Polynomial systems, sparsity, and applications

Matías R. Bender

November 9, 2021

Short CV

- Since 2019, PostDoc at Institut für Mathematik, TU Berlin.
  (Mentor: P. Bürgisser)
- In 2019: PhD in Informatics from Sorbonne Université.
  (Advisors: J.-C. Faugère & E. Tsigaridas)
- In 2015, Bsc+Msc in Computer Science from Univ. de Buenos Aires.

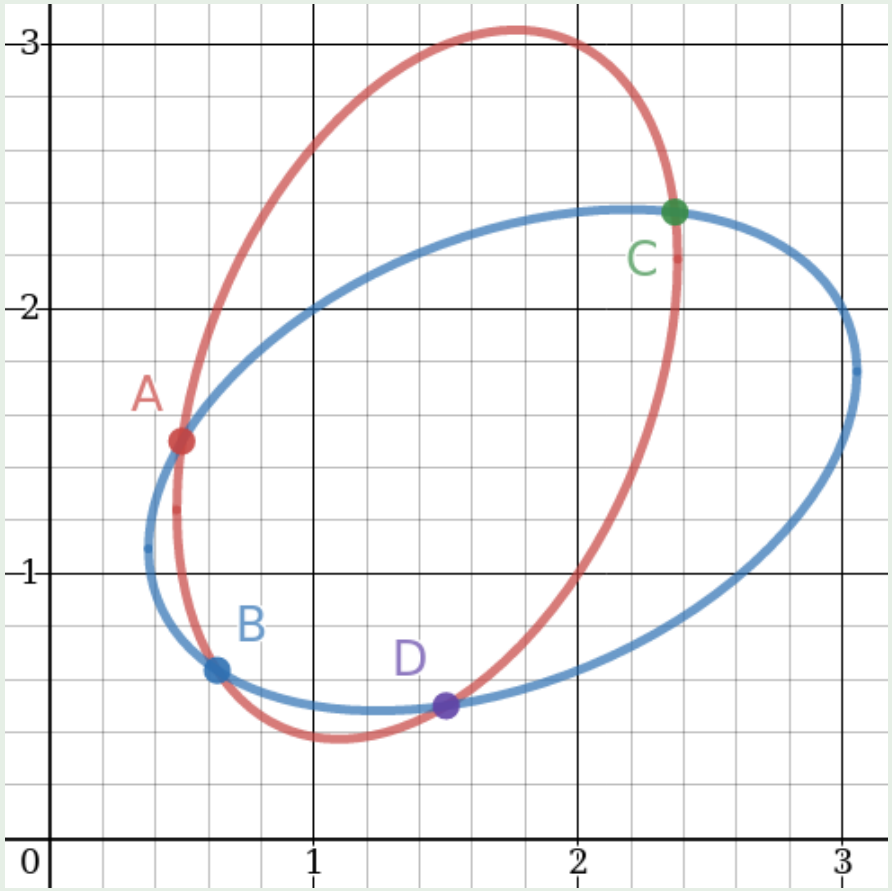
Supported by the ERC under the European's Horizon 2020 research and innovation programme (grant no. 787840)

I work in computer algebra. I focus on

I work in computer algebra. I focus on

Solving structured polynomial systems

- Solving polynomial systems of degree at most d in $\mathbb{C}[x_1, \ldots, x_n]$.



$$\begin{cases} 2x^2 - xy + y^2 - 4x - 2y + 3 = 0 \\ x^2 - xy + 2y^2 - 2x - 4y + 3 = 0 \end{cases}$$

$$\boxed{\begin{array}{c} d = 2 \\ n = 2 \end{array}}$$

I work in computer algebra. I focus on

Solving structured polynomial systems

- Solving polynomial systems of degree at most d in $\mathbb{C}[x_1, \ldots, x_n]$.

$$\begin{cases} 2x^2 - xy + y^2 - 4x - 2y + 3 = 0 \\ x^2 - xy + 2y^2 - 2x - 4y + 3 = 0 \end{cases} \qquad \boxed{\begin{array}{c} d = 2 \\ n = 2 \end{array}}$$

- Different methods: symbolic and numeric

I work in computer algebra. I focus on

Solving structured polynomial systems

- Solving polynomial systems of degree at most d in $\mathbb{C}[x_1, \ldots, x_n]$.

