Harmonic hierarchies for polynomial optimization.

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Polynomial Optimization:

Denote by:

- $R := \mathbb{R}[x_1, \ldots, x_n]$ the ring of n-variate polynomials with real coefficients.
- $F \in R_{2k}$ a homogeneous polynomial of degree $2k$.
- $S \subseteq \mathbb{R}^n$ the unit sphere.

Definition.

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This is a fundamental problem for several reasons: It is a model for **global**, **non-convex** optimization problems and has a **wealth of applications** initiated by work of J.B. Lasserre and P. Parrilo in the early 2000's ([20C93 according to 2020 MSC])

A solution strategy:

Assume $F(x_1, \ldots, x_n)$ is homogeneous and of even degree $2k$.

$$\alpha^* := \min_{x \in S} F(x)$$

Lemma.

The following equality holds:

$$\alpha^* := \max\left\{\lambda \in \mathbb{R} : F(x) - \lambda \|x\|_2^{2k} \geq 0 \text{ on } S\right\}$$

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Nonnegative polynomials

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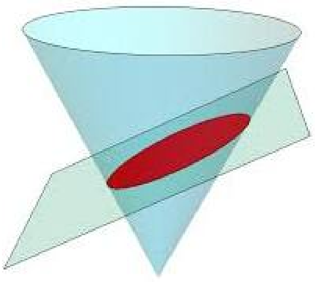

The set $P_{2k} \subseteq R_{2k}$ is a closed convex cone.

Hierarchies:

A natural approach is to construct **hierarchies** approximating P_{2k},

Definition.

A hierarchy is a collection of convex cones $(A_s)_{s\in\mathbb{N}}$ satisfying the following properties:

1. *The cones A_s have fast membership algorithms.*
2. $A_s \subseteq P$
3. *The sets A_s converge to P in the sense that*

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Useful because if

$$\alpha_s := \sup\left\{\lambda \in \mathbb{R} : F(x) - \lambda\|x\|^{2k} \in A_s\right\}$$

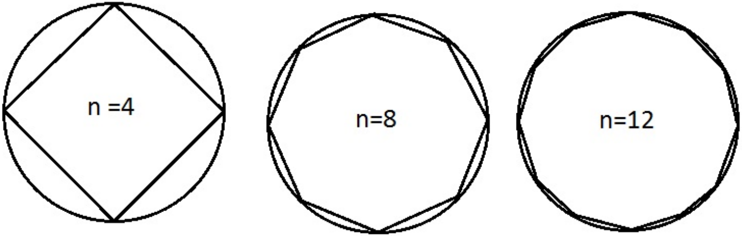
then the α_s are easily computable, $\alpha_s \leq \alpha^*$ and $\lim_{s\to\infty} \alpha_s = \alpha^*$.

Hierarchies

There exist several well-known hierarchies for P_{2k}

1. SOS hierarchies: [Parrilo],[Lasserre],[Peña, Vera, Zuluaga],[D. Henrion],...
2. SONC, SIGNOMIALS: [T. De Wolff], [V. Chandrasekaran],...
3. Polyhedral hierarchies:
   - Polya-type [Peña, Vera, Zuluaga],
   - DSOS, SDSOS: [A.A. Ahmadi, Majumdar],[A.A.Ahmadi, G. Hall],...

The purpose of this talk is to introduce several new **polyhedral hierarchies** for approximating P and to give **quantitative bounds on the rate at which they converge**.



We call them **Harmonic Hierarchies** because the harmonic analysis on spheres (or equivalently the representation theory of $SO(n)$) plays a key role in their definition.

Outline

1. Motivation.
2. Three basic ingredients:
    1. Quadrature rules on spheres.
    2. Polynomial averaging operators.
    3. Harmonic expansion on spheres.
3. The **Construction** of Harmonic hierarchies for P_{2k}.
4. A Convergence **Theorem**.
5. Some simple polynomial optimization **examples** (computed with our Julia package for optimization via harmonic hierarchies).

1. Quadrature rules

Definition.

