Harmonic hierarchies for polynomial optimization.

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Polynomial Optimization:

Denote by:
- \( R := \mathbb{R}[x_1, \ldots, x_n] \) the ring of \( n \)-variate polynomials with real coefficients.
- \( F \in R_{2k} \) a homogeneous polynomial of degree \( 2k \).
- \( S \subseteq \mathbb{R}^n \) the unit sphere.

Definition.

The **polynomial optimization problem** asks us to find

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This is a fundamental problem for several reasons: It is a model for global, non-convex optimization problems and has a wealth of applications initiated by work of J.B. Lasserre and P. Parrilo in the early 2000’s ([20C93 according to 2020 MSC])
A solution strategy:

Assume $F(x_1, \ldots, x_n)$ is homogeneous and of even degree $2k$.

$$\alpha^* := \min_{x \in S} F(x)$$

Lemma.

The following equality holds:

$$\alpha^* := \max \left\{ \lambda \in \mathbb{R} : F(x) - \lambda \|x\|_2^{2k} \geq 0 \text{ on } S \right\}$$
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Definition.

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The set $P_{2k} \subseteq R_{2k}$ is a closed convex cone.
A natural approach is to construct hierarchies approximating $P_{2k}$,

**Definition.**

A hierarchy is a collection of convex cones $(A_s)_{s \in \mathbb{N}}$ satisfying the following properties:

1. The cones $A_s$ have fast membership algorithms.
2. $A_s \subseteq P$
3. The sets $A_s$ converge to $P$ in the sense that

$$P^\circ \subseteq \bigcup_{s} A_s \subseteq P$$
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Useful because if

\[
\alpha_s := \sup \left\{ \lambda \in \mathbb{R} : F(x) - \lambda \|x\|^2k \in A_s \right\}
\]

then the $\alpha_s$ are easily computable, $\alpha_s \leq \alpha^*$ and $\lim_{s \to \infty} \alpha_s = \alpha^*$. 
There exist several well-known hierarchies for $P_{2k}$

1. **SOS hierarchies:** [Parrilo], [Lasserre], [Peña, Vera, Zuluaga], [D. Henrion], ...

2. **SONC, SIGNOMIALS:** [T. De Wolff], [V. Chandrasekaran], ...

3. **Polyhedral hierarchies:**
   - Polya-type [Peña, Vera, Zuluaga],
   - DSOS, SDSOS: [A.A. Ahmadi, Majumdar], [A.A. Ahmadi, G. Hall], ...
The purpose of this talk is to introduce several new **polyhedral hierarchies** for approximating $P$ and to give **quantitative bounds on the rate at which they converge**.

We call them **Harmonic Hierarchies** because the harmonic analysis on spheres (or equivalently the representation theory of $SO(n)$) plays a key role in their definition.
Motivation.

Three basic ingredients:
1. Quadrature rules on spheres.
2. Polynomial averaging operators.
3. Harmonic expansion on spheres.

The **Construction** of Harmonic hierarchies for $P_{2k}$.

A Convergence **Theorem**.

Some simple polynomial optimization **examples** (computed with our Julia package for optimization via harmonic hierarchies).
1. Quadrature rules

Definition.

A **quadrature rule** of strength $2t$ on the sphere $S \subseteq \mathbb{R}^n$ is a pair $(X, W)$ where $X \subseteq S$ is a finite set and $W : X \rightarrow \mathbb{R}_{>0}$ is a function for which the following equality holds,

$$\forall f \in R_{2t} \left( \int_S f(y) \, d\mu(y) = \sum_{x \in X} W(x) f(x) \right)$$

where the integral on the left-hand side is taken with respect to the $(n-1)$-dimensional volume measure $\mu$ on the sphere $S$. 

**Example**:

$$\forall f \in R_{2t} \left( \int_S f(x, y) \, d\mu(y) = 2t+1 \sum_{j=0}^{2(2t+1)} 2\pi^{2(t+1)} f(x_j) \right)$$

where the $x_j$ are the vertices of a regular $2(2t+1)$-gon.
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**Example:**

$$\forall f \in R_{2t} \left( \int_{S^1} f(x, y) d\mu(y) = \sum_{j=0}^{2t+1} \frac{2\pi}{2(t + 1)} f(x_j) \right)$$

where the $x_j$ are the vertices of a regular $2(t + 1)$-gon.
The main invariant of a quadrature rule \((X, W)\) is size \(|X|\). It is known that the minimal size \(\nu_{2k}\) of a quadrature rule of weight \(2k\) satisfies
\[
\dim(R_k) \leq \nu_{2k} \leq \dim(R_{2k}) + 1
\]
rules of minimal size are known only in few cases.
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**Example: (Gaussian Quadrature)**

