Reducing non-negativity over general semialgebraic sets to non-negativity over simple sets

Olga Kuryatnikova

ERASMUS UNIVERSITEIT ROTTERDAM

Joint work with Juan C. Vera (Tilburg University),
Luis F. Zuluaga (Lehigh University)

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Polynomial optimization and non-negativity

- Let $p, g_1, ..., g_m$ be polynomials

$$\begin{aligned} & \inf_x \; p(x) \text{ s.t. } g_1(x) \geq 0, \ldots, g_m(x) \geq 0 \\ = & \sup_\lambda \; \lambda \text{ s.t. } p(x) - \lambda \geq 0 \text{ for all } x \text{ such that } g_1(x) \geq 0, \ldots, g_m(x) \geq 0 \end{aligned}$$

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  $\rightarrow$ lower bounds for optimization

Certificates of non-negativity

[Artin '1927], [Pólya '1928], [Krivine '1964]-[Stengle '1974], [Handelman '1988], [Schmüdgen '1991], [Putinar '1993], [Reznick '1995], [Putinar and Vasilescu '1999], [Lasserre '2001,2006,2015], [Powers '04], [Peña, Vera, Zuluaga '2005], [Waki et al. '2006], [Nie et al. '06], [Demmel et al. '07], [Marshall '10], [Nguyen and Powers '10], [Nie '10], [Jeyakumar et al. '2014], [Dickinson and Povh '2015, 2018], [Ahmadi and Hall '2017], [Ahmadi and Majumdar '2017], [Dressler et al. '2017], [Peña, Vera, Zuluaga '2017], [Wang '2018], [Mai, Lasserre, Magron '2020, 2022]

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Ideal solution: an automatic procedure to generate such certificates for a problem and solver at hand.

Observations about simple sets

- There are more certificates for simple sets (box, simplex, ball)
- Both lower and upper bounds when optimizing over simple sets
- Often easy to find a simple set that contains a given set

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- Lift initial set and polynomial to $\mathbb{R}^{n+m}$
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The process is automatic and universal, c.f. [Schweighofer '2002]

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- Universal lifting approach for unbounded sets
- Certificates and PO for unbounded sets

Main result for compact sets

Notation: $\mathbb{R}_d[x]$ are polynomials of degree $\leq d$ with real coefficients; $\mathcal{P}_d(S) \subset \mathbb{R}_d[x]$ (resp. $\mathcal{P}_d^+(S) \subset \mathbb{R}_d[x]$) are polynomials non-negative (resp.positive) on S; and e is the vector of all ones.

Theorem 1 (K.,Vera, Zuluaga)

Let $S = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_m(x) \geq 0\}$ *be non-empty and compact. Let* $T \subset \mathbb{R}^n \times \mathbb{R}^m_+$ *be any compact set such that* $\{(x, g(x)) : x \in S\} \subseteq T$. *For* $p \in \mathcal{P}^+(S)$ *there is* $F \in \mathcal{P}^+_{d_{\max}}(T)$ *such that*

$$p(x) = F(x, g(x)),$$

where $d_{\max} = \max\{2 \deg g_1, \ldots, 2 \deg g_m, \deg p\}$.

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Proof: lift S to $\mathbb{R}^{n+m}$ via equalities obtaining a set $\{x \in \mathbb{R}^n, u \in \mathbb{R}^m_+ : (g_1(x) - u_1)^2 = 0, \ldots, (g_m(x) - u_m)^2 = 0\} \cap T$ and use a representation from [Peña, Vera, Zuluaga '2008].

Box and simplex as simple sets

Lift each set to the same set type in a higher dimension

- **Box:** Let L, M be such that $S \subseteq \{x \in \mathbb{R}^n : L \leq x \leq M\}$:
  $T = \{(x, u) : L \leq x \leq M,\ 0 \leq u \leq \hat{M}\}$
  for any $\hat{M}_j \geq \|g_j\|(1 + e^\top \max\{|M|, |L|\})^{\deg g_j}$, $j \in \{1, \ldots, m\}$.

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- **Simplex:** Let L, M be such that $S \subseteq \{x \in \mathbb{R}^n : x \geq L, e^\top x \leq M\}$:
  $T = \{(x, u) : L \leq x,\ 0 \leq u,\ e^\top x + e^\top u \leq \hat{M}\}$
  for any $\hat{M} \geq M + (1 + M + e^\top(|L| - L))^{\deg g_j} \|g_j\|$.

