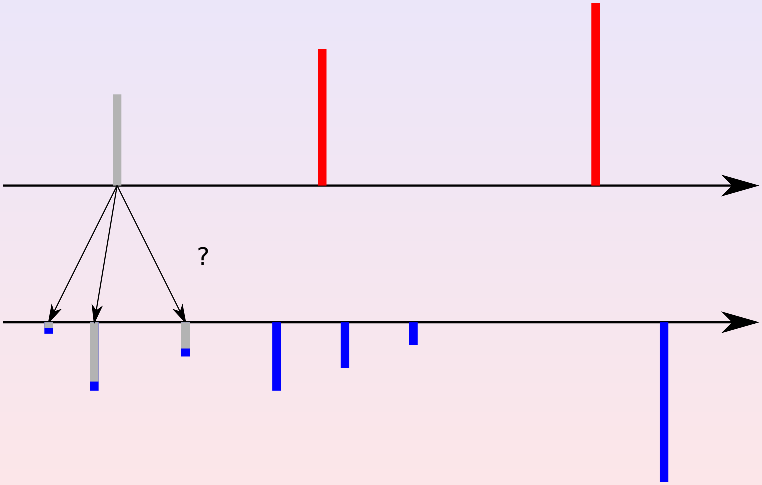

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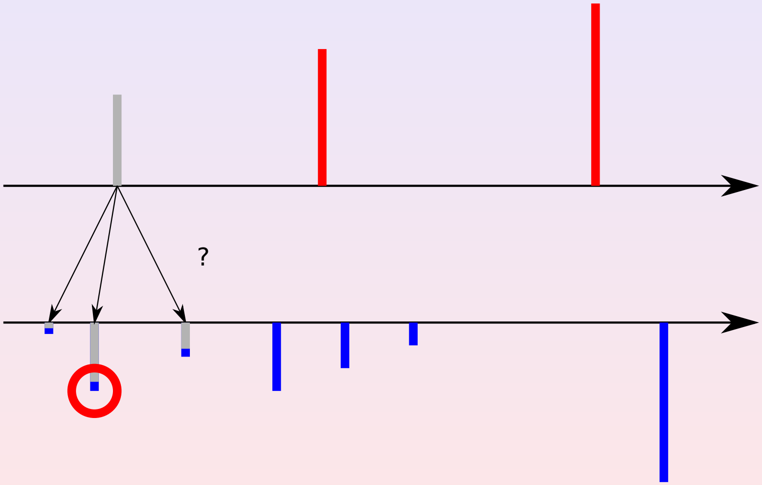

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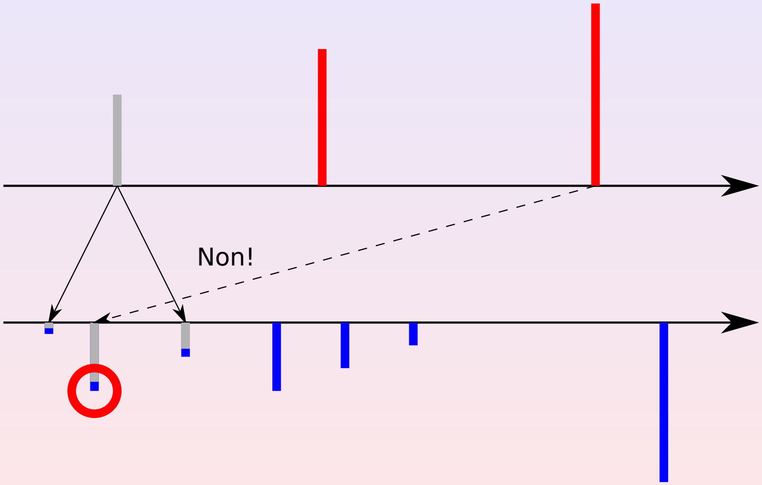