If \(X\) consists of the \(k\) roots of the Jacobi polynomials \(P_k^{(\alpha, \beta)}(t)\) there are weights \(W\) which lead to a quadrature rule of strength \(2k - 1\) on \([-1, 1]\),

\[
\int_{-1}^{1} f(t)(1 - t)^a(1 + t)^b \, dt = \sum_{x \in X} W(x)f(x)
\]
1. Quadrature rules

**Theorem. (Cristancho)**

For every integer \( t \) there exists a quadrature rule of strength \( 2t \) in the sphere \( S \subseteq \mathbb{R}^n \) supported in \( 2(t + 1)^{n-1} \) points.
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Theorems (and code!) of Alex Townsend makes these computations possible for high degrees.
The blue polyhedra $Q_s$:

Given quadrature rules $(X_d, W_d)$ of strength $2d$ for every $d$ and an integer $2k$
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**Definition.**

Let \(Q_s\) be the polyhedron

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Q_s := \{ F \in \mathbb{R}_{2k} : \forall x \in X_{2(k+s)} (F(x) \geq 0) \}
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This polyhedron has $\leq 2(k + s + 1)^{n-1}$ facets.
2. Averaging polynomials

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\textbf{Definition.}

Let $g(t)$ be a univariate polynomial which is nonnegative on $[-1, 1]$. Define $\Gamma_g : \mathbb{R} \to \mathbb{R}$ via $\Gamma_g(f(x)) = h(x)$ where

$$h(x) = \int_{S} g(\langle x, y \rangle)f(y) \, d\mu(y)$$

where $\mu$ is the $(n-1)$-dimensional volume measure.
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We will map the polyhedron \(Q_s\) into \(P\) with our averaging map \(\Gamma_g\)
Lemma. (-, Cristancho)

If \( f \in Q_s \) then \( \Gamma_s(f) \) is a nonnegative polynomial.
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If $f \in Q_s$ then $\Gamma_s(f)$ is a nonnegative polynomial.

Proof.

Recall $\Gamma_s(f(x)) = h(x)$.

$$h(x) = \int_{S} \frac{\langle x, y \rangle^{2s}}{N} f(y) d\mu(y)$$
Lemma. (-, Cristancho)

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Recall $\Gamma_s(f(x)) = h(x)$.

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\begin{align*}
h(x) &= \int_S \frac{\langle x, y \rangle^{2s}}{N} f(y) d\mu(y) \\
\int_S \frac{\langle x, y \rangle^{2s}}{N} f(y) d\mu(y) &= \sum_{j=1}^{X_{s+k}} \frac{\langle x, y_j \rangle^{2s}}{N} f(y_j) w_j
\end{align*}
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Since $f \in Q_s$ the expression is a sum of even powers of linear forms with nonnegative coefficients.
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Since \( f \in Q_s \) the expression is a sum of even powers of linear forms with nonnegative coefficients.
The maps \( \Gamma_g \) have the following remarkable symmetry property

**Lemma.**

For every \( T \in SO(n) \) we have

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\Gamma_g (f(Tx)) = h(Tx)
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As a result, due to the machinery of representation theory, all the maps $\Gamma_g$ become simultaneously diagonal in a natural basis.
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**Lemma.**
For every $T \in SO(n)$ we have

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As a result, due to the machinery of representation theory, all the maps $\Gamma_g$ become simultaneously diagonal in a natural basis. This basis allows us to understand the operators $\Gamma_g$ and allows us to compute them for very high-degrees in practice.
Every homogeneous polynomial \( f \in R_{2k} \) can be written uniquely in its harmonic expansion as

\[
f = \|x\|^{2k}f_0 + \|x\|^{2(k-1)}f_2 + \|x\|^{2(k-2)}f_4 + \cdots + f_{2k}
\]

where the \( f_{2j} \) are homogeneous harmonic polynomials (i.e. \( \Delta f_{2j} \equiv 0 \)) of degree \( 2j \).
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where the $f_{2j}$ are homogeneous harmonic polynomials (i.e. $\Delta f_{2j} \equiv 0$) of degree $2j$.

Equivalently, the homogeneous polynomials decompose, as $SO(n)$-representations as:

$$R_{2k} = H_{2k} \oplus \|x\|^{2} H_{2(k-1)} \oplus \|x\|^{4} H_{2(k-2)} \oplus \cdots \oplus \|x\|^{2k} H_{0}$$

where the $H_j$ are the vector spaces of homogeneous harmonic polynomials of degree $j$. 
3. Harmonic expansions

Every homogeneous polynomial $f \in R_{2k}$ can be written uniquely in its harmonic expansion as

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where the $H_j$ are the vector spaces of homogeneous harmonic polynomials of degree $j$.

*This is the decomposition of $R_{2k}$ into pairwise non-isomorphic irreducible representations of $SO(n)$.*
In this decomposition the operator $\Gamma_g$ becomes diagonal...