$$\begin{cases} 2x^2 - xy + y^2 - 4x - 2y + 3 = 0 \\ x^2 - xy + 2y^2 - 2x - 4y + 3 = 0 \end{cases} \qquad \boxed{\begin{array}{c} d = 2 \\ n = 2 \end{array}}$$

- Different methods: symbolic and numeric

  (Today: Gröbner basis, resultants, numerical linear algebra approaches)

I work in computer algebra. I focus on

Solving structured polynomial systems

- Solving polynomial systems of degree at most d in $\mathbb{C}[x_1, \ldots, x_n]$.

$$\begin{cases} 2x^2 - xy + y^2 - 4x - 2y + 3 = 0 \\ x^2 - xy + 2y^2 - 2x - 4y + 3 = 0 \end{cases} \qquad \boxed{\begin{array}{c} d = 2 \\ n = 2 \end{array}}$$

- Different methods: symbolic and numeric

  (Today: Gröbner basis, resultants, numerical linear algebra approaches)

- It is an intrinsically hard problem $\rightarrow$ Complexity $d^{O(n)}$.

I work in computer algebra. I focus on

Solving structured polynomial systems

- Solving polynomial systems of degree at most d in $\mathbb{C}[x_1, \ldots, x_n]$.

Dense
$$\begin{cases} 2x^2 - xy + y^2 - 4x - 2y + 3 = 0 \\ x^2 - xy + 2y^2 - 2x - 4y + 3 = 0 \end{cases} \qquad \boxed{\begin{array}{c} d = 2 \\ n = 2 \end{array}}$$

- Different methods: symbolic and numeric

  (Today: Gröbner basis, resultants, numerical linear algebra approaches)

- It is an intrinsically hard problem $\rightarrow$ Complexity $d^{O(n)}$.

- In applications $\rightarrow$ the problems have some structure.

Sparse
$$\begin{cases} -xy - 4x + 3 = 0 \\ 2y^2 - 2x - 4y + 3 = 0 \end{cases}$$

I work in computer algebra. I focus on

Solving structured polynomial systems

- Solving polynomial systems of degree at most d in $\mathbb{C}[x_1, \ldots, x_n]$.

Dense
$$\begin{cases} 2x^2 - xy + y^2 - 4x - 2y + 3 = 0 \\ x^2 - xy + 2y^2 - 2x - 4y + 3 = 0 \end{cases} \qquad \boxed{\begin{array}{c} d = 2 \\ n = 2 \end{array}}$$

- Different methods: symbolic and numeric

  (Today: Gröbner basis, resultants, numerical linear algebra approaches)

- It is an intrinsically hard problem $\rightarrow$ Complexity $d^{O(n)}$.

- In applications $\rightarrow$ the problems have some structure.

Sparse
$$\begin{cases} -xy - 4x + 3 = 0 \\ 2y^2 - 2x - 4y + 3 = 0 \end{cases}$$

- To improve complexity $\rightarrow$ Exploit sparsity.

I work in computer algebra. I focus on

Solving structured polynomial systems

- Solving polynomial systems of degree at most d in $\mathbb{C}[x_1, \ldots, x_n]$.

Dense
$$\begin{cases} 2x^2 - xy + y^2 - 4x - 2y + 3 = 0 \\ x^2 - xy + 2y^2 - 2x - 4y + 3 = 0 \end{cases} \qquad \boxed{\begin{array}{c} d = 2 \\ n = 2 \end{array}}$$

- Different methods: symbolic and numeric

  (Today: Gröbner basis, resultants, numerical linear algebra approaches)

- It is an intrinsically hard problem $\rightarrow$ Complexity $d^{O(n)}$.

- In applications $\rightarrow$ the problems have some structure.

Sparse
$$\begin{cases} -xy - 4x + 3 = 0 \\ 2y^2 - 2x - 4y + 3 = 0 \end{cases}$$

- To improve complexity $\rightarrow$ Exploit sparsity.

Applications

Tensor decomposition, Topological data analysis, Computational biology.

Gröbner basics

- $\mathbb{C}[\boldsymbol{x}] = \mathbb{C}[x_1, \ldots, x_n]$, polynomial ring in n indeterminates over $\mathbb{C}$.
- Polynomial $\rightarrow \sum_\alpha c_\alpha \boldsymbol{x}^\alpha \in \mathbb{C}[\boldsymbol{x}]$.
- Monomial $\rightarrow \boldsymbol{x}^\alpha$, for $\alpha \in \mathbb{N}^n$.