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Do not lift a ball to a ball, can use a more efficient strategy

Optimization on compact sets

$$\inf_{x} \; p(x) \text{ s.t. } x \in S \quad = \quad \sup_{\lambda, F} \; \lambda \text{ s.t. } \begin{aligned} & p(x) - \lambda = F(x, g(x)), \\ & F \in \mathcal{P}_{d_{\max}}(T). \end{aligned}$$

- **Lower bound:** use any *certificate* of non-negativity on T
  $\rightarrow$ certificates of non-negativity on box and simplex

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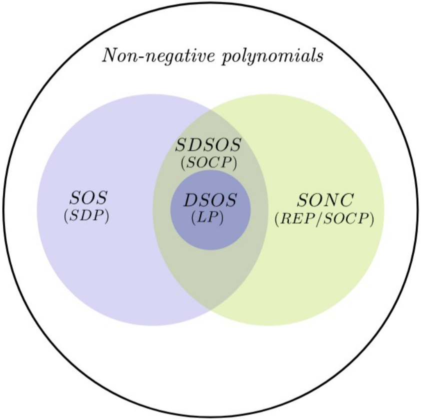
- **Lower bound:** use any *certificate* of non-negativity on T
  $\rightarrow$ certificates of non-negativity on box and simplex
- **Upper bound:** use any *outer* approximation for $\mathcal{P}_{d_{\max}}(T)$
  $\rightarrow$ Simplex: outer hierarchies for copositive tensors
  $\rightarrow$ Box, simplex: integral-based hierarchy [Lasserre '2011, 2020]

Deriving new certificates

- Go beyond SOS
- Try to use polynomials with few terms (sparse in some sense)
- Combination with correlative or term sparsity possible



Use Box: Non-SOS Schmudgen-type certificates

Proposition 1 (Schmudgen-type Certificates)

Let $S = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_m(x) \geq 0\} \subseteq \{x \in \mathbb{R}^n : L \leq x \leq U\}$. *For* $p \in \mathcal{P}^+(S)$ *there are* $r \geq 0$ *and* $\sigma_{\alpha,\beta,\gamma} \in \mathcal{K}$ *such that*

$$p(x) = \sum_{(\alpha,\beta,\gamma) \in \mathbb{N}_r^{2n+m}} \sigma_{\alpha,\beta,\gamma}(x)(x-L)^\alpha (U-x)^\beta g(x)^\gamma,$$

and $\mathcal{K}$ *can be any class of non-negative polynomials containing non-negative constants.*

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Possible $\mathcal{K}$ with fewer terms than SOS:

- $\mathbb{R}_+$, c.f. [Dickinson and Povh '2018]
- DSOS, SDSOS, c.f. [Kuang et al. '2017]
- SONC, c.f. [Dressler, Iliman, de Wolff '2017]

Use simplex: Semi-sparse Putinar-type Certificates

Proposition 2 (Putinar-type Certificates via Simplex)

Let $S = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_m(x) \geq 0\} \subseteq \{x : L \leq x, e^\top x \leq M\}$. *Let* $\hat{g}(x) = (x - L, g(x), \widehat{M} - e^\top x - e^\top g(x))$. *For* $p \in \mathcal{P}^+(S)$ *there are SOS* τ_1, τ_2 *and univariate SOS* $\sigma_1, \ldots, \sigma_{n+m+1}$ *such that*

$$p(x) = \tau_1(x) + \tau_2(x)(\widehat{M}^2 - \|\hat{g}(x)\|^2) + \sum_{j=1}^{n+m+1} \sigma_j(\hat{g}(x)_j)\hat{g}(x)_j.$$

- Can view $\sigma_j(\hat{g}(x)_j)$ as SOS in variables of g_j if needed ("automatic" correlative sparsity)
- Can additionally use correlative or term sparsity for τ_1, τ_2

Ball as a simple set

- Define a ball as $\mathcal{B}_r := \{x \in \mathbb{R}^n : \|x\| \leq r\}$
- Iteratively apply box and ball as simple sets

Proposition 3 (Thm.4 in [Roebers, Vera, Zuluaga ' 2021])