**Lemma. (Funk-Hecke)**

Assume $g(t) = \sum_{j=0}^{n} \lambda_{2j}^g \phi_{2j}(t)$ is the unique expression of $g(t)$ as linear combination of (suitably normalized) Gegenbauer polynomials. If

$$f = \|x\|^{2k} f_0 + \|x\|^{2(k-1)} f_2 + \|x\|^{2(k-2)} f_4 + \cdots + f_{2k}$$

is the unique harmonic expansion for $f \in R_{2k}$ then we have

$$\hat{\Gamma}_g(f) = \lambda_0^g \|x\|^{2k} f_0 + \lambda_2^g \|x\|^{2(k-1)} f_2 + \lambda_4^g \|x\|^{2(k-2)} f_4 + \cdots + \lambda_{2k}^g f_{2k}.$$
3. Harmonic expansions

\[ \hat{\Gamma}_g(f) = \lambda_0^g \|x\|^{2k} f_0 + \lambda_2^g \|x\|^{2(k-1)} f_2 + \lambda_4^g \|x\|^{2(k-2)} f_4 + \cdots + \lambda_{2k}^g f_{2k}. \]

**Definition.**

The *Frobenius threshold* of the map \( \hat{\Gamma}_g \) in degree \( 2k \) is the number

\[ \tau_{2k}(g) := \sqrt{\sum_{j=0}^{k} \dim(H_{2j}) \left( \frac{1}{\lambda_{2j}^g} - 1 \right)^2} \]
Construction. Harmonic Hierarchies for $P_{2k}$

Given:

1. Quadrature rules $(X_t, W_t)$ of strength $t$ on $S$ for every even integer $t$.

2. A sequence of even univariate polynomials $(g_s(t))_{s \in \mathbb{N}}$ which are nonnegative on the interval $[-1, 1]$.

Definition.

The **Linear Harmonic Hierarchy** $(A_s)_{s \in \mathbb{N}}$ in degree $2k$ determined by (1) and (2) is given by

$$A_s := \hat{\Gamma}_g (Q_s) \subseteq P_{2k}$$
A quantitative convergence Theorem

Theorem. (\ (-, Cristancho)\)

The following statements hold:

1. If \( f \in R_{2k} \) satisfies the inequality
   
   \[
   \min_{x \in X_{2(k + ds)}} f(x) > \frac{\|f\|_2}{\sqrt{\mu(S)}} \tau_{2k}(g_s),
   \]

   then \( f \in A_s \).

2. If \( \lim_{s \to \infty} \tau_{2k}(g_s) = 0 \), then every strictly positive polynomial is contained in some \( A_s \) and in particular the hierarchy is convergent in the sense that the following inclusions hold

   \[
   P_{2k}^0 \subseteq \bigcup_{s=0}^\infty A_s \subseteq P_{2k}
   \]
Example:

Let \( g_s(t) = \frac{t^{2s}}{\int_S y_1^{2s} \, dy_1} \).

Corollary.

The inequality

\[
\frac{1 + \frac{n}{2}}{s} + O \left( \frac{1}{s^2} \right) \leq \tau_{2k}(g_s) \leq \frac{k^2 + \frac{kn}{2}}{s} + O \left( \frac{1}{s^2} \right)
\]

holds. In particular, the resulting linear harmonic hierarchy converges.
Remark.

*The previous Corollary implies Polya’s Theorem (Reznick’s proof explained by Blekherman).*
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There exist better $g_s$ which can ensure convergence rates $\simeq 1/s^2$ [Kang-Fawzi]. Bounds for harmonic hierarchies (linear) are as good as the best bound for SOS-hierarchies (SDP).

Theorem. (-, Cristancho)

The problem of minimizing $\tau_{2k}(g)$ among all valid $g(t)$ of degree $\leq 2s$ is a convex optimization problem.
Polynomial optimization examples

The Motzkin polynomial and the Robinson form are nonnegative ternary sextics with zeroes. We minimize them with harmonic hierarchies...

\[
m(x, y, z) = x^4 y^2 + x^2 y^4 + z^6 - 3x^2 y^2 z^2
\]

\[
r(x, y, z) = x^6 + y^6 + z^6 - x^4 y^2 - x^4 z^2 - y^4 z^2 - x^2 z^4 - y^2 z^4 + 3x^2 y^2 z^2
\]
Pure powers vs Kang-Fawzi averaging...

\[ m(x, y, z) = x^4 y^2 + x^2 y^4 + z^6 - 3x^2 y^2 z^2 \]
Since $A_s \subseteq P_{2k}$ the duals $A_s^* \supseteq P_{2k}^*$ define converging harmonic hierarchies for moments.

Establishing the practical performance of harmonic hierarchies in areas of interest (and comparison with other hierarchies) is the subject of ongoing work...