Version tropicale du théorème de Putinar.

Dualité et applications.

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Brainstorming days on measure and polynomial optimization

Outline

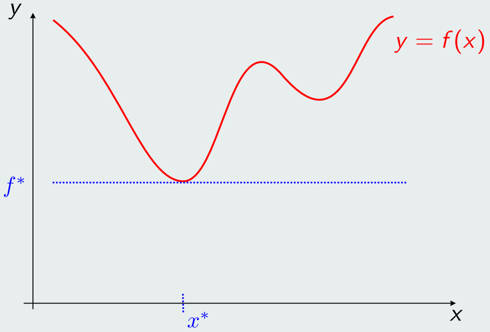
Mathematical programming problem

Let f be a continuous real-valued function and consider

$$f^* = \inf_{x \in K} f(x) \tag{1}$$

where $K = \{x \in \mathbb{R}^n \mid g_j(x) \geq 0, j = 1, \ldots, m\}$ is compact.

Example



Proposition

$$f^* = \sup_{\lambda \in \mathbb{R}} \{\lambda \text{ such that } f(x) - \lambda \geq 0, \ \forall x \in K\}. \qquad (2)$$

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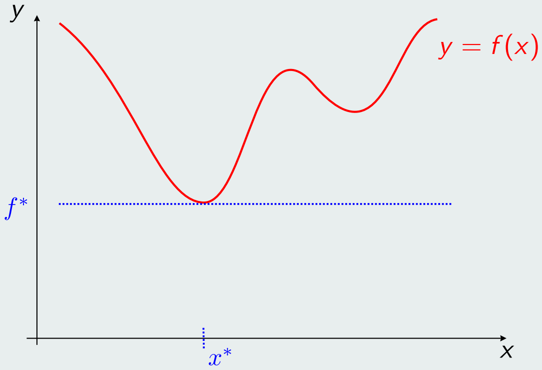

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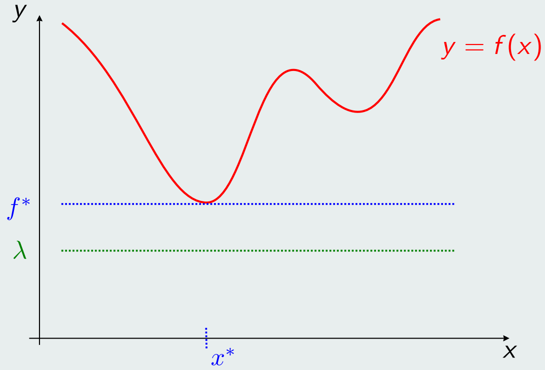

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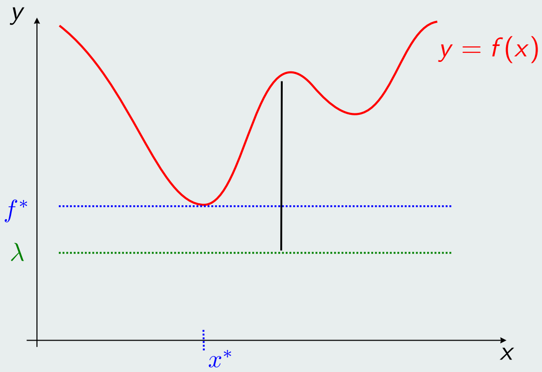

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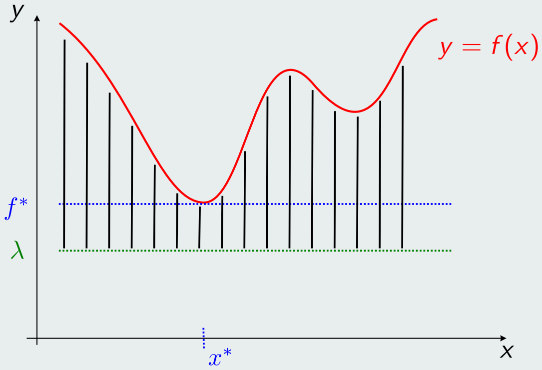

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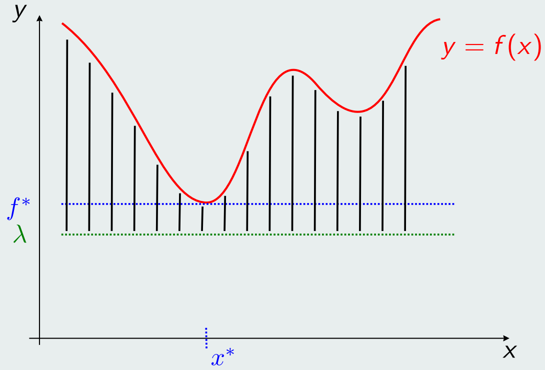

