On the ensemble controllability of quantum systems

BRAINPOP Seminar, LAAS

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14/06/2021

Joint work with Mario Sigalotti, Ugo Boscain, Rémi Robin

inventeurs du monde numérique

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Section 1

Introduction

Quantum control

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Quantum control

The goal of quantum control is to achieve transition of states in quantum mechanical systems thanks to an external field (laser, magnetic field).
Several applications:

- To induce chemical reactions
- Quantum sensing, spectroscopy, magnetic resonance phenomenons
- High-fidelity quantum gates in superconducting quantum processors

Controllability Problem

Schrödinger Equation:

$$i\frac{d\psi}{dt}(t) = H(u(t))\psi(t), \quad \psi(0) = \psi_0, \ \|\psi\| = 1, u(t) \in \mathbb{R}^k,$$

where H is a n-dimensional Hermitian matrix.

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Main questions:

- Controllability of the Schrödinger Equation? Controllability between eigenstates?
- Design of explicit control laws. Our tool: **ADIABATIC CONTROL** (Boscain, Chittaro, Mason, Sigalotti '12)
- Find geometric criterions on the spectrum of Hamiltonians which guarantee controllability.

Ensemble Controllability Problem

Parametric Schrödinger Equation (Unknown parameter or ensemble of quantum systems): For $z \in [z_0, z_1]$,

$$i\frac{d\psi^z}{dt}(t) = H_z(u(t))\psi^z(t), \quad \psi(0) = \psi_0^z,\ \|\psi^z\| = 1, u(t) \in \mathbb{R}^2, H_z \in S_n(\mathbb{R}).$$

Find u independent of z achieving transitions for every $z \in [z_0, z_1]$.

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- Extension of the adiabatic techniques?
  - (A., Boscain, Sigalotti, SICON'18), (A., Boscain, Sigalotti, MCRF'20),
  - (A., Boscain, Sigalotti, CDC'19), (A., Boscain, Sigalotti, Accepted in Automatica), (A., Boscain, Robin, Sigalotti, Submitted)

Section 2

Adiabatic ensemble control with two real controls

Adiabatic control

$$i\frac{d\psi_\epsilon}{dt}(t) = H(u(\epsilon t))\psi_\epsilon(t), \quad \psi_\epsilon(0) = \psi_0, \ \|\psi_\epsilon\| = 1, u(t) \in \mathbb{R}^k,$$

where H is a n-dimensional Hermitian matrix, (λ_j, ϕ_j) associated eigenpairs.

Theorem (Adiabatic Theorem)

Assume that for every $u \in K$ ($K \subset \mathbb{R}^k$ compact), $\lambda_j(u)$ is simple. Consider a C^2 path $(u(t))_{t\in[0,1]}$ in K. Then the solution of the Schrödinger equation $|\langle\psi_\epsilon(0), \phi_j(u(0))\rangle| = 1$ satisfies

$$\left\|\psi_\epsilon(\frac{1}{\epsilon}) - e^{i\eta}\phi_j(u(1))\right\| \leq C\epsilon,$$

$\eta = \eta(\epsilon) \in \mathbb{R}$, $C > 0$ *independent of* ϵ.

Conical intersections: climbing phenomenon

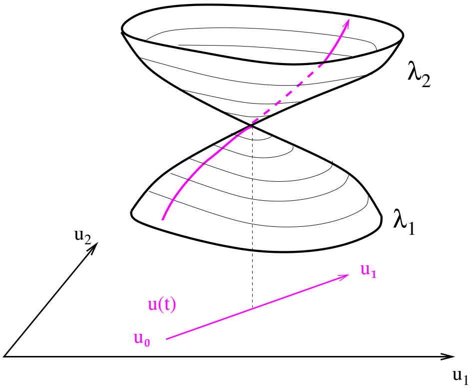

Figure: Conical intersection as a function of (u_1, u_2).

