On the exactness of Lasserre’s relaxation for polynomial optimization with equality constraints

Zheng Qu

University of Hong Kong

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joint work with Z. Hua (HKU)
Polynomial Optimization with Equality Constraints

Consider

\[ f_\ast = \min_{x \in \mathbb{R}^n} f(x) \]

s.t. \( g_1(x) = g_2(x) = \ldots = g_n(x) = 0, \]

where \( f, g_1, \ldots, g_n \) are polynomial functions of \( n \) variables. Lasserre’s SOS relaxation of order \( d \) corresponds to solving:

\[ f_d = \max_{c \in \mathbb{R}} c \]

s.t. \( f - c \in \langle g \rangle_{2d} + \sum_{\mathbb{R}[x_1, \ldots, x_n]_{\leq 2d}} \]

where

\[ \langle g \rangle_{2d} := \left\{ \sum_{i=1}^{n} \lambda_i g_i : \deg(\lambda_i) \leq 2d - \deg(g_i) \quad \text{for } i = 1, \ldots, n \right\}, \]

and \( \sum_{\mathbb{R}[x_1, \ldots, x_n]_{\leq 2d}} \) is the set of SOS polynomials of degree at most \( 2d \).
The \(d\)-SDP Exactness

Lasserre’s relaxation of order \(d\):

\[
 f_d = \max_{c \in \mathbb{R}} \quad c \\
\text{s.t.} \quad f - c \in \langle g \rangle_{2d} + \Sigma \mathbb{R}[x_1, \ldots, x_n]_{\leq 2d}^2 
\]

(3)

- Problem (3) can be formulated as a convex SDP problem.
- If for some \(d\), we have

\[
 f - f_* \in \langle g \rangle_{2d} + \Sigma \mathbb{R}[x_1, \ldots, x_n]_{\leq 2d}^2 .
\]

then

\[
 f_d = f_* .
\]

- We say that (1) is \(d\)-SDP exact if (4) holds.
Complete Intersection Assumption

Assumption

We have \( \dim_{\mathbb{R}} (\mathbb{R}[x_1, \ldots, x_n]/\langle g \rangle) = \prod_{i=1}^{n} \deg(g_i) \), where the \( \dim_{\mathbb{R}}(A) \) is the dimension of \( A \) as an \( \mathbb{R} \)-vector space.

It implies:

- The complex variety
  \[ V(g) := \{ x \in \mathbb{C}^n : g_1(x) = g_2(x) = \ldots = g_n(x) = 0 \}, \]
  is finite.
- There is no solution at infinity:
  \[
  \{ x = [x_0 : x_1 : \cdots : x_n] \in \mathbb{P}^n : \bar{g}_1(x_0, x_1, \ldots, x_n) = \cdots = \bar{g}_n(x_0, x_1, \ldots, x_n) = 0 \} \\
  \subset \{ x \in \mathbb{P}^n : x_0 = 1 \} 
  \]
Examples

Example (Binary polynomial programming)

\[
\min_{x \in \mathbb{R}^n} f(x) \\
\text{s.t. } g_i(x) := x_i^2 - 1 = 0, \quad \forall i \in \{1, \ldots, n\}
\]

Example (Grid polynomial programming)

\[
\min_{x \in \mathbb{R}^n} f(x) \\
\text{s.t. } g_i(x) := \prod_{j=1}^{d_i} (x_i - a_{ij}) = 0, \quad \forall i \in \{1, \ldots, n\}
\]
Examples

Example

\[
g_1(x_1, x_2) = x_2(a(x_2 - 1) - x_1(b - 1)), \\
g_2(x_1, x_2) = x_1(x_2(a - 1) - b(x_1 - 1)).
\]

Example

\[
g_1(x_1, x_2) = x_2(a(x_2 - 1) - x_1(b - 1)), \\
g_2(x_1, x_2) = x_1(x_1 + x_2 - 1).
\]

Every algebraic set in affine \(n\)-space is the intersection of \(n\) hypersurfaces [Eisenbud and Evans 1973].

