Spectrahedral representations of hyperbolic plane curves

BrainPOP - CNRS LAAS, Toulouse

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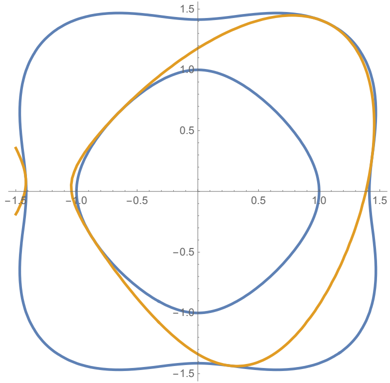

Context

Convex semialgebraic set : $\Lambda_+(f,e) = \{a \in \mathbb{R}^n : f(a) \geq 0, \ldots\}$

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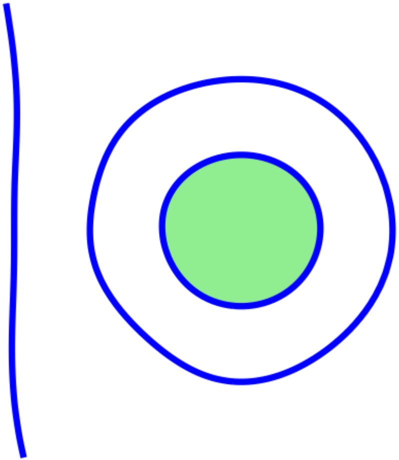

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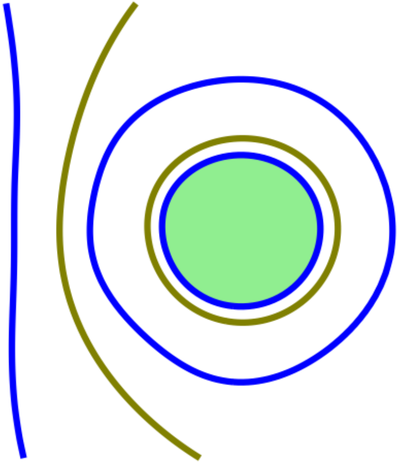

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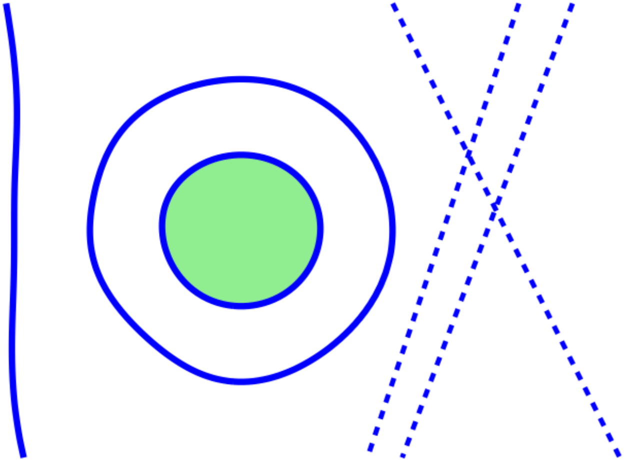
Hyperbolic curve : $V(f) = \{a \in \mathbb{R}^n : f(a) = 0\}$

Second curve : $g \in \mathbb{R}[x]_{\deg f - 1}$. Extra-factor : $\ell_1 \cdot \ell_2 \cdots \ell_s$



Hyperbolic polynomials

A homogeneous polynomial $f \in \mathbb{R}[x]_d$ is called *hyperbolic in direction* e if

- $f(e) \neq 0$
- The characteristic polynomial $f(te - a)$ is real rooted, for every $a \in \mathbb{R}^n$.

It is *hyperbolic* whenever such a direction e exists.

Examples:

- $f = \ell_1 \ell_2 \cdots \ell_d$, with $\ell_i \in \mathbb{R}[x]_1$, is hyperbolic
- $f = \det(X)$, with $X = (x_{ij})$ symmetric, is hyperbolic in direction I
- More generally, $f = \det(x_1 A_1 + \cdots + x_n A_n)$, with $e_1 A_1 + \cdots + x_n A_n \succ 0$, is hyperbolic in direction $e = (e_1, \ldots, e_n)$
- More generally, $f^k = \det(x_1 A_1 + \cdots + x_n A_n)$, with $e_1 A_1 + \cdots + x_n A_n \succ 0$, for some $k \in \mathbb{N}$, implies f hyperbolic in direction $e = (e_1, \ldots, e_n)$

often called "determinantal" and "weakly determinantal".