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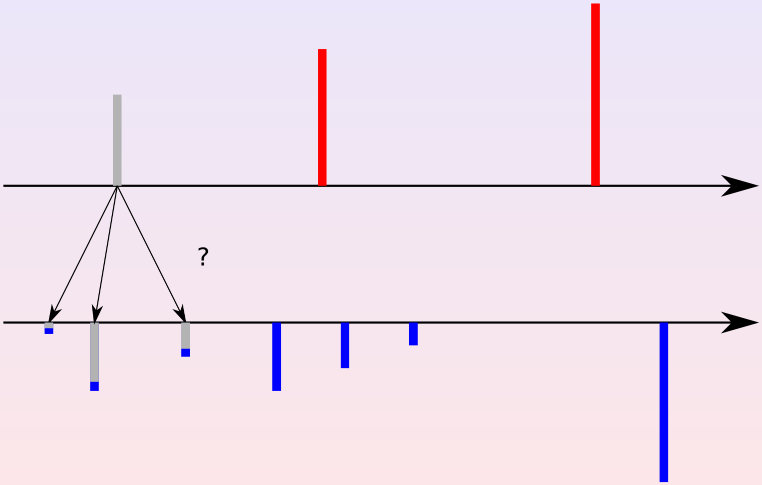

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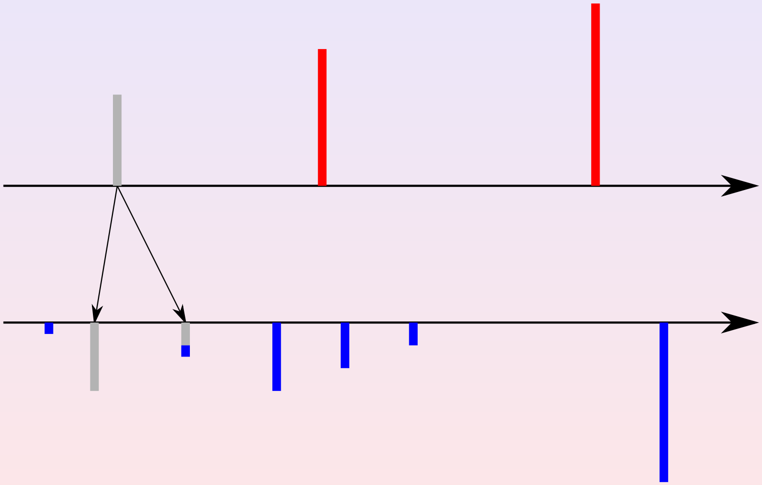

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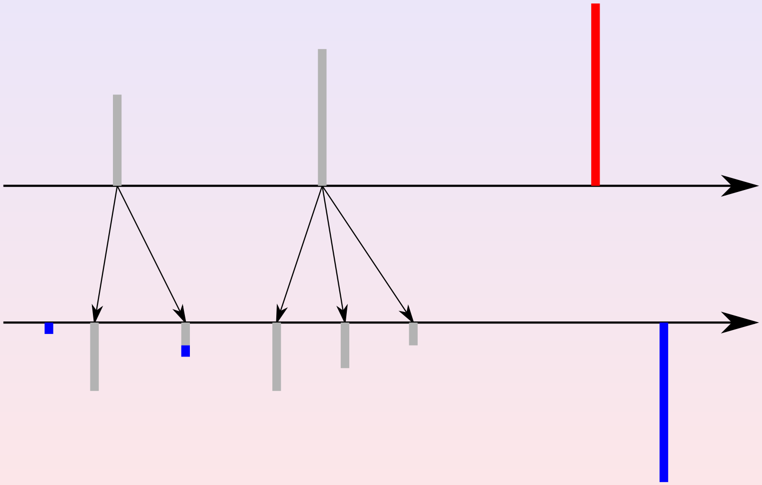