For any finite set \(V \subset \mathbb{R}^n\), there are polynomials \(g_1, \ldots, g_n \in \mathbb{R}[x_1, \ldots, x_n]\) satisfying the complete intersection assumption such that

\[
V = \{x \in \mathbb{C}^n : g_1(x) = \ldots = g_n(x) = 0\}
\]
Complete Intersection Assumption

If \( \dim \mathbb{R}[x_1, \ldots, x_n]/\langle g \rangle = \prod_{i=1}^{n} \deg(g_i) \) and 
\( \dim \mathbb{R}[y_1, \ldots, y_m]/\langle h \rangle = \prod_{i=1}^{m} \deg(h_i) \), then 
\( \dim \mathbb{R}[x_1, \ldots, x_n, y_1, \ldots, y_m]/\langle g, h \rangle = \left(\prod_{i=1}^{n} \deg(g_i)\right) \left(\prod_{i=1}^{m} \deg(h_i)\right) \). Ex:

\[
\begin{align*}
g_1(x) &= x_2(a(x_2 - 1) - x_1(b - 1)) \\
g_2(x) &= x_1(x_2(a - 1) - b(x_1 - 1)) \\
g_3(x) &= x_4(c(x_4 - 1) - x_3(d - 1)) \\
g_4(x) &= x_3(x_4(c - 1) - d(x_3 - 1))
\end{align*}
\]
Related Work

Lasserre’s SOS relaxation was introduced for more general optimization problem model:

\[
    f_\ast = \min_{x \in \mathbb{R}^n} f(x) \\
    \text{s.t.} \quad g_1(x) = g_2(x) = \ldots = g_m(x) = 0, g_{m+1} \leq 0, \ldots g_{m+k} \leq 0
\]

For the more general model:

- Parrilo [Parrilo 2002]: \( d \)-SDP exactness holds for some integer \( d \) when the ideal generated by \((g_1, \ldots, g_m)\) is radical.
- Laurent [Laurent 2007]: \( f_d = f_\ast \) for some \( d \leq \max(d_B + d_+, \max(\deg(g_1), \ldots, \deg(g_m))/2) \) if \( \{g_1, \ldots, g_m\} \) forms a Gröbner basis. Here \( d_B \) is the maximum degree of a polynomial in a basis of \( \mathbb{R}[x_1, \ldots, x_n]/(g_1, \ldots, g_m) \).

\( d \)-SDP exactness holds for optimization over grid:

- Lasserre [Lasserre 2002]: \( d \leq \sum_{i=1}^n \deg(g_i) - n + \max(\deg(g_1), \ldots, \deg(g_n))/2 \).
- Laurent [Laurent 2007]: 
  \( d \leq \max \left( \sum_{i=1}^n \deg(g_i) - n, \max(\deg(g_1), \ldots, \deg(g_n))/2 \right) \).

\( d \)-SDP exactness holds for binary optimization:

- Fawzi et al [Fawzi et al. 2016]: \( d \leq \lceil n/2 \rceil \) if \( f \) is a quadratic homogeneous.
- Sakaue et al. [Sakaue et al. 2017]: \( d \leq \lceil (n + \deg(f) - 1)/2 \rceil \).
- Sakaue et al. [Sakaue et al. 2017]: \( d \leq \lceil (n + \deg(f) - 2)/2 \rceil \) if \( f \) only contains monomials of even degree.
Taking Square Root in the Quotient Ring

- Denote:

$$V(g) = \{ x \in \mathbb{C}^n : g_1(x) = g_2(x) = \ldots = g_n(x) = 0 \},$$

and $$V_{\mathbb{R}}(g) = V(g) \cap \mathbb{R}^n.$$  

- $$x \in V(g)$$ is **singular** if its multiplicity is $$> 1.$$  

- Denote by $$\phi$$ the standard ring homomorphism from $$\mathbb{R}[x_1, \ldots, x_n]$$ to the quotient ring $$\mathbb{R}[x_1, \ldots, x_n]/\langle g \rangle.$$  

**Proposition**

Let $$p \in \mathbb{R}[x_1, \ldots, x_n]$$ be a polynomial such that $$p(x) \geq 0$$ for any $$x \in V_{\mathbb{R}}(g)$$ and $$p(x) \neq 0$$ for any singular point $$x \in V(g).$$ Then there is $$q \in \mathbb{R}[x_1, \ldots, x_n]/\langle g \rangle$$ such that

$$\phi(p) = q^2.$$
Proof Outline

The complete intersection assumption implies that the quotient ring

$$A := \mathbb{R}[x_1, \ldots, x_n]/\langle g \rangle$$

is an Artin ring.