A **quadrature rule** *of strength* $2t$ *on the sphere* $S \subseteq \mathbb{R}^n$ *is a pair* (X, W) *where* $X \subseteq S$ *is a finite set and* $W : X \to \mathbb{R}_{>0}$ *is a function for which the following equality holds,*

$$\forall f \in R_{2t} \left(\int_S f(y) d\mu(y) = \sum_{x \in X} W(x) f(x) \right)$$

where the integral on the left-hand side is taken with respect to the $(n-1)$*-dimensional volume measure* μ *on the sphere* S*.*

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Example:

$$\forall f \in R_{2t} \left(\int_{S^1} f(x, y) d\mu(y) = \sum_{j=0}^{2t+1} \frac{2\pi}{2(t+1)} f(x_j) \right)$$

where the x_j *are the vertices of a regular* $2(t+1)$*-gon.*

The main invariant of a quadrature rule (X, W) is *size* $|X|$. It is known that the minimal size ν_{2k} of a quadrature rule of weight $2k$ satisfies

$$\dim(R_k) \leq \nu_{2k} \leq \dim(R_{2k}) + 1$$

rules of minimal size are known only in few cases.

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Example: (Gaussian Quadrature)

If X consists of the k roots of the Jacobi polynomials $P_k^{(\alpha,\beta)}(t)$ there are weights W which lead to a quadrature rule of strength $2k-1$ on $[-1,1]$,

$$\int_{-1}^{1} f(t)(1-t)^a(1+t)^b dt = \sum_{x\in X} W(x)f(x)$$

1. Quadrature rules

Theorem. (-,Cristancho)

For every integer t there exists a quadrature rule of strength $2t$ in the sphere $S \subseteq \mathbb{R}^n$ supported in $2(t+1)^{n-1}$ points.

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Theorems (and code!) of **Alex Townsend** *makes these computations possible for high degrees.*

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This polyhedron has $\leq 2(k+s+1)^{n-1}$ facets.

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Definition.

Let $g(t)$ be a univariate polynomial which is nonnegative on $[-1,1]$. Define $\Gamma_g : R \to R$ via $\Gamma_g(f(x)) = h(x)$ where

$$h(x) = \int_S g(\langle x, y \rangle) f(y) d\mu(y)$$

where μ is the $(n-1)$-dimensional volume measure.

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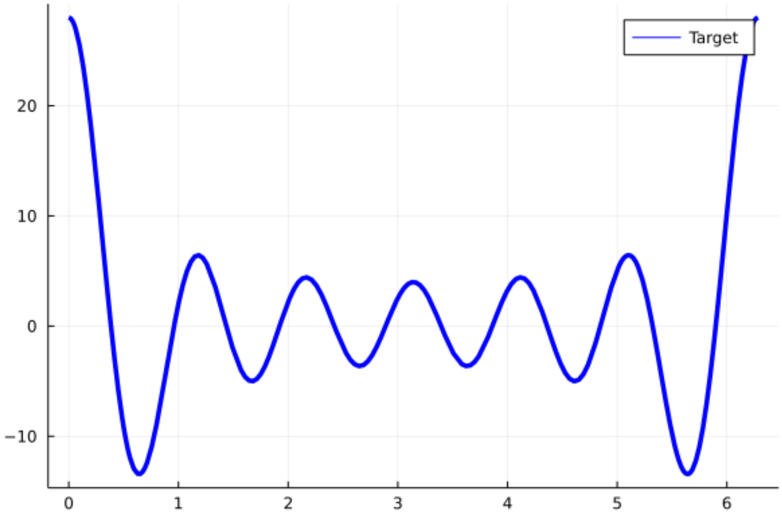

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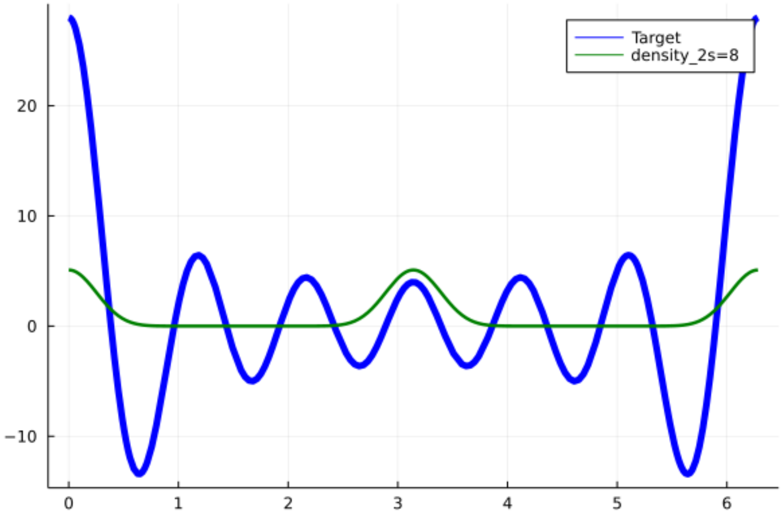