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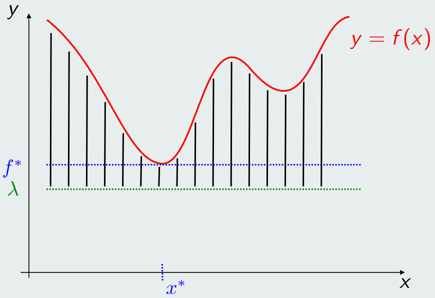


Remarks

- (2) is a linear programming problem with infinite number of constraints.
- (2) is the dual of another one (end of the talk).
- Deciding whether a given real valued function is non-negative is fundamental.

Definition

A certificate of positivity for a real polynomial is an algebraic identity that gives an immediate proof of a positivity condition for the polynomial.

Example

Let

$$f(x, y) = 4x^4 + 4x^3y - 2x^2y^2 + 10y^4,$$

f can be written

$$f(x, y) = (2x^2 - 3y^2 + xy)^2 + (y^2 + 3xy)^2.$$

f is a sum of squares, therefore f is non-negative.

Definition

We denote by $\Sigma[x]$ the set of sum of squares (also denoted SOS).

Definition

In real algebraic geometry, a Positivstellensatz (German for "positive-locus-theorem") is a characterization of polynomials that are positive on a semialgebraic set K.

Theorem (Positivstellensatz dimension 1)

A real polynomial $p \in \mathbb{R}[x]$ is non-negative on $\mathbb{R}$ if and only

$$f = \sigma_0 \text{ with } \sigma_0 \in \Sigma[x].$$

Remarks

The previous theorem can be generalized in different ways :

- for some dimension and degrees,
- with constraints on the variables x (i.e. $x \in K$ with K a basic semi-algebraic set).

Theorem (Positivstellensatz dimension 1 with constraints)

Let $p \in \mathbb{R}[x]$, p non-negative on $K = [-1, 1]$ if and only if

$$p = \sigma_0 + (1 - x^2)\sigma_1 \text{ with } \sigma_0, \sigma_1 \in \Sigma[x].$$

Remark

$K = [-1, 1] = \{x \in \mathbb{R} \mid 1 - x^2 \geq 0\}$

Example

$f(x) = -x^4 - 2x^3 + x^2 + 2x + 1$ is non-negative on $[-1, 1]$ since
$f(x) = \sigma_0 + (1 - x^2)\sigma_1$ with $\sigma_0 = x^2, \sigma_1 = (1 + x)^2$

Definition

A subset K of $\mathbb{R}^n$ is a basic semi-algebraic set if it can be written as

$$K = \bigcap_{j=1}^{m} \{x \in \mathbb{R}^n; g_j(x) \geq 0\} \tag{3}$$

with $g_j \in \mathbb{R}[x_1, \ldots, x_n]$.

Examples

1. $K_1 = [-1, 1]$ is a semi-algebraic set since $K_1 = \{x \in \mathbb{R} \mid 1 - x^2 \geq 0\}$.
2. The non-negative orthant in $\mathbb{R}^n$,
3. The cone of positive semi-definite matrices,
4. $K_2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 - 4 \leq 0 \text{ or } x \geq 0\}$ is a semi-algebraic set but NOT basic.

Putinar Theorem

Let $K = \{g_j \geq 0\}$ be a basic semi-algebraic set satisfying a compactness hypothesis α. $\forall x \in K, f(x) > 0$ if and only if

$$f = \sigma_0 + \sum_j \sigma_j g_j \text{ with } \sigma_i \in \Sigma[x].$$

Remarks

- Bounds on the degrees of σ_j,
- As $\sigma \in \Sigma[x] \Leftrightarrow Q(\sigma) \succeq 0$, to decide whether a polynomial f is positive on K can be written as a convex problem.

Definition

The tropical semiring $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ is the semiring equipped with two binary operations :

- $x \oplus y = \max\{x, y\}$,
- $x \otimes y = x + y$.

Examples

- $2 \oplus 3 = 3$,
- $2 \otimes 3 = 5$.