- Structure theorem for Artin rings.

$$A \simeq \prod_{i=1}^{s} A_{m_i}.$$

Here $m_1, \ldots, m_s$ are the maximal ideals of $A$.

- Existence of square root on each Artinian local ring $A_m$.

Given $p \in A_m$, there is $q \in A_m$ such that $p = q^2$ if

1. if $A/m = \mathbb{R}$ then $\rho_m(p) \geq 0$ and $\rho_m(p) > 0$ when $\dim_{\mathbb{R}} A_m > 1$,

2. if $A/m = \mathbb{C}$ and $\dim_{\mathbb{C}} A_m > 1$ then $\rho_m(p) \neq 0$. 

Exactness of Lasserre's relaxation
Two Surjectivity Results

Define

\[ n := \sum_{i=1}^{n} \deg(g_i) - n. \]

**Proposition**

For any \( q \in \mathbb{R}[x_1, \ldots, x_n]/\langle g \rangle \), there is \( h \in \mathbb{R}[x_1, \ldots, x_n] \) with \( \deg(h) \leq n \) such that

\[ \phi(h) = q. \]

**Proposition**

Let \( d \geq \max(\deg(g_1), \ldots, \deg(g_n)) \). For any \( h \in \langle g \rangle \) with \( \deg(h) \leq d \), there are \( \lambda_1, \ldots, \lambda_n \in \mathbb{R}[x_1, \ldots, x_n] \) with \( \deg(\lambda_i) \leq d - \deg(g_i), \; i = 1, \ldots, n \) such that

\[ h = \lambda_1 g_1 + \ldots + \lambda_n g_n. \]
Degree Bound on the $d$-SDP Exactness

Suppose that $2n \geq \max(\deg(g_1), \ldots, \deg(g_n))$. If $\deg(f) \leq 2n$ and $f(x) \neq f_*$ for any singular point $x \in V(g)$, then there are $h, \lambda_1, \ldots, \lambda_n \in \mathbb{R}[x_1, \ldots, x_n]$ with $\deg(h) \leq n$, $\deg(\lambda_i) \leq 2n - \deg(g_i)$, $\forall i = 1, \ldots, n$ such that

$$f - f_* = h^2 + \sum_{i=1}^{n} \lambda_i g_i$$

Theorem

If $f(x) \neq f_*$ for any singular point $x \in V(g)$, then the problem (1) is $d$-SDP exact for $d \geq \max(n, \max(\deg(g_1), \ldots, \deg(g_n)) / 2, \deg(f)/2)$.

Corollary

We have $f_d = f_*$ for any $d \geq \max(n, \max(\deg(g_1), \ldots, \deg(g_n)) / 2, \deg(f)/2)$. 

Exactness of Lasserre’s relaxation
Sheaf Cohomology

- Projective space: $\mathbb{P}^n$
- Graded ring: $S = \mathbb{R}[x_0, x_1, \ldots, x_n] = \bigoplus_{d \geq 0} S_d. \ (S_d \cong \mathbb{R}[x_1, \ldots, x_n]_{\leq d})$
- Homogenization of $g_1, \ldots, g_n$: $\bar{g}_1, \ldots, \bar{g}_n \in S$
- Graded ideal: $I = (\bar{g}_1, \ldots, \bar{g}_n) = \bigoplus_{d \geq 0} I_d$
- Sheaf associated to $S(d)$: $\mathcal{O}(d)$
- Ideal sheaf associated to $I(d)$: $\mathcal{I}(d)$
- Spectrum of $\mathbb{R}[x_1, \ldots, x_n]/\langle g \rangle$: $K$

The short exact sequence of sheaves

\[ 0 \rightarrow \mathcal{I}(d) \rightarrow \mathcal{O}(d) \rightarrow \mathcal{O}_K \rightarrow 0 \]

induces a long exact sequence of cohomology groups

\[ 0 \rightarrow H^0(\mathbb{P}^n, \mathcal{I}(d)) \rightarrow H^0(\mathbb{P}^n, \mathcal{O}(d)) \rightarrow H^0(\mathbb{P}^n, \mathcal{O}_K) \rightarrow H^1(\mathbb{P}^n, \mathcal{I}(d)) \rightarrow H^1(\mathbb{P}^n, \mathcal{O}(d)) \]
Koszul Resolution of Ideal Sheaf

