Time-Dependent Moments and the Heat Equation

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ArXiv 2108.03505: joint work with Raúl E. Curto moment:

$$s_\alpha := \int_{\mathbb{R}^n} x^\alpha \, \mathrm{d}\mu(x) \tag{$*$}$$

moment sequence:

$$s = (s_\alpha)_{\alpha\in\mathbb{N}_0^n} \text{ moment sequence} \quad \Leftrightarrow \quad \exists \mu \geq 0 : (*)$$

Riesz functional: $L_s : \mathbb{R}[x_1, \dots, x_n] \to \mathbb{R}$ with

$$L_s(x^\alpha) := s_\alpha$$

criteria:

$$s \text{ moment sequence} \quad \Leftrightarrow \quad L_s(p) \geq 0 \text{ for all } p \in \mathbb{R}[x_1, \dots, x_n] \text{ with } p \geq 0$$

- translation $\mu \mapsto \mu(\cdot - c)$

$$\int x^\alpha \, \mathrm{d}\mu(x-c) = \int (x+c)^\alpha \, \mathrm{d}\mu(x)$$

- rotation and scaling $\mu \mapsto \mu(R\,\cdot\,)$

$$\int x^\alpha \, \mathrm{d}\mu(Rx) = \int (R^{-1}x)^\alpha \, \mathrm{d}\mu(x)$$

- translation + rotation + scaling

Question:

- are there other transformations and what do they preserve or accomplish?

Requirement:

- transformation must be known for the moments and the measure

heat equation:
$$\partial_t u(x,t) = \Delta u(x,t) = (\partial_1^2 + \cdots + \partial_n^2) u(x,t)$$
$$u(x,0) = u_0(x)$$

heat kernel: $\Theta_t(x) := \frac{1}{(4\pi t)^{n/2}} \cdot e^{-\frac{|x|^2}{4t}} \quad = \quad \begin{cases} \text{Dirac delta } \delta_0 & t = 0 \\ \text{Gaussian} & t > 0 \end{cases}$

unique solution: $u(x,t) = (\Theta_t * u_0)(x)$

Schwartz functions[1] $f \in \mathcal{S}(\mathbb{R}^n)$: $\|x^\alpha \cdot \partial^\beta f(x)\|_\infty < \infty$ for all $\alpha, \beta \in \mathbb{N}_0^n$

$$u_0 \in \mathcal{S}(\mathbb{R}^n) \quad \Rightarrow \quad u(\,\cdot\,,t) \in \mathcal{S}(\mathbb{R}^n) \text{ for all } t \geq 0$$

[1] multi-index notation: $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ and $\partial^\beta := \partial_1^{\beta_1} \partial_2^{\beta_2} \cdots \partial_n^{\beta_n}$

$$\partial_t u = \Delta u = \partial_x^2 u(x,t)$$

$$s_\alpha(t) = \int_{\mathbb{R}} x^\alpha \cdot u(x,t)\,\mathrm{d}x \qquad (\alpha \in \mathbb{N}_0)$$

induction:

$$\partial_t s_\alpha(t) = \alpha \cdot (\alpha - 1) \cdot s_{\alpha-2}(t)$$

$$\begin{aligned}
\partial_t s_0(t) &= 0 && \Rightarrow && s_0(t) = s_0(0)\\
\partial_t s_1(t) &= 0 && \Rightarrow && s_1(t) = s_1(0)\\
\partial_t s_2(t) &= 2s_0(t) && \Rightarrow && s_2(t) = s_2(0) + 2s_0(0)\cdot t\\
\partial_t s_3(t) &= 6s_1(t) && \Rightarrow && s_3(t) = s_3(0) + 6s_1(0)\cdot t\\
\partial_t s_4(t) &= 12s_2(t) && \Rightarrow && s_4(t) = s_4(0) + 12s_2(0)\cdot t + 12s_0(0)\cdot t^2\\
\partial_t s_5(t) &= 20s_2(t) && \Rightarrow && s_5(t) = s_5(0) + 20s_3(0)\cdot t + 60s_1(0)\cdot t^2\\
& && \vdots &&
\end{aligned}$$