[L.P. Yatsenko, S.Guérin, H.R. Jauslin], [U. Boscain, F. Chittaro, P. Mason, M. Sigalotti]

Singularities of the spectrum: conical intersections

Definition (Conical intersection)

$u \in \mathbb{R}^2$ is a **conical intersection** *between the eigenvalues λ_j and λ_{j+1} if $\lambda_{j+1}(u) = \lambda_j(u)$ has a multiplicity equal to 2 and there exists a constant $c > 0$ such that $\forall v \in \mathbb{R}^2$ and $t > 0$ small enough:*

$$\frac{d}{dt}\left(\lambda_{j+1}(u+tv) - \lambda_j(u+tv)\right)\bigg|_{t=0} > c.$$

Proposition

Consider a C^2 path at a conical intersection at speed ϵ: $u := u(\epsilon t)$. If $|\langle \psi_0, \phi_j(u(0))\rangle| = 1$, then $|\langle \psi(\frac{1}{\epsilon}), \phi_{j+1}(u(1))\rangle| = 1 + O(\sqrt{\epsilon})$.

Theorem (Boscain, Gauthier, Rossi, Sigalotti, 2015)

Assume that the spectrum of $H(\cdot)$ is conically connected. Then the Schrödinger Equation is controllable.

Proposition (Genericity)

Let $H_0 \in S_n(\mathbb{R})$; then, generically with respect to $(H_1, H_2) \in S_n(\mathbb{R})^2$, all intersections of eigenvalues of $H_0 + u_1 H_1 + u_2 H_2$ are conical, i.e. there exists an open and dense subset of $S_n(\mathbb{R})^2$ such that all intersections of eigenvalues of $H_0 + u_1 H_1 + u_2 H_2$ are conical.

Ensemble control

Parametric Schrödinger Equation:

$$i\frac{d\psi^z}{dt}(t) = H_z(u(t))\psi^z(t), \quad \psi^z(0) = \psi_0^z,\ \|\psi^z\| = 1, u \in \mathbb{R}^2, z \in \mathbb{R}, H_z \in S_n(\mathbb{R}).$$

Definition

The system is ensemble approximately controllable between eigenstates uniformly w.r.t. $z \in [z_0, z_1]$ if $\forall \epsilon > 0, j, k \in \{1, \ldots, n\}$ and $\forall \hat{u}, \bar{u} \in \mathbb{R}^2$ s.t. $\lambda_j^z(\hat{u})$ and $\lambda_k^z(\bar{u})$ are simple for every $z \in [z_0, z_1]$, $\exists u : [0, T] \to \mathbb{R}^2$ such that for every z, the solution with initial condition $\psi^z(0) = \phi_j^z(\hat{u})$ satisfies $||\psi^z(T) - e^{i\theta}\phi_k^z(\bar{u})|| < \epsilon$.

Principle of our ensemble control strategy

Assumption: We know the set $\pi = \{(u,v) \in \mathbb{R}^2 \mid \exists z \in \mathbb{R},\ \lambda_j^z(u,v) = \lambda_{j+1}^z(u,v)\}$ and we have regularity informations on $Z = \{(u,v,z) \in \mathbb{R}^3 \mid \lambda_j^z(u,v) = \lambda_{j+1}^z(u,v)\}$.
Method: Find a suitable connected component γ of Z and design a control law following slowly the curve $\pi(\gamma)$ so as to get transitions for every $z \in [z_0, z_1]$.

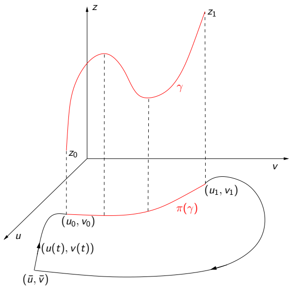

Figure: Control strategy

Ensemble control in the conical case

Let U be an open set of $\mathbb{R}^2$. For $j \in \{1 \dots n\}$,

$$\gamma_j = \{(u, v, z) \in \mathrm{U} \times [z_0, z_1] \mid \lambda_j(u, v, z) = \lambda_{j+1}(u, v, z)\},$$

where $\gamma_0 = \gamma_n = \emptyset$. Denote the projection of $(u, v, z) \in \mathbb{R}^3$ onto the (u, v)-component by π.