If \( \dim \mathbb{R}[x_1, \ldots, x_n]/\langle g \rangle = \prod_{i=1}^{n} \deg(g_i) \), then

\[
\{ x = [x_0 : x_1 : \cdots : x_n] \in \mathbb{P}^n : \bar{g}_1(x_0, x_1, \ldots, x_n) = \cdots = \bar{g}_n(x_0, x_1, \ldots, x_n) = 0 \} \\
\subset \{ x \in \mathbb{P}^n : x_0 = 1 \}
\]

\[\text{Spec}(\mathbb{C}[x_1, \ldots, x_n]/\langle g \rangle_{\mathbb{C}})\] is a complete intersection in \( \mathbb{P}^n_{\mathbb{C}} \). There is a Koszul Resolution of the ideal sheaf \( I \) associated to \( I \) [Eisenbud 1995]:

\[
0 \rightarrow \bigwedge^n E \overset{\delta_n}{\rightarrow} \cdots \overset{\delta_i}{\rightarrow} \bigwedge^i E \overset{\delta_i}{\rightarrow} \cdots \overset{\delta_1}{\rightarrow} E \overset{\delta_1}{\rightarrow} I \rightarrow 0. \tag{5}
\]

where \( E := \mathcal{O}(-d_{g_1}) \oplus \cdots \oplus \mathcal{O}(-d_{g_n}) \) and \( \bigwedge^i E \) denotes the \( i \)th exterior power of \( E \).

Based on (5) we deduce

\[ H^1(\mathbb{P}^n, I(d)) = 0, \quad \forall d \geq n. \]
Dimension Counting from Koszul Resolution of Ideal

There is a Koszul Resolution of the graded ideal \( I \):

\[
0 \longrightarrow S(-d_{g_1} - \cdots - d_{g_n}) \overset{\delta_n}{\longrightarrow} \cdots \longrightarrow S(-d_{g_1}) \oplus \cdots \oplus S(-d_{g_n}) \overset{\delta_1}{\longrightarrow} I \longrightarrow 0
\]

where \( \delta_1(\lambda_1, \ldots, \lambda_n) = \lambda_1 \bar{g}_1 + \cdots + \lambda_n \bar{g}_n \).

Compute Hilbert series using the above graded resolution:

\[
\chi_{S/I}(t) = \sum_{i=0}^{n} (-1)^i \frac{t^{\sum_{j \in I} d_{g_j}}}{(1-t)^{n+1}} = \prod_{i=1}^{n} \left( \sum_{j=1}^{d_{g_i}} t^{j-1} \right) \frac{1}{1-t}.
\]

Let \( c_d = |\{(k_1, \ldots, k_n) \in \mathbb{N}^n : k_1 + \cdots + k_n = d, \; k_i \in [0, d_{g_i} - 1], \; \forall i \in \{1, \ldots, n\}\}| \)

We have:

\[
\dim \left( \phi \left( \mathbb{R}[x_1, \ldots, x_n]_{\leq d} \right) \right) = \sum_{i=0}^{d} c_d.
\]
Obstruction Map

\[ \cdots \longrightarrow H^0(\mathbb{P}^n, \mathcal{O}(d)) \longrightarrow H^0(\mathbb{P}^n, \mathcal{O}_K) \xrightarrow{\Phi_d} H^1(\mathbb{P}^n, \mathcal{I}(d)) \longrightarrow 0 \]

\[ \ker(\Phi_d) = \phi(\mathbb{R}[x_1, \ldots, x_n]_{\leq d}) \]

- I.e., \( \ker(\Phi_d) \) is isomorphic to the set of polynomials in \( \mathbb{R}[x_1, \ldots, x_n]/\langle g \rangle \) which can be lifted to a polynomial of degree at most \( d \).
- If \( V(g) = V_\mathbb{R}(g) \) and all the solutions are of multiplicity one, then \( H^0(\mathbb{P}^n, \mathcal{O}_K) \) corresponds to the space of functions on \( V_\mathbb{R}(g) \).