$n = 1$:

$$s_{2k}(t) = \sum_{j=0}^{k} \frac{(2k)! \cdot s_{2k-2j}(0)}{(2k-2j)! \cdot j!} \cdot t^j \qquad s_{2k+1}(t) = \sum_{j=0}^{k} \frac{(2k+1)! \cdot s_{2k+1-2j}(0)}{(2k+1-2j)! \cdot j!} \cdot t^j$$

Lemma

$\alpha \in \mathbb{N}_0^n$:

$$s_\alpha \in \mathbb{R}[t] \quad \textit{with} \quad \deg s_\alpha = \left\lfloor \frac{\alpha_1}{2} \right\rfloor + \cdots + \left\lfloor \frac{\alpha_n}{2} \right\rfloor$$

Definition

$d \in \mathbb{N} \cup \{\infty\}$ *and* $s = (s_\alpha(0))_{|\alpha| \leq d}$ *sequence*[2]*:*

$$\mathfrak{p}_{s,\alpha}(t) := s_\alpha(t) \qquad \textit{and} \qquad \mathfrak{p}_s := (\mathfrak{p}_{s,\alpha})_{|\alpha| \leq d} \subset \mathbb{R}[t]$$

[2] not necessarily a moment sequence

Corollary

- A) $\mathfrak{p}_s(0) = s$
- B) $\mathfrak{p}_{a\cdot s+b\cdot s'} = a\cdot\mathfrak{p}_s + b\cdot\mathfrak{p}_{s'}$
- C) $\mathfrak{p}_{\mathfrak{p}_s(t_1)}(t_2) = \mathfrak{p}_s(t_1+t_2)$

Proof. C) semi-group property $\Theta_{t_1} * (\Theta_{t_2} * f) = \Theta_{t_1+t_2}$.[3] □

Theorem

$k\in\mathbb{N}$, $N\in\mathbb{N}\cup\{\infty\}$, $y_i\in\mathbb{R}^n$, $c_i\in\mathbb{R}$, $t_1,\dots,t_k\in[0,\infty)$. $\tau := \min_i t_i$.

$$\begin{array}{ccc} s \textit{ represented by} & & \mathfrak{p}_s(t) \textit{ represented by} \\ \mu_0(x) := \sum_{i=0}^k c_i\cdot\Theta_{t_i}(x+y_i) & \xrightarrow{t\in[-\tau,\infty)} & \mu_t(x) := \sum_{i=0}^k c_i\cdot\Theta_{t_i+t}(x+y_i) \end{array}$$

Proof. Convolution with the heat kernel. □

[3] here actually a group isomorphic to $(\mathbb{R},+)$

$d \in \mathbb{N} \cup \{\infty\}$: $\mathcal{S}_d :=$ moment cone ($=$ set of all moment sequences $|\alpha| \leq d$).

Definition

$$\mathfrak{I}_s := \{t \in \mathbb{R} \,|\, \mathfrak{p}_s(t) \in \mathcal{S}_d\}$$

Theorem

$s \in \mathcal{S}_d \setminus \{0\}$. $\exists \mathfrak{d}_s \in \left[0, \frac{s_{2e_1}(0)+\dots+s_{2e_n}(0)}{2n\cdot s_0(0)}\right]$ *(we call it heat distance to the boundary of $\mathcal{S}_d$):*

$$d \in \mathbb{N}: \qquad \mathfrak{I}_s = [-\mathfrak{d}_s, \infty) \quad \text{or} \quad (-\mathfrak{d}_s, \infty)$$

$$d = \infty: \qquad \mathfrak{I}_s = [-\mathfrak{d}_s, \infty).$$

Proof. $\underline{d \in \mathbb{N}}$: s represented by atomic measure $\Rightarrow$ $\mathfrak{p}_s(t)$ represented by Gaussian mixture for $t > 0$. Since $\mathcal{S}_d$ not closed $\Rightarrow$ $[-\mathfrak{d}_s, \infty)$ or $(-\mathfrak{d}_s, \infty)$.[4]

$\underline{d = \infty}$: $s|_k$ truncation $\Rightarrow$ $(-\mathfrak{d}_s, \infty) \subseteq \bigcap_{k\in\mathbb{N}} (-\mathfrak{d}_{s|_k}, \infty)$.

