Approximating regions of attraction of a sparse polynomial differential system

POP brainstorming
MAC team

Matteo Tacchi, C. Cardozo, D. Henrion, J.B. Lasserre

2nd year PhD student at RTE & LAAS-CNRS

Friday, September 11th

Outline

Outline

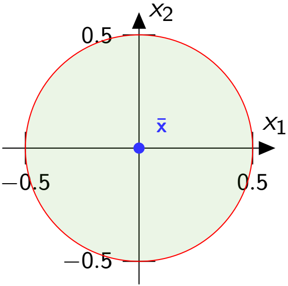
A radial system

$$\begin{cases} \dot{x}_1 = (x_1^2 + x_2^2 - \frac{1}{4})\ x_1 \\ \dot{x}_2 = (x_1^2 + x_2^2 - \frac{1}{4})\ x_2 \end{cases}$$

summarized into $\dot{\mathbf{x}} = \left(|\mathbf{x}|^2 - \frac{1}{4} \right) \mathbf{x}$

Stability analysis

- $\bar{\mathbf{x}} = (0,0)$ locally asymptotically stable equilibrium point
- $D(0, \frac{1}{2}) := \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < \frac{1}{4}\}$ ROA of $\bar{\mathbf{x}}$
- $C(0, \frac{1}{2}) := \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = \frac{1}{4}\}$ set of unstable EPs



A sparse system

$$\begin{cases} \dot{x}_1 = (x_1^2 + x_2^2 - \frac{1}{4})\, x_1 \\ \dot{x}_2 = (x_2^2 + {\color{red}x_3^2} - \frac{1}{4})\, x_2 \\ \dot{x}_3 = (x_2^2 + x_3^2 - \frac{1}{4})\, x_3 \end{cases} \qquad \text{summarized into } \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

Stability analysis

- $V(\mathbf{x}) := |\mathbf{x}|^2 = x_1^2 + x_2^2 + x_3^2$ Lyapunov function (V & $-\dot{V}$ p.d.) on

$$\mathcal{D} := \left\{ \mathbf{x} \in \mathbb{R}^3 : x_1^2 + x_2^2 < \frac{1}{4}, x_2^2 + x_3^2 < \frac{1}{4} \right\}$$

$\Rightarrow$ $\bar{\mathbf{x}} = (0,0,0)$ locally asymptotically stable equilibrium point

- $\mathcal{D}$ is positively invariant $\Rightarrow$ it is included in the ROA of $\bar{\mathbf{x}}$

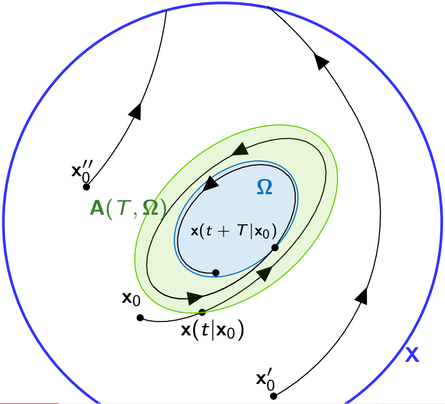
$\rightarrow$ what is the ROA of $\bar{\mathbf{x}}$?

Finite time ROA

Problem

Let $\mathbf{X} := \{\mathbf{x} \in \mathbb{R}^3 : x_1^2 + x_2^2 < 1, x_2^2 + x_3^2 < 1\}$ be the admissible state set.
Given time horizon $T > 0$ & target set $\mathbf{\Omega} \subset \mathcal{D}$ containing $\bar{\mathbf{x}}$, compute

$$\mathbf{A}(T, \mathbf{\Omega}) := \left\{ \mathbf{x}^0 \in \mathbb{R}^3 : \begin{array}{c} \forall t \in [0,T] \;\; \mathbf{x}(t|\mathbf{x}^0) \in \mathbf{X} \\ \mathbf{x}(T|\mathbf{x}^0) \in \mathbf{\Omega} \end{array} \right\}$$



Finite time ROA

Problem

Let $\mathbf{X} := \left\{\mathbf{x} \in \mathbb{R}^3 : x_1^2 + x_2^2 < 1, x_2^2 + x_3^2 < 1\right\}$ be the admissible state set.
Given time horizon $T > 0$ & target set $\mathbf{\Omega} \subset \mathcal{D}$ containing $\bar{\mathbf{x}}$, compute

$$\mathbf{A}(T, \mathbf{\Omega}) := \left\{\mathbf{x}^0 \in \mathbb{R}^3 : \substack{\forall t \in [0,T]\ \mathbf{x}(t|\mathbf{x}^0) \in \mathbf{X} \\ \mathbf{x}(T|\mathbf{x}^0) \in \mathbf{\Omega}}\right\}$$