Remarks

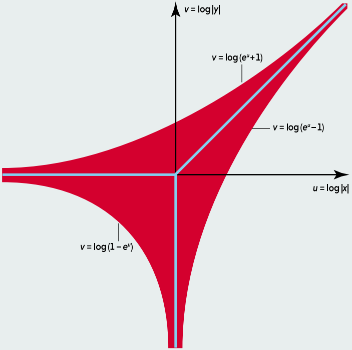
- The operations $\oplus$ and $\otimes$ are referred to as tropical addition and tropical multiplication respectively.
- The unit for $\oplus$ is $-\infty$,
- the unit for $\otimes$ is 0.

Tropical algebraic geometry

Tropical algebraic varieties [a]

a. A. Chambert-Loir, *Quand la géométrie devient tropicale*, Pour la sciences 2018.

Example of a tropical algebraic variety



Dynamic programming

- The Bellman short path algorithm can be seen as a linear dynamical system $x_{n+1} = Ax_n$ in $(\mathbb{R} \cup \{+\infty\}, \min, +)$.
- Optimal control (Hamilton-Jacobi-Bellman) [a]

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a. M. Akian et al, *The Max-Plus Finite Element Method for Solving Deterministic Optimal Control Problems : Basic Properties and Convergence Analysis*, SIAM Journal on Control and Optimization, 2006

Proposition

A real-valued function naturally extends to a valued function in $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$.

Corollary

- The set of real-valued functions is a tropical semiring,
- and one can write $f_1 \oplus f_2$ and $f_1 \otimes f_2$.

Tropical Theorem Putinar

Let $(g_i)_{i=1}^m$ be a family of continuous real-valued functions and $K = \{x \in \mathbb{R}^n \mid g_i(x) \geq 0, i = 1, \dots, m\}$ compact, $\forall x \in K, f(x) > 0$ if and only if

$$f \oplus \bigoplus_{j=1}^{m} -(\sigma_j \otimes g_j) \geq \sigma_0, \text{ with } \sigma_i \in \mathcal{C}^+, \tag{4}$$

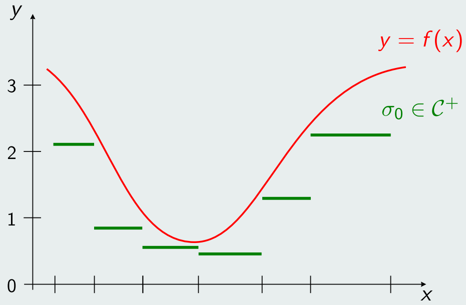
where $\mathcal{C}^+$ is the set of non-negative piecewise constant functions.

Comparison with the "classical" Putinar

$$f = \sigma_0 + \sum_{j=1}^{m} \sigma_j g_j \text{ with } \sigma_i \in \Sigma[x].$$

$$f + \sum_{j=1}^{m} -\sigma_j g_j = \sigma_0 \text{ with } \sigma_i \in \Sigma[x].$$

Example



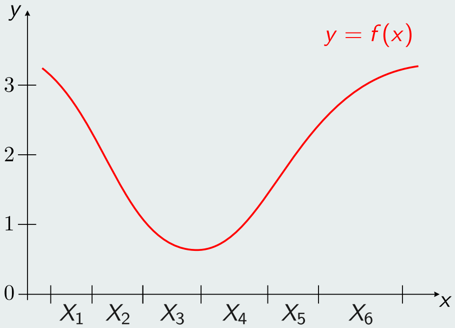
Certificate of positivity

$$f \geq \sigma_0 \text{ and } \sigma_0 \in \mathcal{C}^+, \text{ therefore } f \text{ positive.}$$

Remarks

From the algorithmic point of view, functions σ_j can be obtained throw interval analysis.

Illustration



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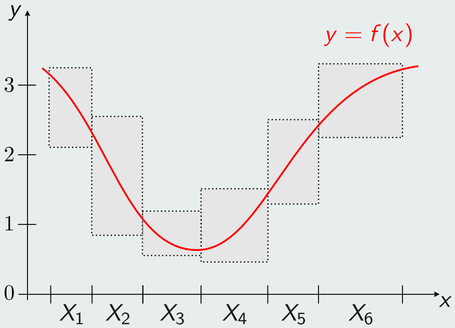

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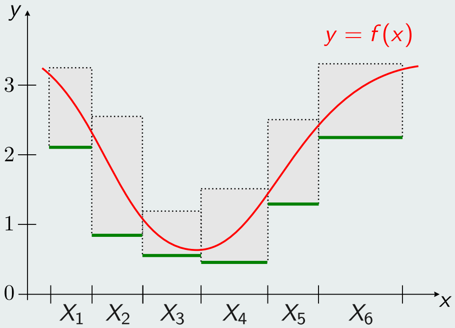