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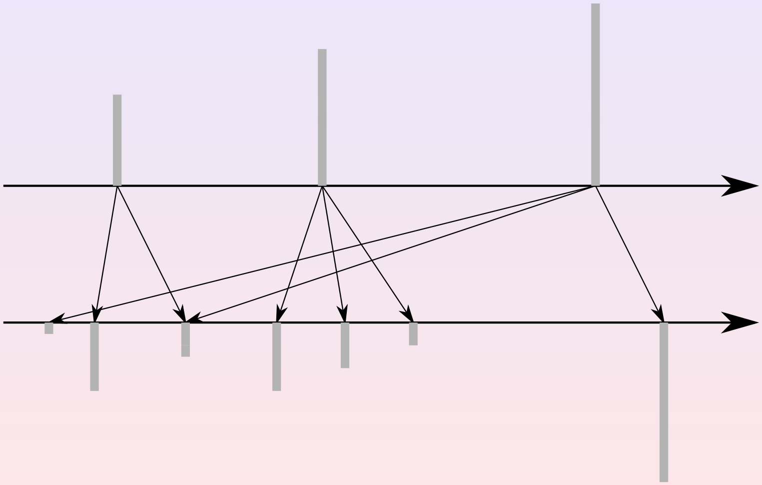

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Notion of Shadow

Associativity of the shadows

Left-curtain transport for uniform measures

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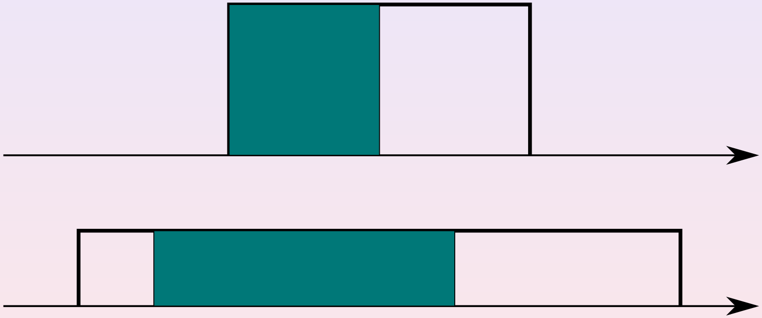

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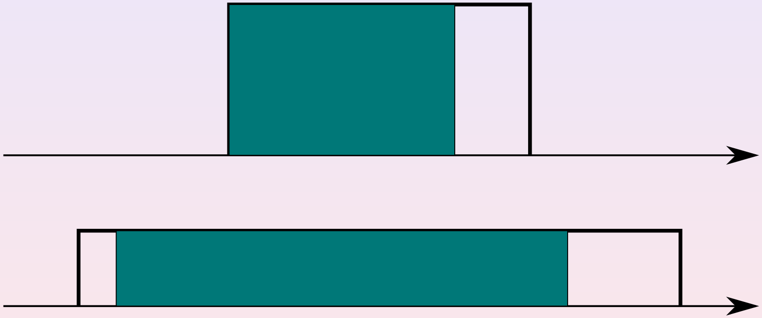

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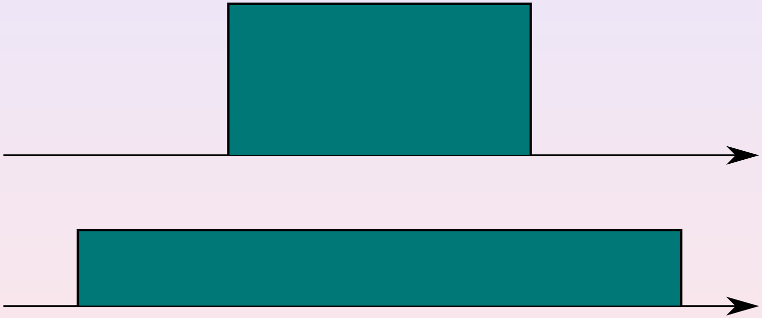

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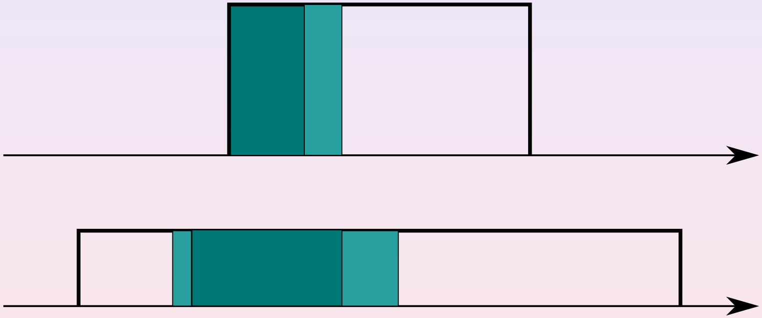