ASSUMPTION A_j. There exist a connected component $\hat{\gamma}_j$ of γ_j and a map $\beta_j : [z_0, z_1] \to U$ such that β_j is a C^3 embedding and

- $\pi(\hat{\gamma}_j)$ is contained in $\mathrm{U} \times [z_0, z_1] \setminus (\pi(\gamma_{j-1}) \cup \pi(\gamma_{j+1}))$;
- $\pi(\hat{\gamma}_j) = \beta_j([z_0, z_1])$;
- for every $z \in [z_0, z_1]$, $\lambda_j(\cdot, z)$ and $\lambda_{j+1}(\cdot, z)$ have a unique intersection on $\pi(\hat{\gamma}_j)$, which is conical and occurs at $\beta_j(z)$.

Moreover the set $U \setminus (\pi(\gamma_{j-1}) \cup \pi(\gamma_j) \cup \pi(\gamma_{j+1}))$ is pathwise connected.

Theorem (A., Boscain, Sigalotti, SICON '18)

Consider a C^3 map $U \times [z_0, z_1] \ni (u, v, z) \mapsto H(u, v, z) \in S_n(\mathbb{R})$. Let assumption A_j be satisfied for every $j \in \{1 \dots n\}$. Then the equation $i\frac{d\psi}{dt} = H(u, v, z)\psi$ is ensemble approximately controllable between eigenstates.

This result is based on an uniform adiabatic result w.r.t. $z \in [z_0, z_1]$.

The error in the adiabatic regime has order $\sqrt{\epsilon}$ on an interval of time of length $\frac{1}{\epsilon}$ for a control path followed at a speed ϵ.

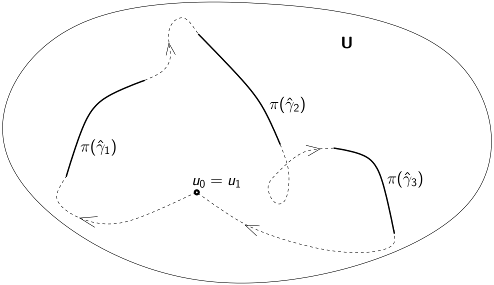

Figure: A control realizing an ensemble transition between $\phi_1^z(u_0, v_0)$ and $\phi_4^z(u_0, v_0)$.

Extensions when semi-conical intersections appear

[A.,Boscain,Sigalotti, MCRF'20]

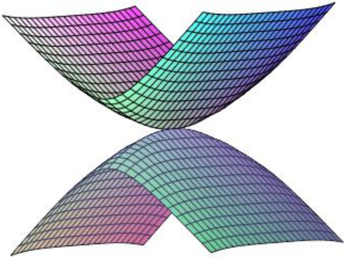

Figure: Semi-conical intersection

Section 3

Extensions to the single control case: RWA

Multiply the controls: general idea

- Usual RWA:
  - Decoupling approximation $H_0 + u_\epsilon(t)H_1 \to \tilde{H}_0 + u_1(t)\tilde{H}_1 + u_2(t)\tilde{H}_2$ up to a change of frame, when the system is driven by a real oscillating control.
  - Known to work well for a small detuning from the resonance frequencies and a small amplitude w.r.t. $||H_0||$.
- Our framework RWA+AA:
  - Find $u_\epsilon(t)$ depending on $u_1(t), u_2(t), H_0, H_1$ such that

$$i\frac{d\psi_\epsilon}{dt}(t) = (H_0 + u_\epsilon(t)H_1)\,\psi_\epsilon(t)$$

    has effective dynamics up to change of frame, when $\epsilon \to 0$:

$$\tilde{H}(u_1(\epsilon t), u_2(\epsilon t)) = \tilde{H}_0 + u_1(\epsilon t)\tilde{H}_1 + u_2(\epsilon t)\tilde{H}_2,$$

    for t in $[0, \frac{1}{\epsilon}]$.
  - Obtain $\tilde{H}$ with good spectral properties, robustness w.r.t. parametric uncertainities?