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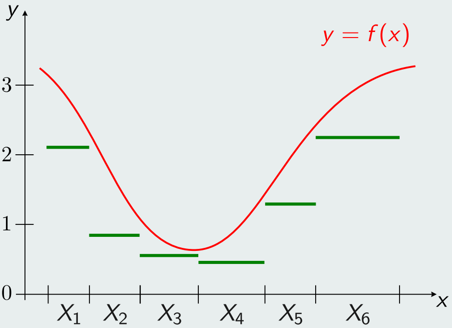

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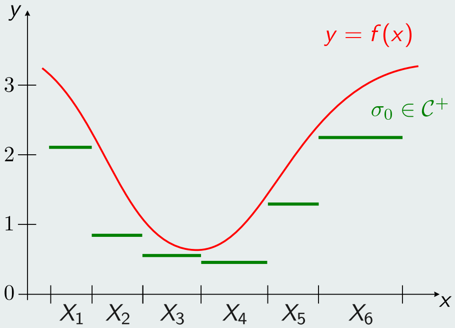
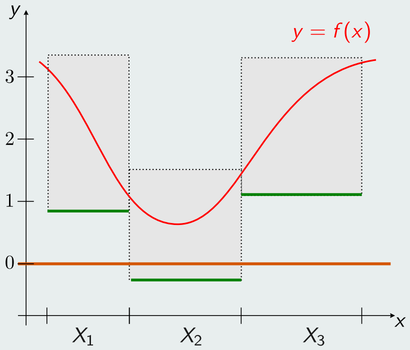


Theorem - Convergence

Suppose

- K is a compact subset of $\mathbb{R}^n$, $f : K \to \mathbb{R}$ positive,
- $[\sigma] : \mathbb{IR} \to \mathbb{IR}$ convergent inclusion function for f ,

then there exists $\epsilon > 0$ s.t. $\forall X_i \subset K, \Delta(X_i) < \epsilon \Rightarrow \underline{\sigma}(X_i) \geq 0$.



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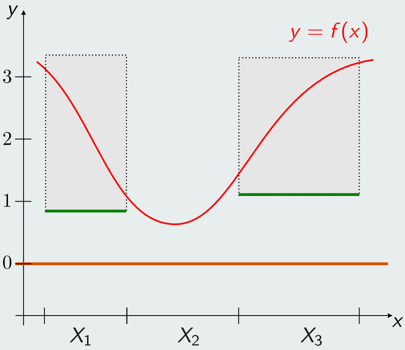

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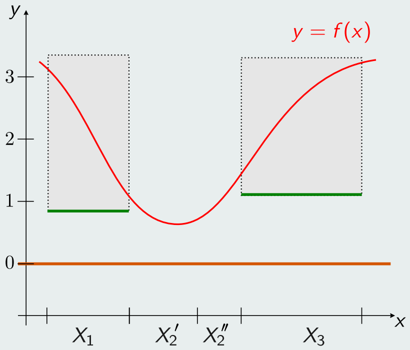

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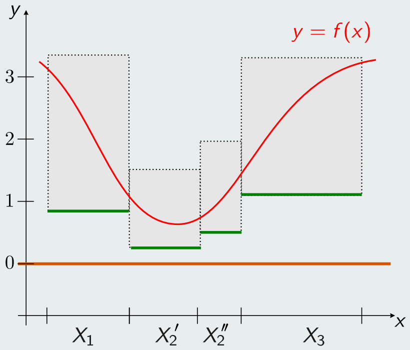

As presented in the introduction,

$$f^* = \sup_{\lambda \in \mathbb{R}} \{\lambda \ \text{ s.t. } f(x) - \lambda \geq 0, \ \forall x \in K\}. \tag{5}$$

Duality

Problem (5) is the dual of the following primal

$$\begin{aligned} \inf_{\mu \in \mathcal{M}(K)} \quad & \int_K f \mathrm{d}\mu, \\ \text{such that} \quad & \int_K \mathrm{d}\mu = 1 \text{ and } \mu \geq 0. \end{aligned}$$

where $\mathcal{M}(K)$ is the set of signed measures with K as support.

Remarks

- Positivstellensatz previously presented can reformulated by duality and let us write constraints on measures (and on their moments),
- On the other side, infinite dimensional convex programming can be relaxed and discretized from this duality ...