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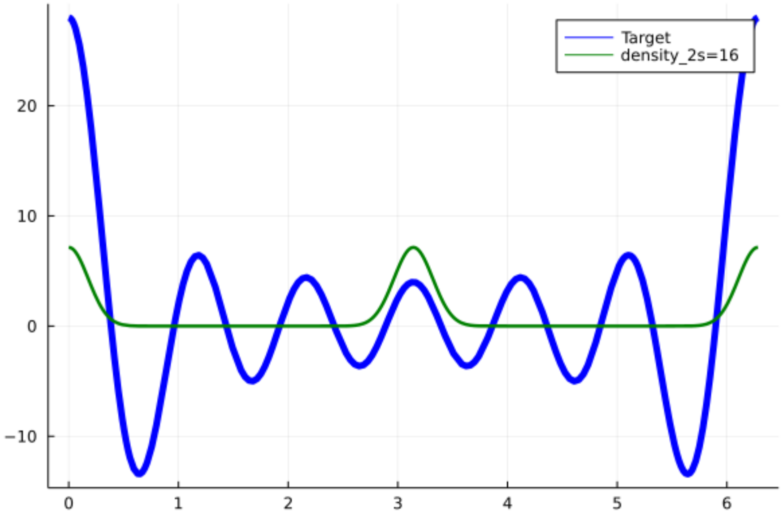

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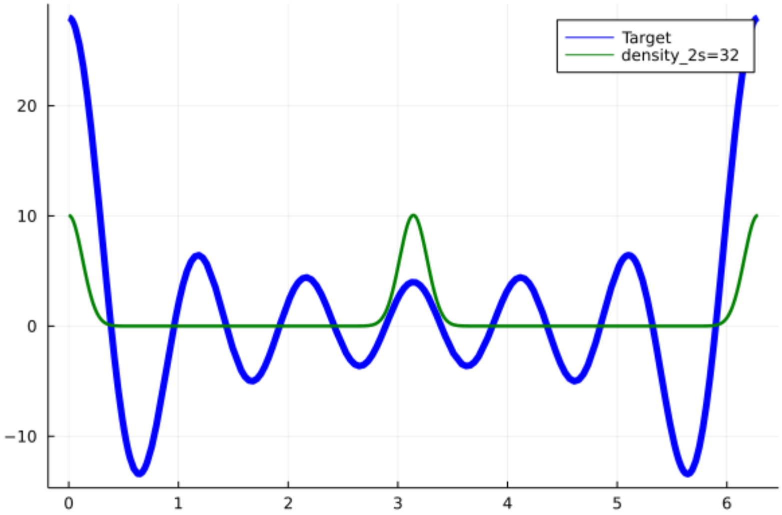

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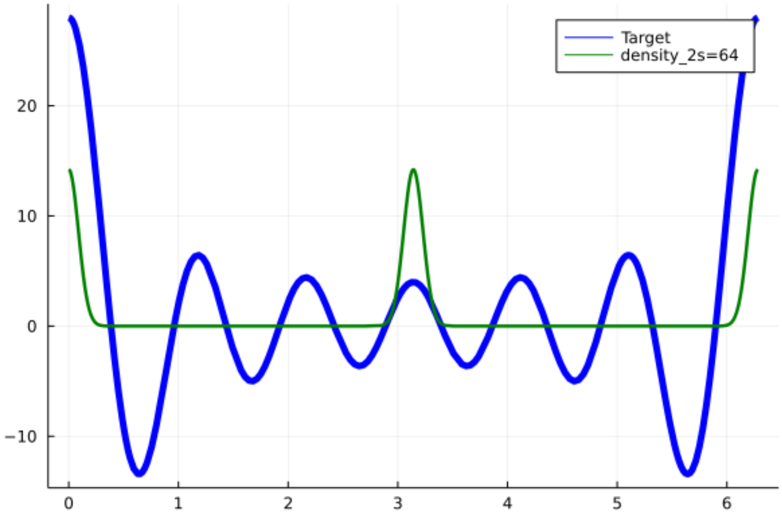

