Optimization over trace polynomials

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Joint work with Igor Klep and Jurij Volcic

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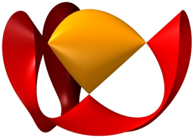

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& sums of product traces T

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$S \subset \operatorname{Sym} \mathbb{T}$ $\quad X_j$ operators from finite von Neumann algebra

Constraints $\mathcal{D}_S = \{\underline{X} = (X_1, \dots, X_n) : s(\underline{X}) \succcurlyeq 0, \quad \forall s \in S\}$

Optimization over $\mathbb{T}$: special cases

- **Eigenvalue** optimization 💡 no traces [Helton-McCallough 04, Navascuez-Pironio-Acin 08]

  💡 Bounded self-adjoint operators on Hilbert spaces $\rightsquigarrow \mathcal{D}_S^\infty$

$$\begin{aligned}\lambda_{\min} &= \inf \{ \langle a(\underline{X})\mathbf{v}, \mathbf{v} \rangle : \underline{X} \in \mathcal{D}_S^\infty, \|\mathbf{v}\| = 1 \} \\ &= \sup \{ \lambda \mid a(\underline{X}) - \lambda \mathbf{I} \succcurlyeq 0 \,, \quad \forall \underline{X} \in \mathcal{D}_S^\infty \}\end{aligned}$$

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- Univ case [Klep-Pascoe-Volcic]: $a \succcurlyeq 0 \Rightarrow a =$ SOHS/SOHS
- Multilinear case [Huber]

Motivation: quantum information

Entanglement in quantum mechanics
$\rightarrow$ **upper bounds** for violation levels of Bell inequalities

[Fukuda-Nechita 14] limit output states with input having specific parameters $\rightarrow$ bounds for generalized tensor traces

[Pozas et al 19] "scalar extension" of NPA hierarchy $\rightarrow$ identify correlations not attainable in entanglement-swapping scenario

NC operators A_i, C_k satisfy causal constraints:

$$\operatorname{tr}(A_{i_1} \cdots A_{i_m} C_{k_1} \cdots C_{k_m}) - \operatorname{tr}(A_{i_1} \cdots A_{i_m}) \operatorname{tr}(C_{k_1} \cdots C_{k_m}) = 0.$$

💡 Additional variables for each $\operatorname{tr}(w)$ but no convergence proof

Contribution

Theorem: T variant of Helton-McCullough Psatz

Let $S \subset \operatorname{Sym} \mathbb{T}$ and $a \in \mathrm{T}$. The Positivstellensatz-induced hierarchy of semidefinite programs produces a convergent increasing sequence with limit $\inf_{\mathcal{D}_S^{\mathrm{II}_1}} a$.

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Theorem: cyclic Positivstellensatz for $\mathbb{T}$

Let $\mathcal{M}^{\mathrm{cyc}}$ be an archimedean cyclic quadratic module & $a \in \operatorname{Sym}\mathbb{T}$. The following are equivalent

(i) $a \succcurlyeq 0$ on $\mathcal{D}_{\mathcal{M}^{\mathrm{cyc}}}$

(ii) $\forall \varepsilon > 0$, there exist SOS $s_1, s_2 \in \mathbb{R}[t]$, $q \in \mathcal{M}^{\mathrm{cyc}}$ such that

$$\operatorname{tr}(ay) + \varepsilon = \operatorname{tr}(s_1(a)y + s_2(a)(1-y)) + q$$

where y is an auxiliary symmetric free variable.