Decoupled Hamiltonian

Consider the system

$$i\frac{d\psi(t)}{dt} = (H_0 + u(t)H_1)\psi(t), \qquad \psi(t) \in \mathbb{C}^n,$$

where the control u takes values in $\mathbb{R}$.
Assume that $H_0 = \mathrm{diag}(E_j)_{j=1}^n$ with $E_1, \dots, E_n \in \mathbb{R}$, and define

$$\Xi = \left\{|E_j - E_k| \mid (j,k) \in \{1,\dots,n\}^2, \, (H_1)_{j,k} \neq 0\right\}$$

For $\sigma \in \Xi$, let $\mathcal{R}_\sigma \subset \{1,\dots,n\}^2$ and $H_1^\sigma \in i\mathfrak{u}(n)$ be defined by

$$\mathcal{R}_\sigma = \{(j,k) \mid E_j - E_k = \sigma, \, (H_1)_{j,k} \neq 0\},$$

$$(H_1^\sigma)_{j,k} = \begin{cases} (H_1)_{j,k} & \text{if } |E_j - E_k| = \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

More precisely, define the *decoupled Hamiltonian* $H_{\mathrm{d}} : \mathbb{R}^n \times \mathbb{R}^{\Xi} \to i\mathfrak{u}(n)$ by

$$H_{\mathrm{d}}(\delta, w) = \sum_{j=1}^{n} \delta_j e_{jj} + \sum_{\sigma \in \Xi} w_\sigma H_1^\sigma,$$

where $\mathbb{R}^{\Xi}$ denotes the set of real vectors $(w_\sigma)_{\sigma \in \Xi}$, and, for every $j, k \in \{1, \dots, n\}$, e_{jk} is the $n \times n$ matrix whose (j,k)-coefficient is equal to 1 and the others are equal to 0.

Let $\alpha > 0$ and define

$$u_\epsilon(t) = \epsilon^\alpha v_0(\epsilon^{\alpha+1} t)$$

$$+ 2\epsilon^\alpha \sum_{\sigma \in \Xi \setminus \{0\}} v_\sigma(\epsilon^{\alpha+1} t) \cos\left(\sigma t + \frac{\phi_\sigma(\epsilon^{\alpha+1} t)}{\epsilon}\right),$$

For $\sigma \in \Xi$,

- choose a smooth $v_\sigma : [0,1] \to \mathbb{R}$, and, if $0 \notin \Xi$, set $v_0 \equiv 0$.
- choose $\phi_\sigma = \varphi_j - \varphi_k$ for $(j,k) \in \mathcal{R}_\sigma$, $\varphi_k(0) = 0$, $k \in \{1, \dots, n\}$.

Time scales:

- AA parameter ϵ
- RWA parameter ϵ^α

Given $\varphi = (\varphi_j)_{j=1}^n$, define $h_{\mathrm{d}}(\tau) = H_{\mathrm{d}}(-\varphi'(\tau), v(\tau)), \qquad \tau \in [0,1]$.

Theorem (A., Boscain, Sigalotti, provisionnally accepted in Automatica)

Set $\psi_0 \in \mathbb{C}^n$, $\alpha > 1$, *and let* $\psi_\epsilon(\cdot)$ *be the solution of*

$$i\frac{d\psi_\epsilon(t)}{dt} = (H_0 + u_\epsilon(t)H_1)\,\psi_\epsilon(t), \qquad 0 \leq t \leq \frac{1}{\epsilon^{\alpha+1}},$$

with $\psi_\epsilon(0) = \psi_0$. *Define* Ψ_ϵ *as the solution of*

$$\frac{d\Psi_\epsilon(s)}{ds} = h_{\mathrm{d}}(\epsilon s)\Psi_\epsilon(s), \qquad 0 \leq s \leq \frac{1}{\epsilon},$$

with $\Psi_\epsilon(0) = \psi_0$. *Assume that* h_{d} *satisfies a* κ*-th order gap condition for some nonnegative integer* κ. *Then*

$$\left\|\psi_\epsilon\left(\frac{\tau}{\epsilon^{\alpha+1}}\right) - V_\epsilon(\tau)\Psi_\epsilon\left(\frac{\tau}{\epsilon}\right)\right\| < c\epsilon^{\min(\frac{1}{\kappa+1},\alpha-1)}$$

for $0 \leq \tau \leq 1$, *where* $V_\epsilon(\tau) = \mathrm{diag}\left(e^{-i(\frac{E_j\tau}{\epsilon^{\alpha+1}}+\frac{\varphi_j(\tau)}{\epsilon})}\right)_{j=1}^n$ *and* $c > 0$ *is independent of* $\tau = \epsilon^{\alpha+1}t \in [0,1]$ *and* $\epsilon > 0$.