Hyperbolicity cone

Let $f \in \mathbb{R}[x]_d$ be hyperbolic in direction e. The set

$$\Lambda_+(f,e) = \{a \in \mathbb{R}^n : ch_a(t) = 0 \Rightarrow t \geq 0\}$$

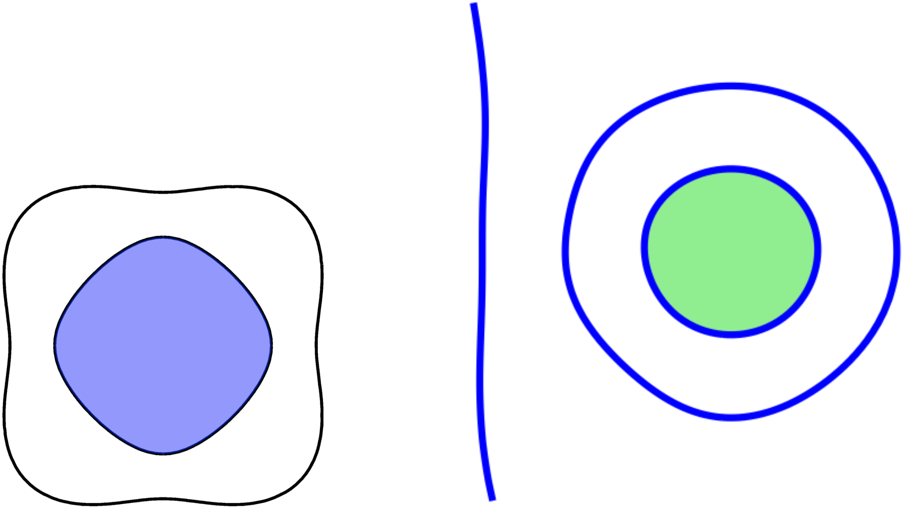
is called the *hyperbolicity cone in direction* e of f.

Equivalent definition: the connected component of $\mathbb{R}^n \setminus V(f)$ containing e.

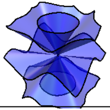
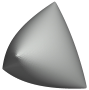
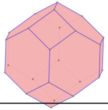
There are "many" hyperbolicity cones:

- Only one pair, if f is irreducible [Kummer, 2018]
- Bound of $2\sum_{k=0}^{n-1}\binom{d-1}{k}$ for large d, or of 2^d for large n, attained for products of linear forms [Theobald et al., 2018]

Hyperbolicity cone



Optimization viewpoint

Feasible set	*name*	**Optimization**	**Polynomial**
	Hyperbolicity Cone	HP	Hyperbolic polynomial
	Spectrahedron	SDP	$f = \det\ A(x)$
	Polyhedron	LP	$f = \prod \ell_i(x)$

Spectrahedral representation

A set $S \subset \mathbb{R}^n$ has a spectrahedral representation if there are real symmetric matrices $A = (A_1, \ldots, A_n)$ such that $S = S_A$ with

$$S_A := \{x \in \mathbb{R}^n : x_1 A_1 + \cdots + x_n A_n \succeq 0\}$$

If $f = \det(X)$, then $\Lambda_+(f, I) = \mathbb{S}^d_+$ (psd matrices), and more generally if $f^k = \det(x_1 A_1 + \cdots + x_n A_n)$, with $e_1 A_1 + \cdots + e_n A_n \succ 0$, then

$$\Lambda_+(f^k, e) = \Lambda_+(f, e) = S_A$$

If f is hyperbolic in direction e, is $\Lambda_+(f, e)$ spectrahedral ?

Generalized Lax conjecture

Geometric version. Let $f \in \mathbb{R}[x]_d$ be hyperbolic in direction e. Then

$$\Lambda_+(f, e) = S_A$$

for some real symmetric matrices $A = (A_1, \dots, A_n)$.

or, equivalently:

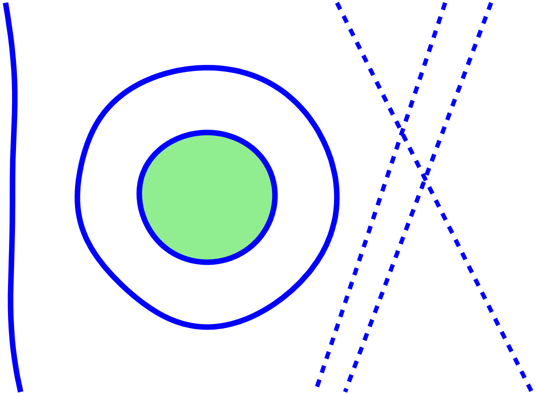
Algebraic version. Let $f \in \mathbb{R}[x]_d$ be hyperbolic in direction e. Then there is $g \in \mathbb{R}[x]_c$ s.t.