Proposition of solution [Korda, Henrion 2012]

For any feasible (v, w) of LP

$$\begin{aligned} &\inf \int_{\mathbf{X}} w(\mathbf{x})\, \mathrm{d}\mathbf{x} \\ &s.t.\ w \geq (v(0,\cdot) + 1)_+ && \text{on } \mathbf{X} \\ &\quad v(T,\cdot) \geq 0 && \text{on } \mathbf{\Omega} \\ &\quad \partial_t v(t,\mathbf{x}) + \nabla v(t,\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) \leq 0 && \forall t \in [0, T], \mathbf{x} \in \mathbf{X} \end{aligned}$$

- $\mathbf{A}(T, \mathbf{\Omega}) \subset \{\mathbf{x} \in \mathbb{R}^3 : v(0, \mathbf{x}) \geq 0\} \subset \{\mathbf{x} \in \mathbb{R}^3 : w(\mathbf{x}) \geq 1\}$
- $\mathrm{vol}\left(\{\mathbf{x} \in \mathbb{R}^3 : w(\mathbf{x}) \geq 1\} \setminus \mathbf{A}(T, \mathbf{\Omega})\right) \longrightarrow 0$ along minimizing sequences

Outline

Exploiting sparsity

"No free lunch" rule

- minimizing sequence computed using Lasserre hierarchy

$\Rightarrow$ reduction to solving LMI / SDP $\Rightarrow$ sensitive to dimension

Lower dimensional rephrased LP

Suppose $\mathbf{\Omega} = \{\mathbf{x} \in \mathbb{R}^3 : (x_1, x_2) \in \mathbf{\Omega}_1 \ \& \ (x_2, x_3) \in \mathbf{\Omega}_2\}$. Consider

Exploiting sparsity

"No free lunch" rule

- minimizing sequence computed using Lasserre hierarchy

$\Rightarrow$ reduction to solving LMI / SDP $\Rightarrow$ sensitive to dimension

Lower dimensional rephrased LP

Suppose $\mathbf{\Omega} = \{\mathbf{x} \in \mathbb{R}^3 : (x_1, x_2) \in \mathbf{\Omega}_1 \ \& \ (x_2, x_3) \in \mathbf{\Omega}_2\}$. Consider

$$\inf_{u, v_i, w_i} \int_{\mathbf{Y}_1} w_1(x_1, x_2) \, \mathrm{d}x_1 \mathrm{d}x_2 + \int_{\mathbf{Y}_2} w_2(x_2, x_3) \, \mathrm{d}x_2 \mathrm{d}x_3$$

$$\begin{aligned}
s.t. \ & w_i \geq (v_i(0, \cdot) + 1)_+ && \text{on } \mathbf{Y}_i \\
& v_i(T, \cdot) \geq 0 && \text{on } \mathbf{\Omega}_i \\
& \partial_t v_2(t, \cdot) + \partial_2 v_2(t, \cdot) f_2 + \partial_3 v_2(t, \cdot) f_3 \leq 0 && \text{on } \mathbf{Y}_2 \ \forall t \in [0, T] \\
& v_1(t, x_1, x_2) = v_{11}(t, x_1) + v_{12}(t, x_2) && \forall (t, x_1, x_2) \in [0, T] \times \mathbf{Y}_1 \\
& \partial_t v_1(t, x_1, x_2) + \partial_1 v_{11}(t, x_1) f_1(x_1, x_2) \leq u(x_2) && \forall (t, x_1, x_2) \in [0, T] \times \mathbf{Y}_1 \\
& u(x_2) + \partial_2 v_{12}(t, x_2) f_2(x_2, x_3) \leq 0 && \forall (t, x_2, x_3) \in [0, T] \times \mathbf{Y}_2
\end{aligned}$$

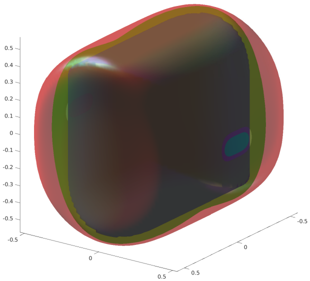
where $\mathbf{Y}_i := \{(x_i, x_{i+1}) \in \mathbb{R}^2 : x_i^2 + x_{i+1}^2 \leq 1\}$, $i = 1, 2$.