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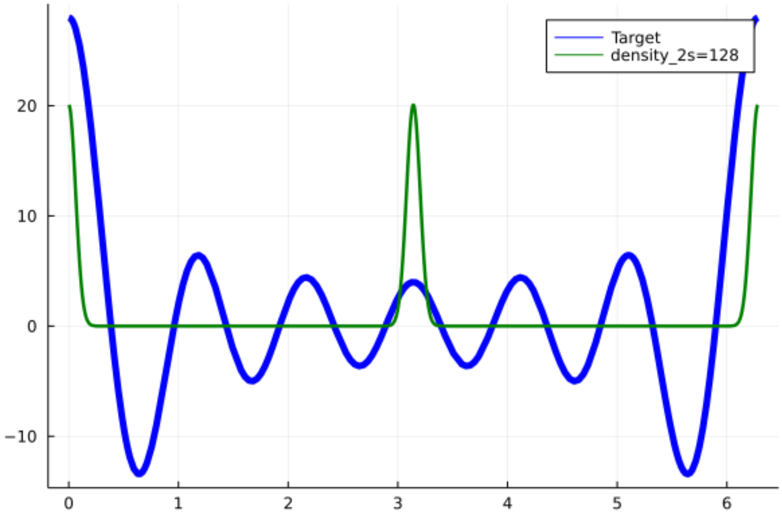

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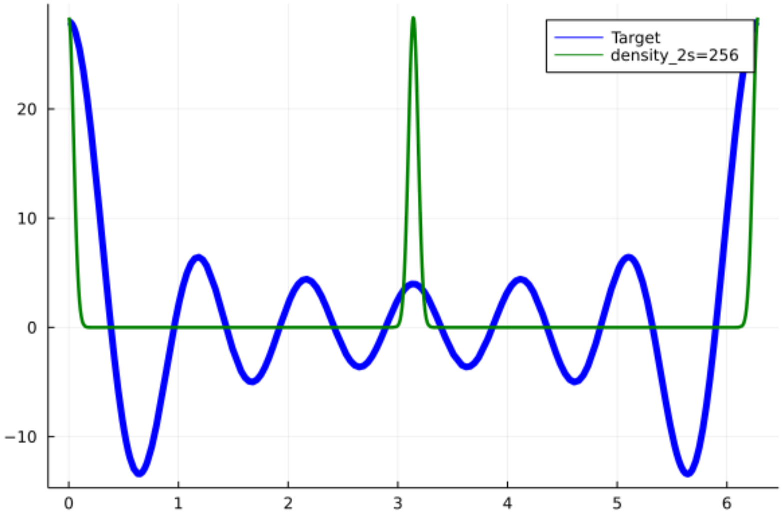

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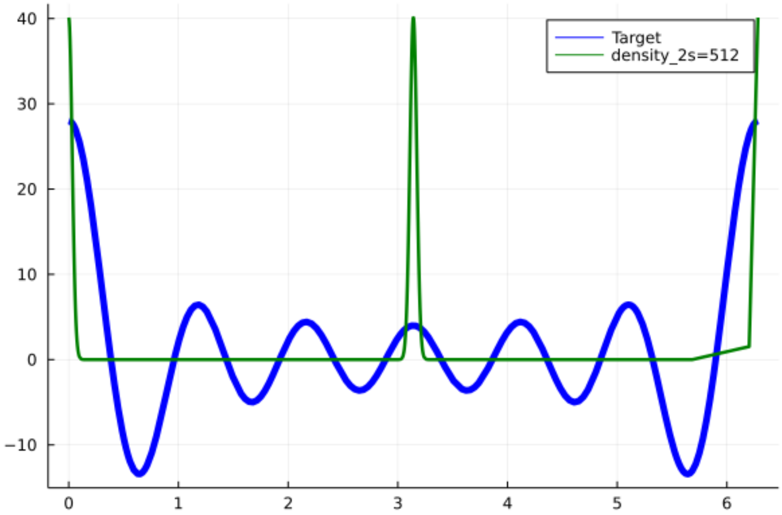