$\operatorname{tr}(ay) + \varepsilon$ is in the module generated by $\mathcal{M}^{\mathrm{cyc}}, y, 1-y$

Moment-sums of squares hierarchies

Non-cyclic Psatz for T

Cyclic Psatz for $\mathbb{T}$

SDP hierarchies

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Commutative Polynomial Optimization

NP hard General Problem: $f^* := \min_{\mathbf{x} \in \mathcal{D}_S} f(\mathbf{x})$

Semialgebraic set $\mathcal{D}_S = \{\mathbf{x} \in \mathbb{R}^n : s_j(\mathbf{x}) \geqslant 0\}$

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$$\overbrace{x_1 x_2}^{f} =$$

$$-\frac{1}{8} + \overbrace{\frac{1}{2}\left(x_1 + x_2 - \frac{1}{2}\right)^2}^{\sigma_0} + \overbrace{\frac{1}{2}}^{\sigma_1} \overbrace{x_1(1 - x_1)}^{s_1} + \overbrace{\frac{1}{2}}^{\sigma_2} \overbrace{x_2(1 - x_2)}^{s_2}$$

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Quadratic module: $\mathcal{M}(S)_r = \left\{ \sigma_0 + \sum_j \sigma_j s_j, \ \deg \sigma_j \, s_j \leqslant 2r \right\}$

Hierarchies for Polynomial Optimization

Hierarchy of SDP relaxations: $\lambda_r := \sup_{\lambda} \Big\{ \lambda : f - \lambda \in \mathcal{M}(S)_r \Big\}$

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Can be computed with SDP solvers (`CSDP`, `SDPA`, `MOSEK`)

"No Free Lunch" Rule: $\binom{n+2r}{n}$ SDP variables

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Theorem: NC Putinar's Psatz [Helton-McCullough 02]

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$a - \lambda \mathbf{1} = \sum_i h_i^\star h_i + \sum_j \sum_i t_{ji}^\star s_j t_{ji}$ with h_i, t_{ji} of **bounded** degrees

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$$\begin{aligned} = \sup \quad & m \\ \text{s.t.} \quad & \mathrm{tr}(a(\underline{X}) - m) \geqslant 0\,, \quad \forall \underline{X} \in \mathcal{D}_S \end{aligned}$$

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$\mathrm{tr}^{\mathrm{II}_1}_{\min}$ = minimal trace over the union of type $-\mathrm{II}_1$ vN algebras

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Converging hierarchy with cyclic quadratic modules:

💡 replace "$\mathrm{tr}(a - m) \geqslant 0$ on $\mathcal{D}_S^{\mathrm{II}_1}$" by $a - m\mathbf{1} \in \mathcal{M}^{\mathrm{cyc}}(S)_r$

$\mathcal{M}^{\mathrm{cyc}}(S)_r$ = polynomials with same trace as some from $\mathcal{M}(S)_r$

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Non-cyclic Psatz for $\mathbb{T}$

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Kadison-Dubois representation theorem

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Theorem: Kadison-Dubois [Marshall 08]

Given an Archimedean quadratic module $\mathcal{M} \subseteq \mathsf{T}$ & $a \in \mathsf{T}$:

$$\forall \varphi \in \chi_{\mathcal{M}} \quad \varphi(a) \geqslant 0 \qquad \Leftrightarrow \qquad \forall \varepsilon > 0 \quad a + \varepsilon \in \mathcal{M}$$

Non-cyclic Psatz for T

For $S \subseteq \mathsf{T}$, "augment" S with traces of hermitian squares:

$$S(N) = S \cup \{\mathrm{tr}(pp^\star) \mid p \in \mathbb{R}\langle \underline{x} \rangle\} \cup \{N^k - \mathrm{tr}(x_j^{2k}) \mid k \in \mathbb{N}\} \subseteq \mathsf{T}$$

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$$S[N] = S \cup \{N - x_j^2\} \subset \mathbb{T}$$

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Lemma [Klep-M.-Volcic 20]

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Proof

By induction: $\forall w \in \langle \underline{x} \rangle$, $m \pm \mathrm{tr}(w) \in \mathcal{M}(S(N))$ for some $m > 0$

$$w = x_j^{2k} \implies N^k + 1 + 2\,\mathrm{tr}(x_j^k) = \left(N^k - \mathrm{tr}(x_j^{2k})\right) + \mathrm{tr}\left((x_j^k + 1)^2\right)$$