Actual ROA estimation

Theory: outer approximation

$$\mathbf{A}(T,\mathbf{\Omega}) \subset \{\mathbf{x} \in \mathbb{R}^3 : v_i(0, x_i, x_{i+1}) \geq 0, i = 1 \ \& \ 2\}$$

Numerics (sparse, dense, $\mathcal{D}$)



comparison of the degree 10 relaxations

degree	dense	sparse
4	4	4
6	24	10
8	334	83
10	5542	440
12	-	1865

comparison of the CPU times

Outline

General sparsity pattern

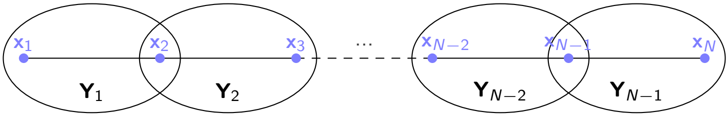
"Linear" sparse differential system

$$\begin{cases} \dot{\mathbf{x}}_i = \mathbf{f}_i(\mathbf{x}_i, \mathbf{x}_{i+1}) & i = 1, \ldots, N-1 \\ \dot{\mathbf{x}}_N = \mathbf{f}_N(\mathbf{x}_{N-1}, \mathbf{x}_N) & \end{cases}$$

$$\mathbf{x}_j \in \mathbf{X}_j \qquad j = 1, \ldots, N$$

$$\mathbf{Y}_i = \mathbf{X}_i + \mathbf{X}_{i+1} \qquad i = 1, \ldots, N-1$$

$$\mathbf{X} = \mathbf{X}_1 \oplus \cdots \oplus \mathbf{X}_N = \mathbf{Y}_1 + \cdots + \mathbf{Y}_{N-1}$$



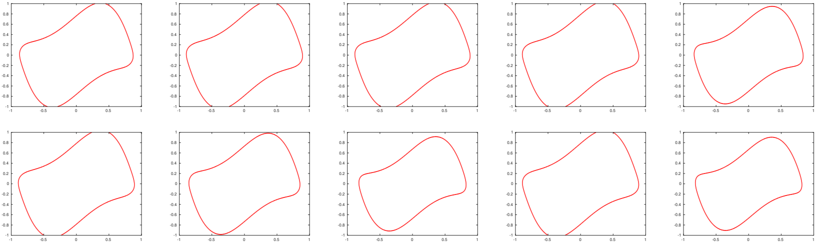
A chain of Van der Pol oscillators

Construction

$$\begin{cases} \dot{y}_i = -2z_i & i = 1,\dots,10 \\ \dot{z}_j = 0.8\ y_j + 10\ (1.44\ y_j^2 - 0.21)\ z_j + \epsilon_j\ z_{j+1}\ y_j & j = 1,\dots,9 \\ \dot{z}_{10} = 0.8\ y_{10} + 10\ (1.44\ y_{10}^2 - 0.21)\ z_{10} \end{cases}$$

$\epsilon_j \in [-0.5, 0.5]$ following a uniform probability, $\mathbf{x}_i = (y_i, z_i)$.

plots: (y_i, z_i) projection of finite time ROA outer approx, $i = 1,\dots,10$



Conclusion and future work

Take-home

Rephrazing Lasserre hierarchy for sparse systems:

- Speed up the computations
- Tackle higher dimensions
- Convergence remains to be studied

Conclusion and future work

Take-home

Rephrazing Lasserre hierarchy for sparse systems:

- Speed up the computations
- Tackle higher dimensions
- Convergence remains to be studied

Future work

- Tackle more general correlative sparsity patterns as in [Tacchi & al. *Exploiting sparsity for semialgebraic volume computation*, 2020]
- Extend the framework to inner stability region approximations:
  [Korda & al. *Inner approximation of the ROA (...)*, IFAC 2013]
  [Oustry & al. *Inner approximation of the MPI set (...)*, CDC 2019]
- Tackle 'time-scale sparsity' as in [Subotić & al. *A Lyapunov framework for nested dynamical systems (...)*, 2020]
- Apply the results to a simple electrical power network

Thank you!