Generalized moment problem

$$\inf_{\mu \in \mathcal{M}^+(X)} \quad \int_X \varphi(x) d\mu$$
$$\text{such that} \quad \int_X \psi(x) d\mu \leq \gamma_\psi, \forall \psi \in \Gamma. \tag{6}$$

Problem (6) can used to formalize and solve other problems :

- Finding the global minimum of a function on a subset of $\mathbb{R}^n$,
- Computing the optimal value of the Kantorovitch transport problem,
- Computing the optimal value of an optimal control problem,
- Computing an upper bound on $\mu(S)$ over all measures μ satisfying some moment conditions,
- Evaluating an ergodic criterion associated with a Markov chain,
- ...

Jean-Bernard Lasserre. *Moments, Positive Polynomials and Their Applications.* Imperial College Press optimization series. Imperial College Press, 2010.

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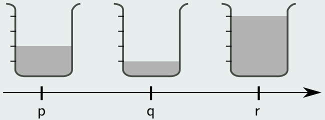
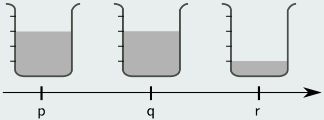
Example



$\mu = (2, 1, 4)$



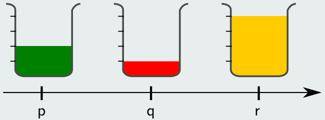
$\nu = (3, 3, 1)$

Transport problem

	3	3	1
4	π_{11}	π_{12}	π_{13}
1	π_{21}	π_{22}	π_{23}
2	π_{31}	π_{32}	π_{33}

s.t. $\begin{cases} \forall i,\ \sum_j \pi_{ij} = \mu_i, \\ \forall j,\ \sum_j \pi_{ij} = \nu_j. \end{cases}$

Example



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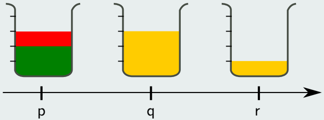


$\nu = (3, 3, 1)$

A solution

	3	3	1
2	2	0	0
1	1	0	0
4	0	3	1

satisfies $\begin{cases} \forall i,\ \sum_j \pi_{ij} = \mu_i, \\ \forall j,\ \sum_j \pi_{ij} = \nu_j. \end{cases}$

Kantorovich optimal transport problem

Let μ and ν be two non negative measures on X and Y.

$$\begin{aligned} \mathcal{T}(\mu,\nu) = \inf_{\pi \in \mathcal{M}^+(X\times Y)} \quad & \int_{X\times Y} c(x,y)\mathrm{d}\pi \\ \text{such that} \quad & \pi_X = \mu, \\ & \pi_Y = \nu. \end{aligned} \tag{7}$$

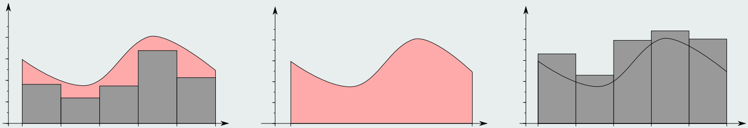
Lemma

Let

- $\varphi : X \to \mathbb{R}$,
- $\{X_i\}_i$ a partition of X,
- $\mu \in \mathcal{M}^+(X)$,
- $\underline{\varphi}_i, \overline{\varphi}_i$ real numbers such that $\forall x \in X_i, \underline{\varphi}_i \leq \varphi(x) \leq \overline{\varphi}_i$,

then

$$\sum_i \underline{\varphi}(X_i)\mu(X_i) \leq \int_X \varphi(x)\mathrm{d}\mu(x) \leq \sum_i \overline{\varphi}(X_i)\mu(X_i).$$



Proposition - Relaxation

- $\mu \in \mathcal{M}^+(X)$, $\nu \in \mathcal{M}^+(Y)$,
- $\{X_i\}_i$, $\{Y_j\}_j$ two partitions of X and Y,
- $\mu(X_i) \in [\underline{\mu}_i, \overline{\mu}_i]$, $\nu(Y_j) \in [\underline{\nu}_j, \overline{\nu}_j]$,
- $\forall (x,y) \in X_i \times Y_j, \underline{c}_{ij} \leq c(x,y)$,