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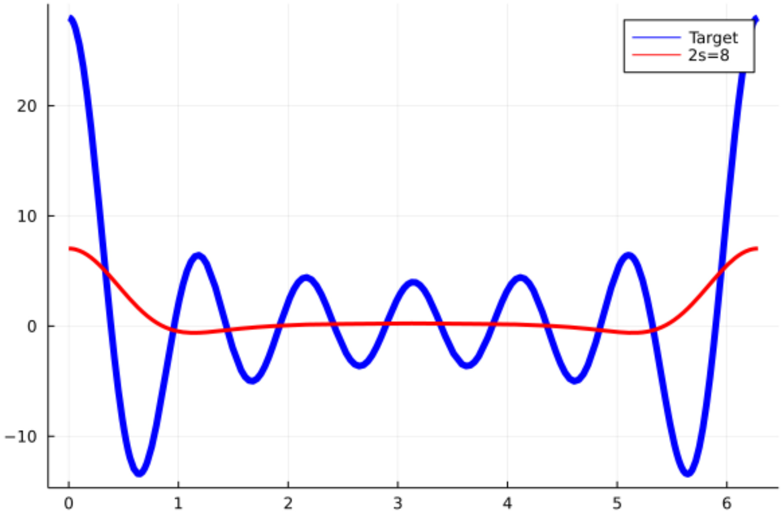

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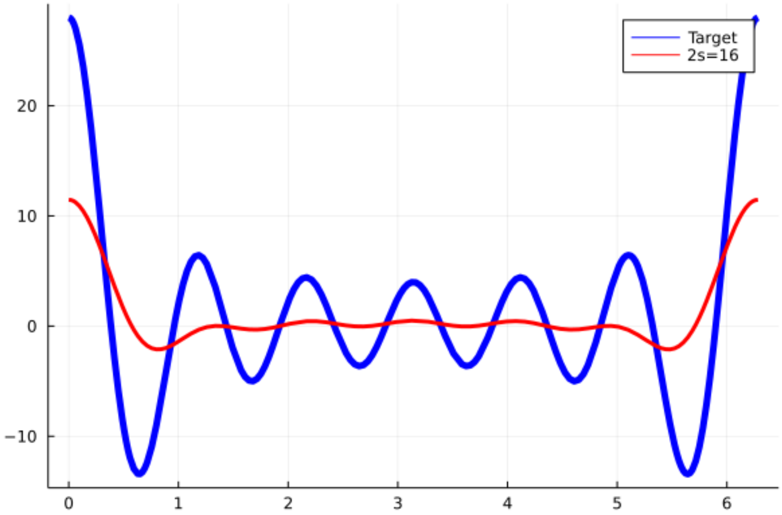

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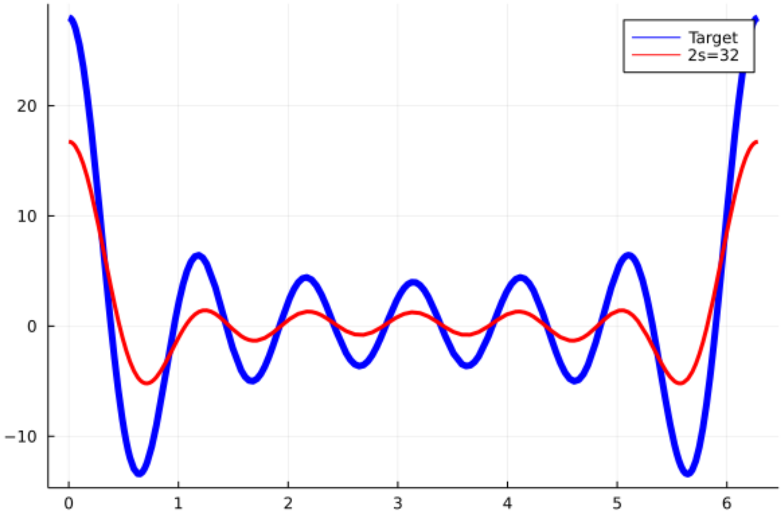

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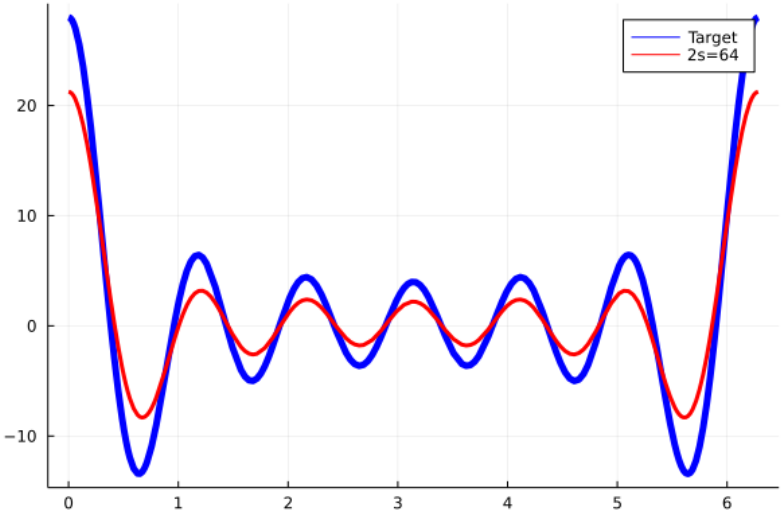