**Example**

Consider \( \mathbb{R}[x] \) and \( g(x) = x^2 - 1 \). Then \( n = 1, \ n = 1 \) and the vanishing locus is \( V = \{1, -1\} \). Take \( \{\frac{x+1}{2}|_V, \frac{1-x}{2}|_V\} \) as a basis of \( H^0(\mathbb{P}^1, \mathcal{O}_K) \). Then

\[ \ker(\Phi_0) = \phi(\mathbb{R}[x]_{\leq 0}) = \{(f_1; f_2) \in \mathbb{R}^2 : f_1 - f_2 = 0\} \]
Example

Consider $\mathbb{R}[x_1, x_2]$, $g_1(x) = x_1^2 - 1$ and $g_2(x) = x_2^2 - 1$. Then $n = 2$, $n = 2$ and the vanishing locus is

$$V = \{(1; 1), (1; -1), (-1; 1), (-1; -1)\}.$$

Take

$$\{(1 - x_1)(1 - x_2)|_V, (1 - x_1)(1 + x_2)|_V, (1 + x_1)(1 - x_2)|_V, (1 + x_1)(1 + x_2)|_V\}$$

as a basis of $H^0(\mathbb{P}^2, \mathcal{O}_K)$. Then

$$\ker (\Phi_0) = \phi(\mathbb{R}[x]_{\leq 0}) = \{f \in \mathbb{R}^4 : f_1 = f_2 = f_3 = f_4\}.$$

and

$$\ker (\Phi_1) = \phi(\mathbb{R}[x]_{\leq 1}) = \{f \in \mathbb{R}^4 : f_1 + f_4 = f_2 + f_3\}.$$
Example

Consider $\mathbb{R}[x_1, \ldots, x_n]$ and $g_i(x) = x_i^2 - 1$, for $i \in \{1, \ldots, n\}$. Then $n = n$ and the vanishing locus is

$$V = \{\pm 1\}^n.$$ 

Take

$$\left\{ \prod_{r \in J} (1 - x_r) \prod_{r \notin J} (1 + x_r) \right\}_{V: J \subset [n]}$$

as a basis of $H^0(\mathbb{P}^n, \mathcal{O}_K)$. Then

$$\ker (\Phi_d) = \bigcap_{J \subset [n], |J| \geq d + 1} \left\{ f \in \mathbb{R}^{2^n} : \sum_{\tilde{J} \subset J} (-1)^{|\tilde{J}|} f_{\tilde{J}} = 0 \right\}.$$
Characterization of the $d$-SDP Exactness

Assumption

The number of distinct real solutions $|V_\mathbb{R}(g)|$ is equal to $\prod_{i=1}^{n} \deg(g_i)$.

This assumption implies that all the solutions are real and of multiplicity one. Let

$$V = \{x^0, \ldots, x^s\} \subset \mathbb{R}^n$$

be the solution set. Note that $s = \prod_{i=1}^{n} \deg(g_i) - 1$.

Theorem

The program (1) is $d$-SDP exact if and only if $\deg(f) \leq 2d$ and

$$(f - f_*)|_V \in \text{conv} \left( \mu \left( \ker (\Phi_d) \right) \right).$$

Here, the map $\mu$ is given by:

$$\mu : p|_V \rightarrow p^2|_V,$$

and $\mu \left( \ker (\Phi_d) \right) = \{p^2|_V : p \in \mathbb{R}[x_1, \ldots, x_n]_{\leq d}\}$. 
Exact Region as Image of Quadratic Map

Denote by $p_i \in \mathbb{R}[x_1, \ldots, x_n]$ the interpolation polynomial at point $x^i$ such that $p_i(x^i) = 1$ and $p_i(x^j) = 0$ for $j \neq i$. Take

$$\{p_0|\nu, \ldots, p_s|\nu\}$$

as a basis of $\phi(\mathbb{R}[x_1, \ldots, x_n]_{\leq d}) = \ker(\Phi_d)$. Then the map $\mu : p|\nu \to p^2|\nu$ can be written as:

$$\mu : (v_0, \ldots, v_s) \in \mathbb{R}^{s+1} \mapsto (v_0^2, \ldots, v_s^2) \in \mathbb{R}^{s+1}_+.$$

so that $\mu(\ker \Phi_d) = \{(q_0(w), \ldots, q_s(w)) : w \in \mathbb{R}^{pq}\}$. Based on a result due to [Barvinok 2013], we show that