- $fg = \det(x_1 A_1 + \cdots + x_n A_n)$ ["generalized determinantal"]
- $\Lambda_+(f, e) \subset \Lambda_+(g, e)$

If such a g exists, then

$$\Lambda_+(f, e) = \Lambda_+(f, e) \cap \Lambda_+(g, e) = \Lambda_+(fg, e) = S_A$$

Our construction



$\Lambda_+(f, e) =$ green region

$f =$ blue curve

$g = \ell_1 \ell_2 \cdots \ell_c$, the "extra factor", whose cone is a polyhedral set that strictly contains $\Lambda_+(f, e)$

Contact curves and interlacers

Let $f \in \mathbb{R}[x,y,z]_d$ be hyperbolic in direction e. Then $h \in \mathbb{R}[x,y,z]_{d-1}$ is a

- ***contact curve*** for f if every intersection point of $V_{\mathbb{C}}(f)$ and $V_{\mathbb{C}}(g)$ has even multiplicity
- ***real contact curve*** for f if every intersection point of $V_{\mathbb{R}}(f)$ and $V_{\mathbb{R}}(g)$ has even multiplicity
- ***interlacer*** of f ***in direction*** e if the roots $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_d$ of $f(te-a)$ and the roots $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_{d-1}$ of $h(te-a)$ interlace perfectly, namely

$$\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \beta_2 \leq \cdots \leq \beta_{d-1} \leq \alpha_d$$

contact curve $>>>$ real contact curve $<<<$ interlacer

Dixon method (variant of Plaumann-Vinzant)

Based on *Cramer's rule* : $\boxed{A \cdot A^{adj} = \det A \cdot I_d}$

Assume $f = \det A$ and $V_{\mathbb{C}}(f)$ is smooth, then

$$\text{co-rank}(A) = \text{rank}(A^{adj}) = 1 \mod \det A.$$

INPUT

f (hyperbolic curve)

OUTPUT

Hermitian matrix A (satisfying $f = \det A$ and $A(e) \succ 0$)

Sketch of the PROCEDURE :

$m_{11} \leftarrow D_e f := e_1 \frac{\partial f}{\partial x} + e_2 \frac{\partial f}{\partial y} + e_3 \frac{\partial f}{\partial z}$ # $D_e f$ special interlacer

split $S \cup \overline{S} = V_{\mathbb{C}}(f) \cap V_{\mathbb{C}}(D_e f)$

extend m_{11} to basis $\langle m_{11} \ldots m_{1d} \rangle$ of polyn. vanishing on S

$m_{jk} \leftarrow$ solve $a_{11} a_{jk} - \overline{a_{1j}} a_{1k} = 0$ mod f for $j \leq k$ # rank $= 1$

$M \leftarrow (m_{jk})$

$A \leftarrow M^{adj} / f^{d-2}$

The cone of interlacers

Precomputing an interlacer is "easier" than precomputing a contact curve.

This is because the set of interlacers is ***tractable***:

(Kummer-Plaumann-Vinzant 2013)

$$Int(f,e) = \{h \in \mathbb{R}[x,y,z]_{d-1} \mid h \text{ interlaces } f \text{ in direction } e\}$$

is a section of the cone of positive polynomials:

$$Int(f,e) = \{h \in \mathbb{R}[x,y,z]_{d-1} \mid W(f,h) := (D_e f)h - f(D_e h) \geq 0\}$$

Thus interlacers can be sampled by computing a SOS decomposition of $W(f,h)$: such relaxation is **exact** if f is **determinantal**.

Main result

Let $f \in \mathbb{R}[x,y,z]_d$ be hyperbolic with respect to e, and let $h \in Int(f,e)$. Let $\ell_1,\ldots,\ell_c$ be the (real) lines joining the pairs of complex intersections of f and h.

Then* there are $A_1, A_2, A_3 \in \mathbb{S}^m$, with $m = (d^2+d)/2 - r$, such that

- $\Lambda_+(f,e) = \{x \in \mathbb{R}^3 : xA_1 + yA_2 + zA_3 \succeq 0\}$
- $f \cdot \ell_1 \cdot \ell_2 \cdots \ell_c = \det(x_1A_1 + x_2A_2 + x_3A_3)$

* up to genericity assumptions on g:

1. no three intersection points of f and h aligned
2. no three of the lines pass through the same point
3. f does not vanish over intersection points of two lines

Our variant

The **main point** is that the extra factor $g = \ell_1 \cdots \ell_c$ corrects the failure of h to be a contact curve, by adding multiplicity to the complex intersections of f and h.

Positive aspects:

1. The multiplier is the simplest we can get : product of linear forms
2. The size of the representation depends on r, the number of real intersections.
3. For maximal r, one gets the Helton-Vinnikov representation

Maximizing r means minimizing the size of the representation, and means that the interlacer is “special”.