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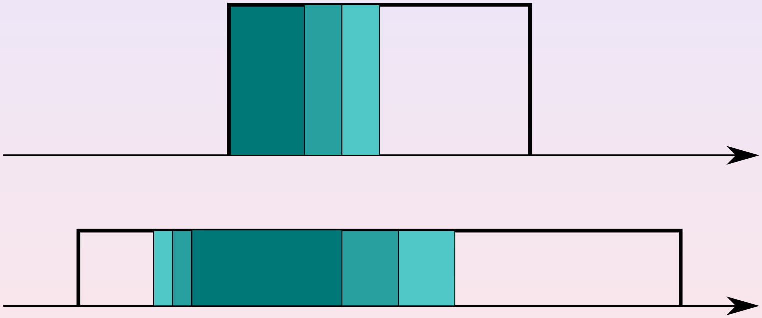

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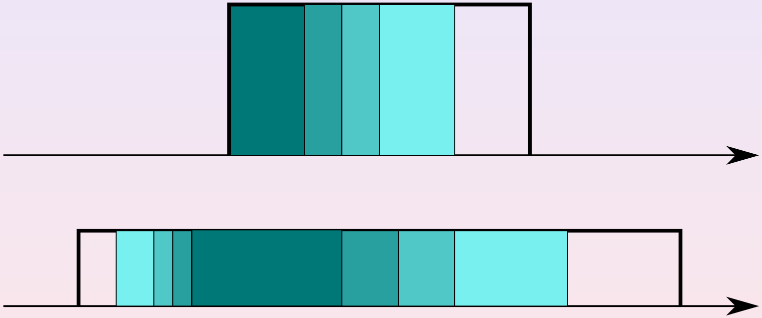

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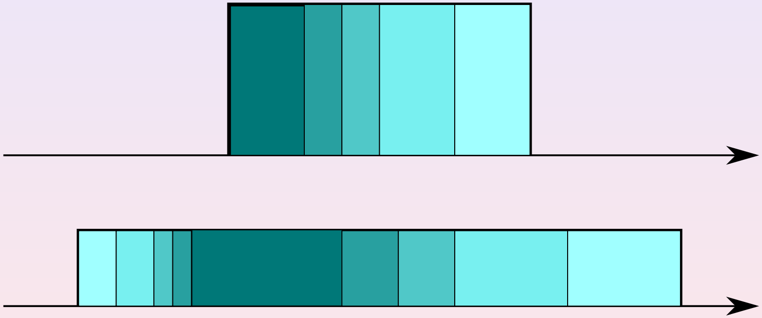

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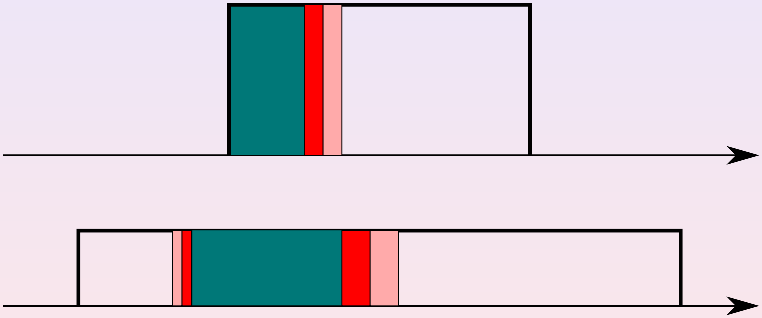

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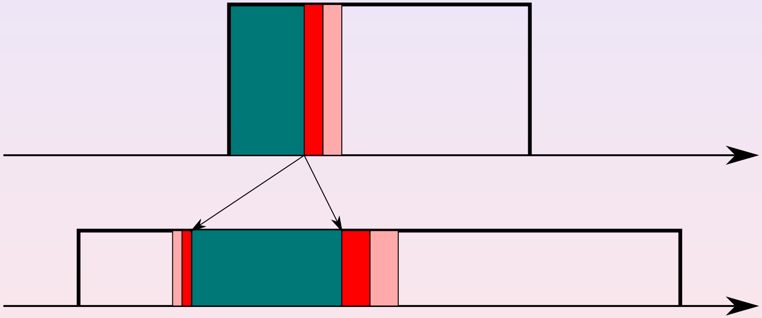