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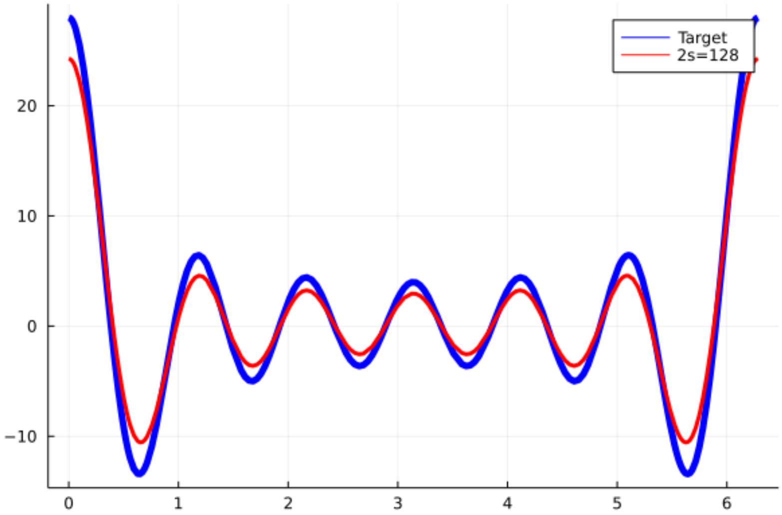

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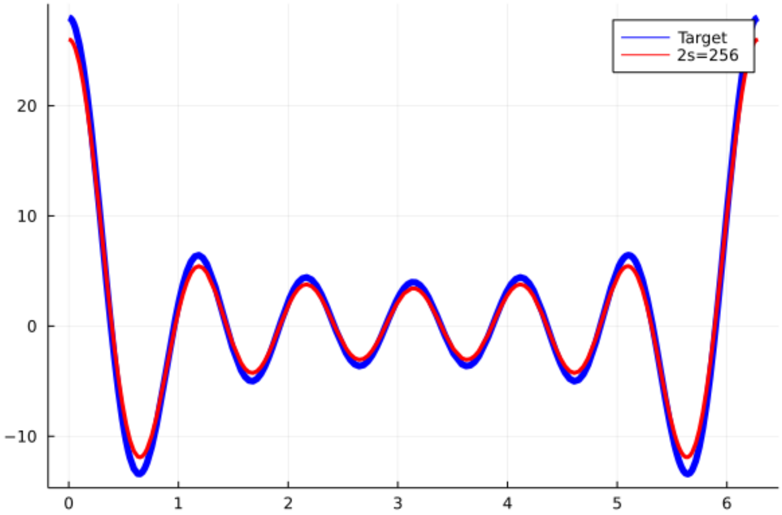

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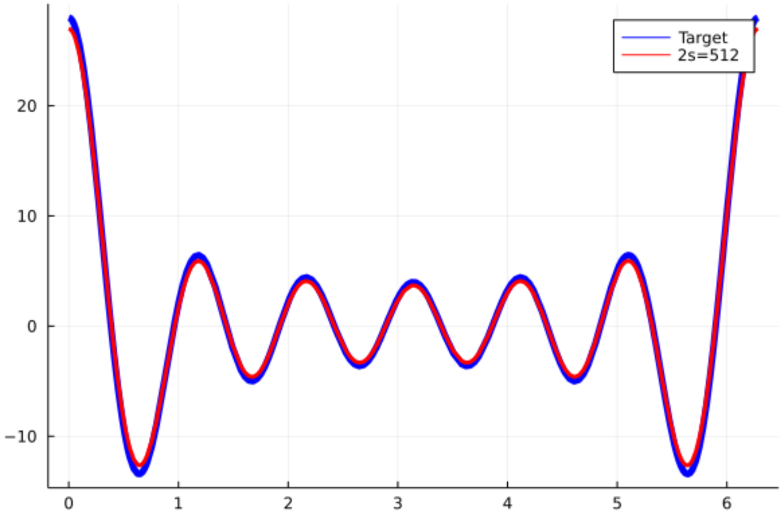