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Theorem: Non-cyclic Psatz for T [Klep-M.-Volcic 20]

$a \geqslant 0$ on $\mathcal{D}_{S[N]}^{\mathrm{II}_1}$ $\quad \Leftrightarrow \quad$ $a + \varepsilon \in \mathcal{M}(S(N))$ for all $\varepsilon > 0$

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Cyclic quadratic modules in $\mathbb{T}$

$\mathcal{M}^{\text{cyc}} \subseteq \operatorname{Sym} \mathbb{T}$ is a cyclic quadratic module if

$1 \in \mathcal{M}^{\text{cyc}}$, $\mathcal{M}^{\text{cyc}} + \mathcal{M}^{\text{cyc}} \subseteq \mathcal{M}^{\text{cyc}}$, $a^\star \mathcal{M}^{\text{cyc}} a \subseteq \mathcal{M}^{\text{cyc}}$ $\forall a \in \mathbb{T}$

$\operatorname{tr}(\mathcal{M}^{\text{cyc}}) \subset \mathcal{M}^{\text{cyc}}$

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$1 \in \mathcal{M}^{\text{cyc}}$, $\mathcal{M}^{\text{cyc}} + \mathcal{M}^{\text{cyc}} \subseteq \mathcal{M}^{\text{cyc}}$, $a^\star \mathcal{M}^{\text{cyc}} a \subseteq \mathcal{M}^{\text{cyc}} \; \forall a \in \mathbb{T}$
$\operatorname{tr}(\mathcal{M}^{\text{cyc}}) \subset \mathcal{M}^{\text{cyc}}$

$\mathcal{M}^{\text{cyc}}(\varnothing)$ = sums of $\operatorname{tr}(h_1 h_1^\star) \cdots \operatorname{tr}(h_\ell h_\ell^\star) h_0 h_0^\star$ for $h_i \in \mathbb{T}$

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Proposition [Klep-M.-Volcic 20]

$\mathcal{M}^{\mathrm{cyc}}$ is archimedean $\Leftrightarrow N - \sum_{i=1}^n x_j^2 \in \mathcal{M}^{\mathrm{cyc}}$ for some $N > 0$

Positivity of elements in T

Theorem [Klep-M.-Volcic 20]

Let $\mathcal{M}^{\text{cyc}} \subseteq \operatorname{Sym} \mathbb{T}$ & $a \in$ T.

$a \geqslant 0$ on $\mathcal{D}^{\text{II}_1}_{\mathcal{M}^{\text{cyc}}} \Leftrightarrow a + \varepsilon \in \mathcal{M}^{\text{cyc}}$ for all $\varepsilon > 0$

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$x_1 \not\geqslant 0$ on $\mathcal{D}^{\text{II}_1}_{\mathcal{M}^{\text{cyc}}}$ but $x_1 + \varepsilon \notin \mathcal{M}^{\text{cyc}}$

Positivity of elements in $\operatorname{Sym}\mathbb{T}$

Proposition [Klep-M.-Volcic 20]

Let $(\mathcal{F}, \tau)$ be a tracial pair and $X = X^* \in \mathcal{F}$. Tfae:

Positivity of elements in Sym $\mathbb{T}$

Proposition [Klep-M.-Volcic 20]

Let $(\mathcal{F}, \tau)$ be a tracial pair and $X = X^* \in \mathcal{F}$. Tfae:

(i) $X \succcurlyeq 0$

(ii) $\tau(XY) \geq 0$ for all positive semidefinite contractions $Y \in \mathcal{F}$

(iii) $\tau(Xp(X)^2) \geq 0$ for all $p \in \mathbb{R}[t]$

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(i) $a \succcurlyeq 0$ on $\mathcal{D}^{\mathrm{II}_1}_{\mathcal{M}^{\mathrm{cyc}}}$

(ii) $\forall \varepsilon > 0$, there exist SOS $s_1, s_2 \in \mathbb{R}[t]$, $q \in \mathcal{M}^{\mathrm{cyc}}$ such that

$$\operatorname{tr}(ay) + \varepsilon = \operatorname{tr}(s_1(a)y + s_2(a)(1-y)) + q$$

where y is an auxiliary symmetric free variable.