Question: how to get interlacers with many real intersections?

Extremal interlacers

An extreme point of $Int(f, e)$ is called an ***extremal interlacer***

It corresponds to interlacers with “many” real intersections with f.

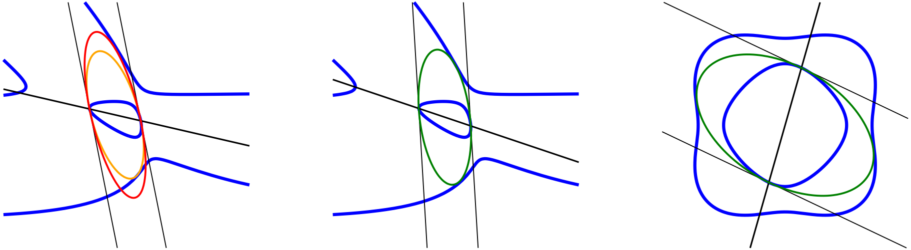
If f is smooth, any extremal interlacer has at least

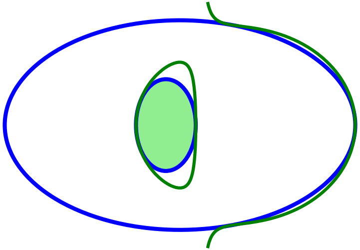
$$\left\lceil \frac{(d+1)d-2}{4} \right\rceil$$

contact points (counted multiplicities).

degree	d	2	3	4	5	6	$\cdots$
lower bound for extr. int.	$\left\lceil \frac{(d+1)d-2}{4} \right\rceil$	1	3	5	7	10	$\cdots$
Bézout	$\frac{d(d-1)}{2}$	1	3	6	10	15	$\cdots$

Quartic curves

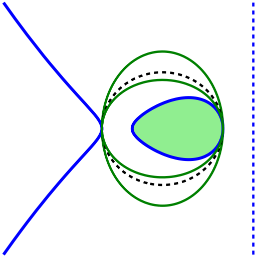




Open question : can we always have **six** intersections for $d = 4$?

One example

The cubic $f = x^3 + 2x^2y - xy^2 - 2y^3 - xz^2$ is hyperbolic with respect to $e = (1, 0, 0)$.



The two green interlacers have coefficients in K with $|K : \mathbb{Q}| = 4$, and the dashed one is rational. The corresponding spectrahedral representation is

$$\frac{24}{125}(2x - y) \cdot f = \det \begin{pmatrix} 5x + 10y & -x - 2y & -4z & 2z \\ -x - 2y & x & 0 & 0 \\ -4z & 0 & 4x + 2y & -2x - 4y \\ 2z & 0 & -2x - 4y & 4x + 2y \end{pmatrix}.$$

Rational representations

We prove that there are (possibly large) **rational** spectrahedral representations (even in the case when there are no rational determinantal representations)

Let $f \in \mathbb{Q}[x, y, z]$ be a hyperbolic curve, with smooth real zero set. Then its hyperbolicity cone has a *rational* spectrahedral representation, of size at most $\binom{d+1}{2}$.

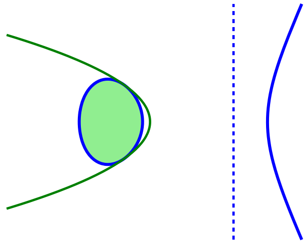
On the other hand, there are hyperbolic curves in $\mathbb{Q}[x, y, z]_d$ that do **not** admit rational *determinantal representations*.

Example

$f = y^2z - (x^3 - 6xz^2 - 3z^3)$ has no rational 3×3 determinantal representation, but it has a rational generalized representation, which yields the spectrahedral representation

$$\Lambda_+(f,e) = \left\{ \begin{pmatrix} 3z & y & -x-z & -3x+z \\ y & -x+2z & 0 & -y \\ -x-z & 0 & z & x+4z \\ -3x+z & -y & x+4z & -x+18z \end{pmatrix} \succeq 0 \right\}.$$

The extra-factor is a line and the interlacer has two contact points:



References

This part is based on

- "Spectrahedral representations of plane hyperbolic curves"
  (M. Kummer, S. Naldi, D. Plaumann) Pac. J. Math. 303(1):243–263 (2019)

Related papers:

- "Determinantal repr. of hyperbolic plane curves: An elementary approach"
  (Plaumann, Vinzant) Journal of Symb. Comp. Vol. 57, 48–60 (2013)
- "Computing Hermitian determinantal representations of hyperbolic curves"
  (Plaumann, Sinn, Speyer, Vinzant) Int. J. Alg. Comp. 25(8):1327-1336 (2015)