**Proposition**

There is a universal constant $0 < \beta \leq 4.8$ such that for any $\nu \in \text{conv}(\mu(\ker(\Phi_d))) \cap \text{Int} \Delta^s$, there is $w \in \mu(\ker(\Phi_d)) \cap \text{Int} \Delta^s$ such that

$$KL(\nu\|w) \leq \beta.$$
Rank-2 Exact Region as Image of Moment map

If \( \deg(f) \leq 2d \) and \( (f - f_*)|_V \in \mu(\ker(\Phi_d)) + \mu(\ker(\Phi_d)) \), then

\[
f - f_* = p^2 + q^2 + \sum_{i=1}^{n} \lambda_i g_i,
\]

The previous \( \mu \) map is the restriction of the moment map:

\[
\mu : (z_0, \ldots, z_s) \in \mathbb{C}^{s+1} \mapsto (|z_0|^2, \ldots, |z_s|^2) \in \mathbb{R}^{s+1}_{+}.
\]

and we can show that

\[
\mu(\ker(\Phi_d) \otimes \mathbb{C}) = \mu(\ker(\Phi_d)) + \mu(\ker(\Phi_d)).
\]

Corollary

The program (1) is \( d \)-SDP exact if and only if \( \deg(f) \leq 2d \) and

\[
(f - f_*)|_V \in \text{conv}(\mu(\ker(\Phi_d) \otimes \mathbb{C})).
\]
The \((n − 1)\)-SDP Exact Region

There are \(a_0, \ldots, a_s \in \mathbb{R}\) such that

\[
\ker(\Phi_{n-1}) \otimes \mathbb{C} = \{ (z_0, \ldots, z_s) \in \mathbb{C}^{s+1} : a_0z_0 + \ldots + a_sz_s = 0 \}.
\]

\(f\) is \((n − 1)\)-SDP exact if and only if

\[
(f - f_*)|_V \in \text{conv} \{ (|z_0|^2, \ldots, |z_s|^2) : a_0z_0 + \ldots + a_sz_s = 0 \}
\]

**Definition (Gelfand et al. 1994)**

Let \(\ell\) be a Laurent polynomial in \(s\) variables over \(\mathbb{C}\). The *amoeba* of \(\ell\), denoted by \(A_{\ell}\), is the image of the zero set

\[
Z_\ell := \{ (z_1, \ldots, z_s) \in (\mathbb{C}^*)^s : \ell(z_1, \ldots, z_s) = 0 \}
\]

under the logarithmic modulus map \(\text{Log}:\)

\[
\text{Log} : (z_1, \ldots, z_s) \mapsto (\log |z_1|, \ldots, \log |z_s|), \; \forall (z_1, \ldots, z_s) \in (\mathbb{C}^*)^s.
\]
The \((n - 1)\)-SDP Exact Region as Amoeba

W.l.o.g. we assume \(a_0 \neq 0\). Let \(\ell\) be the first-order polynomial defined by:

\[
\ell : (z_1, \ldots, z_s) \mapsto a_0 + a_1 z_1 + \ldots + a_s z_s.
\]

The amoeba \(A_\ell\) is known as hyperplane amoeba and was studied by Forsberg et al. [Forsberg et al. 2000]. Also,

\[
\mu \left( \ker(\Phi_{n-1}) \otimes \mathbb{C} \right) \cap \Delta^s \text{ is diffeomorphic to } \overline{\xi \left( A_\ell \right)} \text{ where}
\]

\[
\xi : (w_1, \ldots, w_s) \mapsto \left( \frac{e^{2w_1}}{1 + e^{2w_1} + \cdots + e^{2w_s}}, \ldots, \frac{e^{2w_s}}{1 + e^{2w_1} + \cdots + e^{2w_s}} \right).
\]
Convexity of the Rank-2 Region for $d = n - 1$

Proposition

\[
\mu(\ker(\Phi_{n-1}) \otimes \mathbb{C}) = \text{conv} \left( \mu(\ker(\Phi_{n-1}) \otimes \mathbb{C}) \right) = \left\{ (v_0, \ldots, v_s) \in \mathbb{R}_{+}^{s+1} : 2 \max (|a_0| \sqrt{v_0}, |a_1| \sqrt{v_1}, \ldots, |a_s| \sqrt{v_s}) \leq \sum_{i=0}^{s} |a_i| \sqrt{v_i} \right\}.
\]