$-\mathfrak{d}_s \in \mathfrak{I}_s$: $p \in \mathbb{R}[x_1, \dots, x_n]$, $p \geq 0$ $\Rightarrow$ $L_{\mathfrak{p}_s(-\mathfrak{d}_s)}(p) = \lim_{t \searrow -\mathfrak{d}_s} L_{\mathfrak{p}_s(t)}(p) \geq 0$. $\square$

[4] example for $(-\mathfrak{d}_s, \infty)$?

Definition (heat curve $\mathfrak{C}_s$)

$$\mathfrak{C}_s := \mathfrak{p}_s(\mathfrak{I}_s) \quad \subset \mathcal{S}_d$$

Corollary

$d \in \mathbb{N} \cup \{\infty\}$, $d \geq 2$. *For sequences* s, s' *either*

A. $\mathfrak{C}_s = \mathfrak{C}_{s'}$

or

B. $\mathfrak{C}_s \cap \mathfrak{C}_{s'} = \emptyset$.

Proof. $\mathfrak{p}_{\mathfrak{p}_s(t_1)}(t_2) = \mathfrak{p}_s(t_1 + t_2)$. □

Definition (equivalence relation $\sim_{\mathfrak{p}}$)

$$s \sim_{\mathfrak{p}} s' \quad :\Leftrightarrow \quad \mathfrak{C}_s = \mathfrak{C}_{s'}$$

Theorem

$n, d \in \mathbb{N}$, $d \geq 2$, and

$$\mathcal{C}_d := \bigcup_{s \in \partial\mathcal{S}_d \cap \mathcal{S}_d} \mathfrak{C}_s \subseteq \mathcal{S}_d.$$

Then

$$\partial\mathcal{S}_d \cap \mathcal{S}_d \;\cong\; \mathcal{C}_d/\sim_{\mathfrak{p}}$$

and

$$\mathfrak{p} : (\partial\mathcal{S}_d \cap \mathcal{S}_d) \times [0,\infty) \hookrightarrow\!\!\!\rightarrow \mathcal{C}_d, \;\; (s,t) \mapsto \mathfrak{p}_s(t)$$

is a bijective and polynomial map with the inverse map

$$s \mapsto (\mathfrak{p}_s(-\mathfrak{d}_s), \mathfrak{d}_s).$$

$d = \infty$: what is $\partial\mathcal{S}_\infty$?

Example. $\mathrm{d}\mu(x) = \chi_{[0,1]}(x)\,\mathrm{d}x$ determinate but $L(p) > 0$ for all $p \in \mathbb{R}[x]$ with $p \geq 0$
$\Rightarrow$ μ is a starting point of a heat curve

structure of the heat curves $\mathfrak{C}_s$:

Theorem

$s \in \mathcal{S}_\infty$. There exists a family $\{f_t\}_{t>0}$ of C^∞-functions with

$$f_{t_1+t_2} = \Theta_{t_1} * f_{t_2} = \Theta_{t_2} * f_{t_1}$$

such that

$$\mathfrak{p}_{s,\alpha}(t) = \int x^\alpha \cdot f_t(x)\ \mathrm{d}x,$$

i.e., $\mathfrak{p}_s(t)$ is represented by

$$\mathrm{d}\mu_t(x) = f_t(x)\ \mathrm{d}x.$$

$\mathfrak{C}_s \cong [0,\infty)$ and on $(0,\infty)$ there are always C^∞ absolutely continuous representing measures

We have seen:

1. heat equation induces time-dependent moments $\mathfrak{p}_s(t)$
2. $\mathfrak{p}_s(t)$ moment sequence for all $t \in \mathfrak{I}_s = [-\mathfrak{d}_s, \infty)$ or $(-\mathfrak{d}_s, \infty)$
3. heat curves $\mathfrak{C}_s = \mathfrak{p}_s(\mathfrak{I}_s)$ are equivalence classes
4. all $s' \in \mathfrak{C}_s$ (maybe except the starting point) are represented by C^∞-functions

Question:

- Do $s' \in \mathfrak{C}_s$ share common properties?

s (in)determinate $:\Leftrightarrow$ (not) a unique representing measure

Theorem

$s \in \mathcal{S}_\infty$ indeterminate $\Rightarrow$ $\mathfrak{p}_s(t)$ indeterminate for all $t \geq 0$.