Let $\mathcal{K} \subseteq \mathbb{R}[x], \mathcal{K} \neq \mathbb{R}_+$ be one of the earlier considered non-SOS classes. Let $S = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \dots, g_m(x) \geq 0\} \subset \mathcal{B}_r$. For $p \in \mathcal{P}^+(S)$ there are $\tau_0, \tau_1 \in \mathcal{K}$ and univariate $\sigma_1, \dots, \sigma_m \in \mathcal{K}$ such that

$$
\begin{aligned}
p(x) = {} & \tau_0(r + x_1, \dots, r + x_n, r - x_1, \dots, r - x_n, r^2 - \|x\|^2) \\
& + \tau_1(r + x_1, \dots, r + x_n, r - x_1, \dots, r - x_n, r^2 - \|x\|^2)(r^2 - \|x\|^2) \\
& + \sum_{j=1}^{m} \sigma_j(U_j - g_j(x))g_j(x),
\end{aligned}
$$

where $U_j \geq \max_{x \in \mathcal{B}_r} g_j(x)$ for each $j = 1, \dots, m$.

Ball vs simplex for Putinar-like certificates

- Ball gives a simpler certificate than simplex when $\mathcal{K}$ is SOS
- Can incorporate additional sparsity in both certificates
- With a chordal extension, Proposition 3 gives a certificate with correlative sparsity as in [Lasserre '2006], [Waki et al. '2006]

Generalization for unbounded sets

Notation: $\tilde{p} \in \mathbb{R}_d[x]$ is the homogeneous component of the highest degree in p; $\tilde{S} = \{x \in \mathbb{R}^n : \tilde{g}_1(x) \geq 0, \ldots, \tilde{g}_m(x) \geq 0\}$.

Theorem 2 (K.,Vera, Zuluaga)

Let $S = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_m(x) \geq 0\}$. *For* $p \in \mathcal{P}^+(S)$ *such that* $\tilde{p} \in \mathcal{P}^+(\tilde{S}) \setminus \{0\}$ *there is* $F \in \mathcal{P}^+_{d_{\max}}(\mathbb{R}^{2n+m})$ *such that*

$$(1 + e^\top y + e^\top z)^{d_{\max} - \deg p} p(x) = F(y, z, \mathring{g}_1(y, z), \ldots, \mathring{g}_m(y, z)),$$

where $d_{\max} = \max\{2 \deg g_1, \ldots, 2 \deg g_m, \deg p + \deg p \ \mathit{mod}\ 2\}$ *and* $\mathring{g}_j(y, z) = (1 + e^\top y + e^\top z)^{d_{\max}/2 - \deg g_j} g_j(y - z)$, $j = 1, \ldots, m$.

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If $S \subseteq \mathbb{R}^n_+$, we can set $z = 0$.

Proof: lift S to $\mathbb{R}^{2n+m}$ via equalities and use a representation on unbounded sets with equalities from [Kuryatnikova '2019].

Certificates for unbounded sets

- Certificate on unbounded sets from *any* certificate of tensor-copositivity, e.g., Pólya's theorem for a fixed $r \in \mathbb{R}_+$:

$$q(y,z,r)p(y-z) = \sum_{(\alpha,\beta,\gamma)\in\mathbb{N}^{2n+m}_{r+2d_{\max}}} c_{\alpha,\beta,\gamma}(y-z) y^\alpha z^\beta g(y-z)^\gamma,$$

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- Known fixed denominator q: $\sup_\lambda \lambda$ s.t. $p(x) - \lambda \in \mathcal{P}(S)$
  $\rightarrow$ [Artin '1927], [Krivine '1964]-[Stengle '1974]:
  $q(p-\lambda) = F \implies$ non-linearity in unknowns
  $\rightarrow$ We obtain: $q(p-\lambda) = F \implies$ linear constraints

Optimization over unbounded sets

- Lower bounds: substitute a certificate of copositivity

  → inner approximations of copositive tensors

  [Vera, Pena, Zuluaga '07], [Bundfuss and Dür '2009], [Luo and Qi '2018]
- Tight lower bound under $\tilde{p} \in \mathcal{P}^{+}(\tilde{S}) \setminus \{0\}$

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- Upper bounds: somewhat more loose but possible

  $\rightarrow$ outer approximations of copositive tensors
  [Yildirim '2012], [Dong '2013], [Lasserre '2014]

Questions for further research

- Numerical results for lower and upper bounds
  $\rightarrow$ promising results for related lower bounds in
  [Kuang et al. '2017], [Dickinson and Povh '2018]
  $\rightarrow$ combination with correlative and term sparsity
- Degree bounds: constructive proofs allow to "track" the degree

The talk is based on preprints
`https://arxiv.org/abs/1909.06689`,
`https://arxiv.org/abs/2110.10079`

Thank you for your attention!