Corollary

The program (1) is $(n - 1)$-SDP exact if and only if

\[
2 \max \left( |a_0| \sqrt{f(x^0) - f_*}, |a_1| \sqrt{f(x^1) - f_*}, \ldots, |a_s| \sqrt{f(x^s) - f_*} \right) \leq \sum_{i=0}^{s} |a_i| \sqrt{f(x^i) - f_*}.
\]
Ronkin Function and the Spine of Amoeba

Ronkin function associated with $\ell$:

$$N_\ell(w) = \frac{1}{(2\pi i)^s} \int \log^{-1}(w) \frac{\log |\ell(z_1, \ldots, z_s)|}{z_1 \cdots z_s} \, dz_1 \cdots dz_s, \quad \forall w \in \mathbb{R}^s.$$ 

- $N_\ell$ is affine on each connected component of $A_\ell^c$.

- Piecewise linear approximation of $N_\ell$:
  
  $$S(w) = \max_{\alpha \in G} c_\alpha + \langle \alpha, w \rangle.$$ 

  We have
  
  $$N_\ell(w) = S(w), \quad \forall w \notin A_\ell.$$ 

- Spine: nondifferentiable points of $S$. 

...
A Sufficient Condition for $(n - 1)$-SDP Exactness

**Proposition**

If the maximum in

$$
\left( |a_0| \sqrt{f(x^0) - f_*}, |a_1| \sqrt{f(x^1) - f_*}, \ldots, |a_s| \sqrt{f(x^s) - f_*} \right)
$$

is attained at least twice, then the program (1) is $(n - 1)$-SDP exact.
Estimation of $f_{n-1}$

We assume that
\[ \sum_{j \neq i} |a_j| > |a_i|, \quad \forall i \in \{0, \ldots, s\}. \]

**Theorem**

*If the maximum in*
\[ \left( |a_0| \sqrt{f(x^0) - f_*}, |a_1| \sqrt{f(x^1) - f_*}, \ldots, |a_s| \sqrt{f(x^s) - f_*} \right) \]

*is attained uniquely at* $|a_i| \sqrt{f(x^i)} - f_*$, *then*
\[
\begin{align*}
    f_* - \frac{a_i^2 \max \left( a_i^2 (f(x^i) - f_*) - \left( \sum_{j \neq i} |a_j| \sqrt{f(x^j) - f_*} \right)^2, 0 \right)}{\left( \sum_{j \neq i} |a_j| \right)^2 - a_i^2} & \leq f_{n-1} \leq f_*.
\end{align*}
\]
Applications

Apply the results to the special case when $|a_i| = 1$, $\forall i \in \{0, 1 \ldots, s\}$.

Corollary

*Problem (1) is $d$-SDP exact for any $d \geq n - 1$ if one of the three conditions hold.*

- $\deg(f) \leq n - 1$.
- The maximum of $f$ on the feasible region $V$ is attained at least twice.
- $f$ is a polynomial without constant term and

$$f^* \leq \left(\frac{s + 1}{4} - 1\right)(-f^*),$$

Corollary

*If the maximum value $f^*$ is attained uniquely at $x^i$, then*

$$\max\left(\left(f^* - f_*\right) - \left(\sum_{j=0}^{s} \sqrt{f(x^j) - f_* - \sqrt{f^* - f_*}}\right)^2, 0\right) \leq \frac{f_{n-1}}{s^2 - 1} \leq f_*.$$
Conclusion

- Degree bound of Lasserre’s SOS relaxation under the complete intersection assumption
- Connection of the exact region with moment map image
- Description and approximation of the \((n - 1)\)-SDP exact region with the aid of the theory of amoeba
Degree bound on

\[ f_\ast = \min_{x \in \mathbb{R}^n} f(x) \]

s.t. \( g_1(x) = g_2(x) = \ldots = g_m(x) = 0 \),

when the set of critical points is finite.

Degree bound depending the degree of \( f \).

More exploitation of moment map image, convex hull and Barvinok’s result.

Inner and outer approximation of \( d \)-SDP exact region.
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