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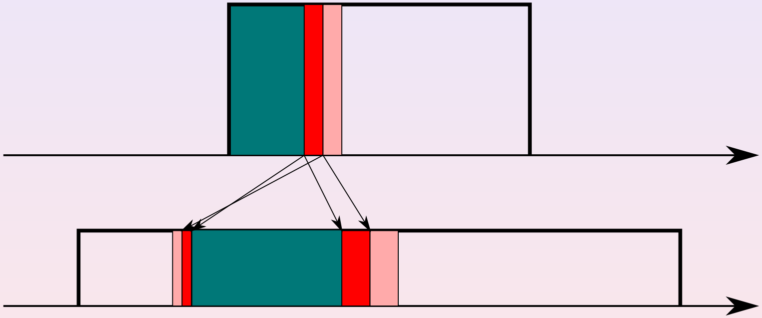

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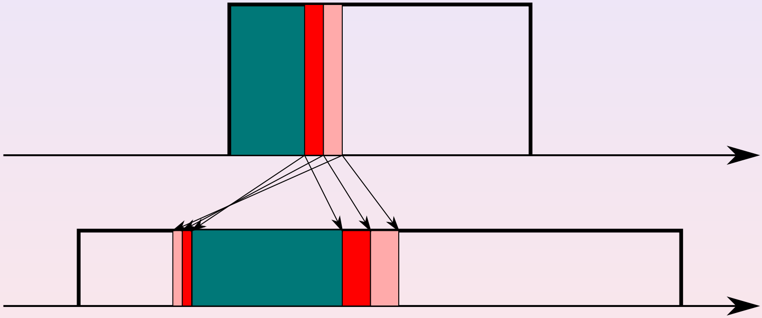
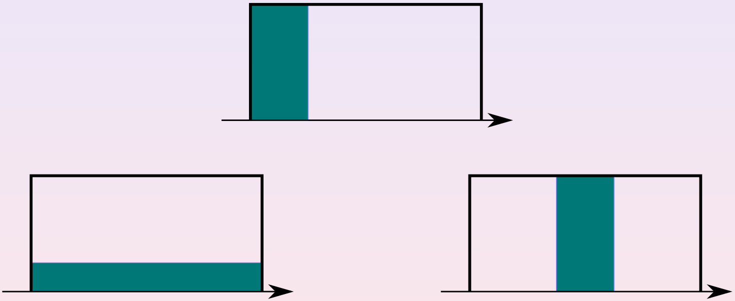


Other parametrizations



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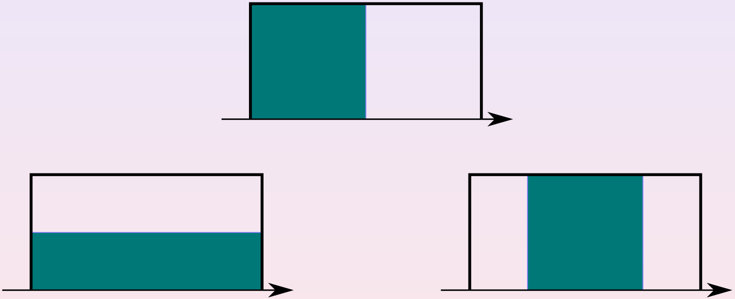

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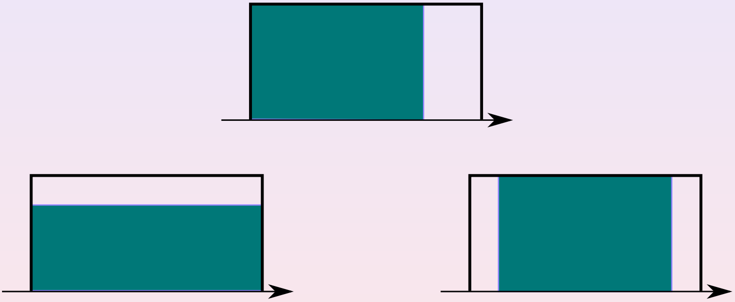

There is a problem

Question

How to transport $\mathbb{P}(X \in \cdot | U = \alpha)$ to $\mathbb{P}(Y \in \cdot | U = \alpha)$?