Proof: $\varepsilon > 0$ two C^∞-functions $f_1 \neq f_2$ in abs. cont. repr. measure, then heat equation. $\square$

Example (indeterminate). $c \in [-1, 1]$.

$$f_c(x) = (1 + c \cdot \sin(2\pi \cdot \ln x)) \cdot \frac{\chi_{(0,\infty)}(x)}{\sqrt{2\pi} \cdot x} \cdot e^{-(\ln x)^2/2} \qquad \Rightarrow \qquad s_k = e^{k^2/2}$$

Corollary

$s \in \mathcal{S}_\infty$ determinate $\Rightarrow$ $\mathfrak{p}_s(t)$ determinate for all $t \in \mathfrak{I}_s \cap (-\infty, 0]$.

Open: Can a determinate sequence become indeterminate on $\mathfrak{C}_s$?

Definition

- (a) $s \in \mathcal{S}_\infty$ *determinate if representing measure* μ *unique*
- (b) $s \in \mathcal{S}_\infty$ *strongly determinate if (a) and* $\mathbb{C}[x_1,\dots,x_n]$ *dense in all* $L^2(\mathbb{R}^n,(1+x_i^2)\,\mathrm{d}\mu)$, $i=1,\dots,n$

Theorem

$s \in \mathcal{S}_\infty$ *with representing measure* μ *such that for an* $\varepsilon > 0$ *we have*

$$\int_{\mathbb{R}^n} e^{\varepsilon\cdot\|x\|}\,\mathrm{d}\mu(x) < \infty.$$

Then all $s' \in \mathfrak{C}_s$ *are strongly determinate (short:* $\mathfrak{C}_s$ *is strongly determinate).*

Proof. $\mathfrak{p}_s(t)$ represented by μ_t with $\varepsilon_t > 0$ and

$$\int_{\mathbb{R}^n} e^{\varepsilon_t\cdot\|x\|}\,\mathrm{d}\mu_t(x) < \infty. \quad \square$$

velocity field $ax := (a_1x_1, \ldots, a_nx_n)$:

$$\partial_t u(x,t) = ax \cdot \nabla u(x,t)$$

solution:

$$u(x,t) = u_0(x_1 e^{a_1 t}, \ldots, x_n e^{a_n t})$$

moments:

$$s_\alpha(t) = s_\alpha(0) \exp\left(-\sum_{i=1}^n a_i(\alpha_i + 1)t\right)$$

preserved:

- (in)determinacy
- Carathéodory number
- being a moment sequence for all $t \in \mathbb{R}$

$$\partial_t u(x,t) = \nu \Delta u(x,t) + ax \cdot \nabla u(x,t)$$

solution:

- no explicit formula since $[\Delta, ax\nabla] \neq 0$
- only implicit solution

moments:

- explicit expression via induction

$$\begin{aligned} s_\alpha(t) = s_\alpha(0) \exp\left(-\sum_{j=1}^n a_j(\alpha_j+1)t\right) \\ + \nu \int_0^t [\alpha_1(\alpha_1-1)\cdot s_{\alpha-2e_1}(\tau) + \ldots + \alpha_n(\alpha_n-1)\cdot s_{\alpha-2e_n}(\tau)] \\ \times \exp\left(\sum_{j=1}^n a_j(\alpha_j+1)\tau\right)\, \mathrm{d}\tau \cdot \exp\left(-\sum_{j=1}^n a_j(\alpha_j+1)t\right) \end{aligned}$$

$$\partial_t u(x,t) = -u(x,t)\partial_x u(x,t)$$

solution:

- no explicit solution
- finite break down!
- with viscosity ($\partial_t u = \nu\partial_x^2 u - u\partial_x u$) no break down (Cole-Hopf transformation 1950/51)

moments:

- $s_{k,p}(t) := \int_{\mathbb{R}} x^k \cdot u(x,t)^p \, \mathrm{d}x$:

$$s_{k,p}(t) = \sum_{i=0}^{k} \frac{s_{k-i,p+i}(0)}{i!} \cdot t^i \cdot \prod_{j=0}^{i-1} \frac{(p+j)\cdot(k-j)}{1+(p+j)^2} \quad \in \mathbb{R}[t].$$

- s (u_0) determinate $\Rightarrow$ time-dependent moments well-defined (unique)
- s (u_0) indeterminate: $s_{k,p}(0)$ different (?)
- how is the finite break down observed trough the moments?

Thank you for your attention!