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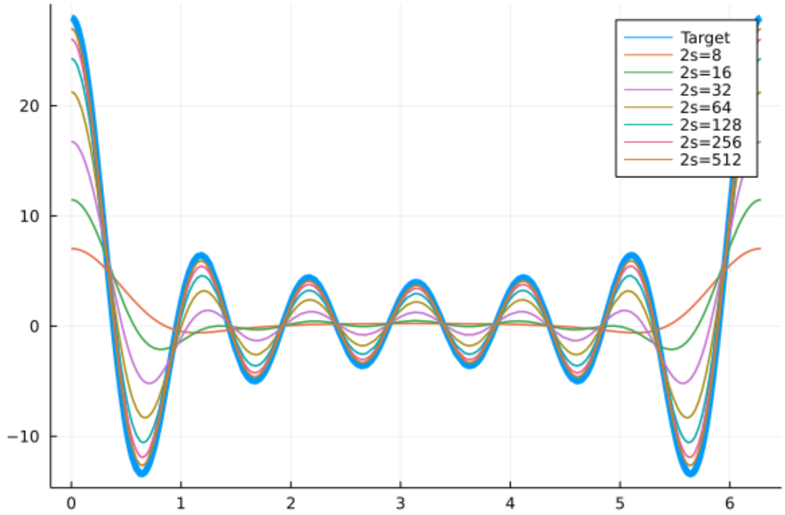

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We will map the polyhedron Q_s into P with our averaging map Γ_g

Lemma. (-,Cristancho)

If $f \in Q_s$ *then* $\Gamma_s(f)$ *is a nonnegative polynomial.*

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Recall $\Gamma_s(f(x)) = h(x)$.

$$h(x) = \int_S \frac{\langle x, y\rangle^{2s}}{N} f(y) d\mu(y)$$

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Since $f \in Q_s$ the expression is a sum of even powers of linear forms with nonnegative coefficients. □

The maps Γ_g have the following remarkable symmetry property

Lemma.

For every $T \in SO(n)$ we have

$$\Gamma_g\left(f(Tx)\right) = h(Tx)$$

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As a result, due to the machinery of representation theory, all the maps Γ_g become simultaneously diagonal in a natural basis. This basis allows us to understand the operators Γ_g and allows us to compute them for very high-degrees in practice.

3.Harmonic expansions

Every homogeneous polynomial $f \in R_{2k}$ *can be written uniquely in its* **harmonic expansion** *as*

$$f = \|x\|^{2k} f_0 + \|x\|^{2(k-1)} f_2 + \|x\|^{2(k-2)} f_4 + \cdots + f_{2k}$$

where the f_{2j} *are homogeneous harmonic polynomials (i.e.* $\Delta f_{2j} \equiv 0$*) of degree* $2j$*.*

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Equivalently, the homogeneous polynomials decompose, as $SO(n)$-representations as:

$$R_{2k} = H_{2k} \oplus \|x\|^2 H_{2(k-1)} \oplus \|x\|^4 H_{2(k-2)} \oplus \cdots \oplus \|x\|^{2k} H_0$$

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This is the decomposition of R_{2k} into pairwise non-isomorphic irreducible representations of $SO(n)$.

3.Harmonic expansions

In this decomposition the operator Γ_g becomes diagonal...

Lemma. (Funk-Hecke)

Assume $g(t) = \sum_{j=0}^{n} \lambda_{2j}^{g} \phi_{2j}(t)$ *is the unique expression of* $g(t)$ *as linear combination of (suitably normalized) Gegenbauer polynomials. If*

$$f = \|x\|^{2k} f_0 + \|x\|^{2(k-1)} f_2 + \|x\|^{2(k-2)} f_4 + \cdots + f_{2k}$$

is the unique harmonic expansion for $f \in R_{2k}$ *then we have*

$$\hat{\Gamma}_g(f) = \lambda_0^g \|x\|^{2k} f_0 + \lambda_2^g \|x\|^{2(k-1)} f_2 + \lambda_4^g \|x\|^{2(k-2)} f_4 + \cdots + \lambda_{2k}^g f_{2k}.$$

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Definition.

The **Frobenius threshold** *of the map* $\hat{\Gamma}_g$ *in degree* $2k$ *is the number*

$$\tau_{2k}(g) := \sqrt{\sum_{j=0}^{k} \dim(H_{2j}) \left(\frac{1}{\lambda_{2j}^g} - 1\right)^2}$$

Construction. Harmonic Hierarchies for P_{2k}

Given:

1. Quadrature rules (X_t, W_t) of strength t on S for every even integer t.
2. A sequence of even univariate polynomials $(g_s(t))_{s\in\mathbb{N}}$ which are nonnegative on the interval $[-1, 1]$.