💡 $\operatorname{tr}(ay) + \varepsilon$ is in the module generated by $\mathcal{M}^{\mathrm{cyc}}, y, 1-y$

Moment-sums of squares hierarchies

Non-cyclic Psatz for $\mathbb{T}$

Cyclic Psatz for $\mathbb{T}$

SDP hierarchies

Tracial words & moment matrices

$\mathbb{T}$-words = $\{\prod_i \operatorname{tr}(u_i) v \mid u_i, v \in \langle \underline{x} \rangle\}$ and T-words

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Tracial degree = up to cyclic equivalence

$\mathbf{W}_r^{\mathbb{T}}$ = vector of $\mathbb{T}$-words of with tracial degree $\leqslant r$

$n = 1$: $\mathbf{W}_2^{\mathbb{T}}$ contains $1, x_1, x_1^2, \operatorname{tr}(x_1), \operatorname{tr}(x_1^2), \operatorname{tr}(x_1)x_1$

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Tracial moment matrix $\mathbf{M}_r^{\mathbb{T}}(L)$ for a linear functional L : T $\to \mathbb{R}$:

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SDP hierarchy for T

Reminder:

$$S(N) = S \cup \{\mathrm{tr}(pp^\star) \mid p \in \mathbb{R}\langle \underline{x} \rangle\} \cup \{N^k - \mathrm{tr}(x_j^{2k}) \mid k \in \mathbb{N}\} \subseteq \mathsf{T}$$

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Elements of $\mathcal{M}(S(N))_r$ are

$$a_1^2 s \quad a_2^2 (N^k - \mathrm{tr}(x_j^{2k})) \quad \mathrm{tr}(f f^\star)$$

for $s \in S$, $a_i \in \mathsf{T}$, $f \in \mathbb{T}$

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& (\mathbf{M}_r^{\mathbb{T}}(L))_{1,1} = 1 \\
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Theorem [Klep-M.-Volcic 20]

There is no duality gap and $a_r \to a_{\min}^{\mathrm{II}_1}$ as $r \to \infty$

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$S \subset \operatorname{Sym} \mathbb{T}$

Reduction from the general trace setting to the pure trace

$$\widetilde{S} = \{\operatorname{tr}(fsf^\star) \mid s \in S,\ f \in \mathbb{T}\} \subset \mathsf{T}$$

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Linear Bell inequalities

CLASSICAL WORLD

$$\psi^*(A_1 \otimes B_1 + A_1 \otimes B_2 + A_2 \otimes B_1 - A_2 \otimes B_2)\psi \leqslant 2$$

for separable states $\psi \in \mathbb{C}^k \otimes \mathbb{C}^k$ and matrices A_j, B_j satisfying $A_j^* = A_j$, $A_j^2 = I$, $B_j^* = B_j$, $B_j^2 = I$

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Polynomial Bell inequalities

COVARIANCES OF QUANTUM CORRELATIONS

$$\mathrm{cov}_\psi(X,Y) = \psi^*(X \otimes Y)\psi - \psi^*(X \otimes I)\psi \cdot \psi^*(I \otimes Y)\psi$$

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for separable states but ... 5 for **one** maximally entangled state

2nd SDP relaxation of the corresponding trace problem outputs 5

$\Rightarrow$ The max value is 5 for **all** maximally entangled state

Conclusion and perspectives

CONVERGING HIERARCHIES to minimize pure trace polynomials

Implementation in progress

Conclusion and perspectives

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Hierarchy for minimal eigenvalue problem?

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APPLICATIONS IN QUANTUM INFORMATION: Werner states

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APPLICATIONS IN QUANTUM INFORMATION: Werner states

Exploiting SPARSITY of cost and constraints?

Thank you for your attention!

`https://homepages.laas.fr/vmagron`

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