Given a nonnegative integer κ, we say that h_d satisfies a κ-th order gap condition if

$$\forall j, l \in \{1, \ldots, n\} \text{ s.t. } j \neq l, \forall \tau \in [0,1], \exists r \in \{0, \ldots, \kappa\} \text{ s.t. } \frac{d^r(\lambda_l - \lambda_j)}{d\tau^r}(\tau) \neq 0$$

and

$$\lambda_p(\tau) \neq \lambda_j(\tau) \text{ for } p \neq j, l.$$

Application: single frequency pulse

Transition for a n-level system between two coupled states 1 and 2, whose energy levels are non-resonant w.r.t. the other levels, i.e. $\mathcal{R}_\sigma = \{(1,2)\}$ for $\sigma = E_2 - E_1$.

- Pulse: $u_\epsilon(t) = 2\epsilon^\alpha v(\epsilon^{\alpha+1}t)\cos\big((E_2 - E_1)t + \frac{1}{\epsilon}\varphi(\epsilon^{\alpha+1}t)\big)$, where $\alpha > 1$, $t \in [0, \frac{1}{\epsilon^{\alpha+1}}]$.

In this case

$$h_d(\varphi', v) = \left(\begin{array}{c|c} \tilde{h}(\varphi', v) & 0 \\ \hline 0 & 0 \end{array}\right),$$

where

$$\tilde{h}(\varphi', v) = \begin{pmatrix} -\frac{\varphi'}{2} & v \\ v & \frac{\varphi'}{2} \end{pmatrix}.$$

The decoupled Hamiltonian h_d has a conical intersection at $v = \varphi' = 0$ ($\lambda_{1/2} = \pm\sqrt{v^2 + \phi'^2/4}$.)

Robustness, ensemble control

$$i\frac{d\psi_\epsilon^z(t)}{dt} = \begin{pmatrix} E & zu_\epsilon(t) \\ zu_\epsilon(t) & -E \end{pmatrix} \psi_\epsilon^z(t)$$

Goal: drive $\psi_\epsilon^z(0) = e_1$ to e_2.

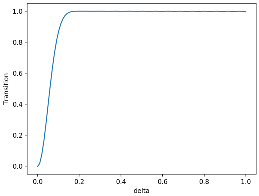

Figure: Fidelity as a function of the amplitude z of controls for two-level systems, with (v, ϕ') chirped pulse.

Robustness

$$i\frac{d\psi_\epsilon^z(t)}{dt} = \begin{pmatrix} E+z & u_\epsilon(t) \\ u_\epsilon(t) & -E-z \end{pmatrix} \psi_\epsilon^z(t)$$

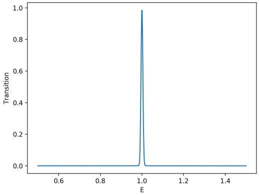

Figure: Fidelity as a function of z for two-level systems, with (v, ϕ') chirped pulse.

- Robust w.r.t. inhomogeneities of H_1^z
- BUT not robust w.r.t. inhomogeneities of the drift term H_0^z.

Conclusion

- Two real controls case:
  - Practical estimation of the singular curve and of the geometry of eigenvalues.
  - Optimization of adiabatic methods, i.e. reduce the control time (or increase their precision) (ongoing works, optimal control problems).
- Single control case:
  - Robustness w.r.t. the drift term (Two-level systems: "Ensemble qubit controllability with a single control via adiabatic and rotating wave approximations." [A., Boscain, Robin, Sigalotti, Submitted]).

THANK YOU FOR YOUR ATTENTION