Definition.

The **Linear Harmonic Hierarchy** $(A_s)_{s\in\mathbb{N}}$ *in degree* $2k$ *determined by* (1) *and* (2) *is given by*

$$A_s := \hat{\Gamma}_{g_s}(Q_s) \subseteq P_{2k}$$

A quantitative convergence Theorem

Theorem. (-,Cristancho)

The following statements hold:

1. *If* $f \in R_{2k}$ *satisfies the inequality*

$$\min_{x \in X_{2(k+d_s)}} f(x) > \frac{\|f\|_2}{\sqrt{\mu(S)}} \tau_{2k}(g_s),$$

   then $f \in A_s$*.*

2. *If* $\lim_{s\to\infty} \tau_{2k}(g_s) = 0$*, then every strictly positive polynomial is contained in some* A_s *and in particular the hierachy is convergent in the sense that the following inclusions hold*

$$P_{2k}^{\circ} \subseteq \bigcup_{s=0}^{\infty} A_s \subseteq P_{2k}$$

An example: pure powers HH

Example:

Let $g_s(t) = \frac{t^{2s}}{\int_S y_1^{2s} dy_1}$.

Corollary.

The inequality

$$\frac{1+\frac{n}{2}}{s} + O\left(\frac{1}{s^2}\right) \leq \tau_{2k}(g_s) \leq \frac{k^2+\frac{kn}{2}}{s} + O\left(\frac{1}{s^2}\right).$$

holds. In particular, the resulting linear harmonic hierarchy converges.

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The previous Corollary implies Polya's Theorem (Reznick's proof explained by Blekherman).

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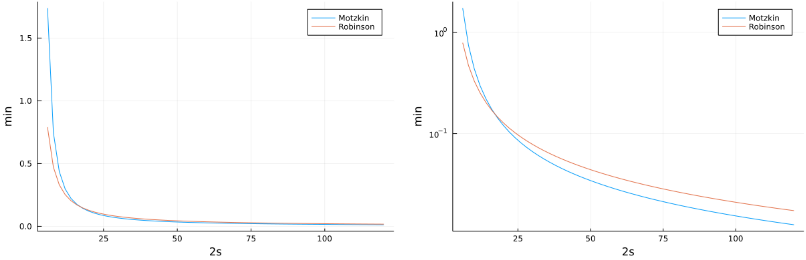
Theorem. (-, Cristancho)

The problem of minimizing $\tau_{2k}(g)$ among all valid $g(t)$ of degree $\leq 2s$ is a convex optimization problem.

Polynomial optimization examples

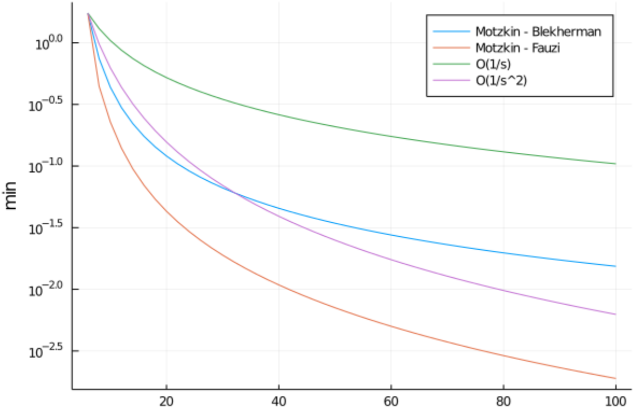
The Motzkin polynomial and the Robinson form are nonnegative ternary sextics with zeroes. We minimize them with harmonic hierarchies...



$$m(x, y, z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2$$

$$r(x, y, z) = x^6 + y^6 + z^6 - x^4y^2 - x^4z^2 - y^4z^2 - x^2z^4 - y^2z^4 + 3x^2y^2z^2$$

Pure powers vs Kang-Fawzi averaging...



$$m(x, y, z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2$$

What didn't fit...

1. Since $A_s \subseteq P_{2k}$ the duals $A_s^* \supseteq P_{2k}^*$ define converging **harmonic hierarchies for moments**.
2. Establishing the practical performance of harmonic hierarchies in areas of interest (and comparison with other hierarchies) is the subject of ongoing work...