Let

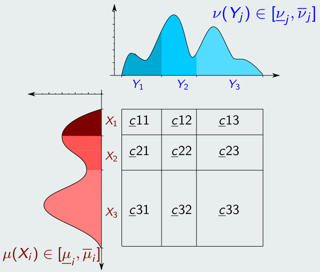
$$\underline{\mathcal{T}} = \min_{\pi_{ij} \in \mathbb{R}^n \otimes \mathbb{R}^m} \sum_{i,j} \underline{c}_{ij} \pi_{ij}$$

$$\text{such that} \quad \forall i,\ \underline{\mu}_i \leq \sum_j \pi_{ij} \leq \overline{\mu}_i, \quad \forall j,\ \underline{\nu}_j \leq \sum_i \pi_{ij} \leq \overline{\nu}_j, \quad \forall i, \forall j,\ \pi_{ij} \geq 0. \tag{8}$$

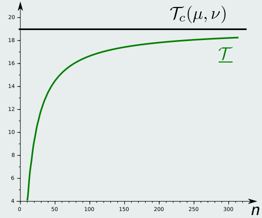
then

$$\underline{\mathcal{T}} \leq \mathcal{T}(\mu, \nu).$$

Spatial discretization



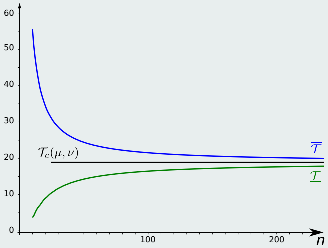
Lower bounds



Theorem - Hierarchy

Let $\mathcal{X}_n$ and $\mathcal{Y}_n$ two monotone sequences of partitions of X and Y, if the functions $[c], [\mu]$ and $[\nu]$ are convergent and monotone then the optimal solutions $\mathcal{T}_n$ of Problem (8) satisfies

$$\underline{\mathcal{T}}_n \uparrow \mathcal{T}(\mu, \nu)$$



N. Delanoue et al. Numerical enclosures of the optimal cost of the Kantorovitch's mass transportation problem. *Computational Optimization and Applications*, 63(3) :855-873, 2016.

Hierarchy - Control optimal

$$J^* = \min_{\mu,\nu \in \mathcal{M}_+} \langle \mu, h\rangle + \langle \nu, H\rangle$$

$$\text{s.t.} \quad \mathcal{L}'(\mu,\nu) = \delta_{(0,x_0)}$$

$\{X_i\}$ a partition of $[0,T] \times X \times U$,
$\{Y_k\}$ a partition of K,
$\mathcal{P} = \{\varphi\}$ a finite familly of functions of t, x.

$$\underline{J} = \min_{\mu_i,\nu_k \in \mathbb{R}^+} \sum_{i\in I} \mu_i \underline{h}_i + \sum_{k\in K} \nu_k \underline{H}_k$$

tel que $\forall \varphi \in \mathcal{P}$ $\displaystyle\sum_{i\in I} \mu_i \underline{\psi}_i + \sum_{k\in K} \nu_k \underline{\varphi}_k \leq \varphi(0,x_0)$

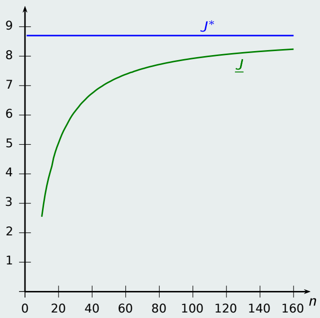
$$\varphi(0,x_0) \leq \sum_{i\in I} \mu_i \overline{\psi}_i + \sum_{k\in K} \nu_k \overline{\varphi}_k,$$

$$\text{avec } \psi = -\frac{\partial\varphi}{\partial t} - \frac{\partial\varphi}{\partial x} f(t,x,u),$$

then

$$\underline{J} \leq J^*.$$

Lower bounds



Nicolas Delanoue, Mehdi Lhommeau, and Sébastien Lagrange. Nonlinear optimal control : A numerical scheme based on occupation measures and interval analysis. *Computational Optimization and Applications, Springer Verlag*, volume 77, pages 307-334, 2020, 10.1007/s10589-020-00198-8

Conclusion

- We have presented a Positivstellensatz based on tropical algebra.
- Interval analysis makes it effective by computing the σ_j functions,
- Some generalized moment problems has been be solved by ideas based on this approach : Kantorovitch transport, Optimal control, Quantum information (Gretsi 2022).

Perspective

- Propose a general approach to be able to discretize any Generalized Moment Problems.
- Our discretization is only spatial, can we reduce the dimension of the discretized problem by not only considering the moments of order 0 of μ on X_i ? Spline ...

Merci pour votre attention.