Beiglböck-J (2021): there is a unique martingale way to do that!

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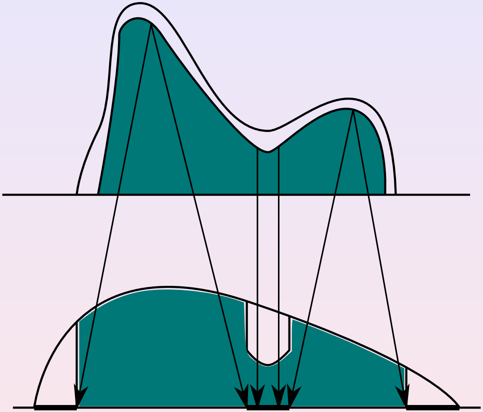

Figure: The Kellerer dilation used to transport shadow increaments

Properties of the shadow transports

1. Also solutions for transport problems (Multi-OT, WOT)
2. Interpretation as Skorkhod embedding.

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Stability as usual

Theorem (J., 2014, two proofs)

The map

$$(\mu,\nu)\in\{\mu\preceq_C\nu\}\longmapsto\pi:=\mathrm{LCurt}(\mu,\nu)\in\mathrm{Mart}(\mu,\nu)$$

is continuous.

Let W be the Kantorovich (=L^1-Wasserstein) distance.

Theorem (J., 2014)

If $S^{\nu}(\mu)$ and $S^{\nu'}(\mu')$ are defined and $\mu(\mathbb{R})=\mu'(\mathbb{R})$, $\nu(\mathbb{R})=\nu'(\mathbb{R})$, then

$$W(S^{\nu}(\mu),S^{\nu'}(\mu'))\leq W(\mu,\mu')+2W(\nu,\nu').$$

More recent stability results

- **Guo – Obłój (19):** Other costs. Stability of the value.
- **Wiesel ('19+):** Stability of the value fonction for general continuous costs
- **Pammer – Backhoff-Veraguas ('22):** Stability for continuous costs c uniform limits of $(c_n)_n$.
- **Beiglböck – Pammer – Jourdain – Margheriti ('21+):** Every optimal π can be approached by $(\pi_n)_n$ almost optimal where $(\mu_n, \nu_n)_n$ are given. Stability for l.s.c continuous costs.

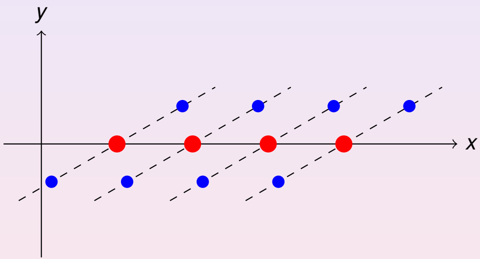
Non stability for dimension $d \geq 2$

- **Ghoussoub - Kim - Lim ('19):** MOT in dimension $d \geq 2$

Theorem (Brückerhoff – J., 2021+)

There exists $\mu_n \to \mu \in \mathcal{P}(\mathbb{R}^2)$, $\nu_n \to \nu \in \mathcal{P}(\mathbb{R}^2)$ and $(\pi_n)_n \in \mathrm{Mart}(\mu_n, \nu_n)$ a sequence of optimal transport plans such that the limit π exists and is not optimal.

The picture-proof



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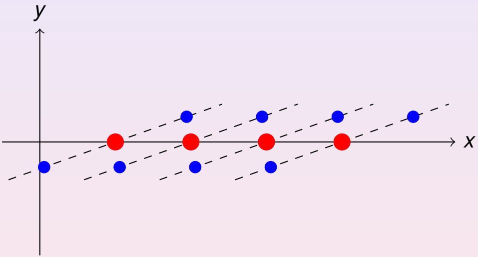

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