Gröbner basics

- $\mathbb{C}[\boldsymbol{x}] = \mathbb{C}[x_1, \ldots, x_n]$, polynomial ring in n indeterminates over $\mathbb{C}$.
- Polynomial $\rightarrow \sum_\alpha c_\alpha \boldsymbol{x}^\alpha \in \mathbb{C}[\boldsymbol{x}]$.
- Monomial $\rightarrow \boldsymbol{x}^\alpha$, for $\alpha \in \mathbb{N}^n$.

- Gröbner basis (GB) of $\langle f_1, \ldots, f_r \rangle := \{\sum_{i=1}^r h_i f_i : h_i \in \mathbb{C}[\boldsymbol{x}]\}$
  $\rightarrow$ set of generators $(g_1, \ldots, g_s)$ with special properties.

$$\langle f_1, \ldots, f_r \rangle = \langle g_1, \ldots, g_s \rangle$$

Gröbner basics

- $\mathbb{C}[\boldsymbol{x}] = \mathbb{C}[x_1, \ldots, x_n]$, polynomial ring in n indeterminates over $\mathbb{C}$.
- Polynomial $\rightarrow \sum_\alpha c_\alpha \boldsymbol{x}^\alpha \in \mathbb{C}[\boldsymbol{x}]$.
- Monomial $\rightarrow \boldsymbol{x}^\alpha$, for $\alpha \in \mathbb{N}^n$.

- Gröbner basis (GB) of $\langle f_1, \ldots, f_r \rangle := \{\sum_{i=1}^r h_i f_i : h_i \in \mathbb{C}[\boldsymbol{x}]\}$
  $\rightarrow$ set of generators $(g_1, \ldots, g_s)$ with special properties.

- Generalize Row Echelon Form for linear systems.

Gröbner basics

- $\mathbb{C}[\boldsymbol{x}] = \mathbb{C}[x_1, \ldots, x_n]$, polynomial ring in n indeterminates over $\mathbb{C}$.
- Polynomial $\rightarrow \sum_\alpha c_\alpha \boldsymbol{x}^\alpha \in \mathbb{C}[\boldsymbol{x}]$.
- Monomial $\rightarrow \boldsymbol{x}^\alpha$, for $\alpha \in \mathbb{N}^n$.

- Gröbner basis (GB) of $\langle f_1, \ldots, f_r \rangle := \{\sum_{i=1}^r h_i f_i : h_i \in \mathbb{C}[\boldsymbol{x}]\}$
  $\rightarrow$ set of generators $(g_1, \ldots, g_s)$ with special properties.

- Generalize Row Echelon Form for linear systems.

- We can use it to compute
  - Ideal membership: Given h', there exists $h_1, \ldots, h_r$ such $h' = \sum_i h_i f_i$?

Gröbner basics

- $\mathbb{C}[\boldsymbol{x}] = \mathbb{C}[x_1, \ldots, x_n]$, polynomial ring in n indeterminates over $\mathbb{C}$.
- Polynomial $\rightarrow \sum_\alpha c_\alpha \boldsymbol{x}^\alpha \in \mathbb{C}[\boldsymbol{x}]$.
- Monomial $\rightarrow \boldsymbol{x}^\alpha$, for $\alpha \in \mathbb{N}^n$.

- Gröbner basis (GB) of $\langle f_1, \ldots, f_r \rangle := \{\sum_{i=1}^r h_i f_i : h_i \in \mathbb{C}[\boldsymbol{x}]\}$
  $\rightarrow$ set of generators $(g_1, \ldots, g_s)$ with special properties.

- Generalize Row Echelon Form for linear systems.

- We can use it to compute
  - Ideal membership: Given h', there exists $h_1, \ldots, h_r$ such $h' = \sum_i h_i f_i$?
  - Certify no common solutions over $\mathbb{C}$: $1 = \sum_i h_i f_i$?

Gröbner basics

- $\mathbb{C}[\boldsymbol{x}] = \mathbb{C}[x_1, \ldots, x_n]$, polynomial ring in n indeterminates over $\mathbb{C}$.
- Polynomial $\rightarrow \sum_\alpha c_\alpha \boldsymbol{x}^\alpha \in \mathbb{C}[\boldsymbol{x}]$.
- Monomial $\rightarrow \boldsymbol{x}^\alpha$, for $\alpha \in \mathbb{N}^n$.

- Gröbner basis (GB) of $\langle f_1, \ldots, f_r \rangle := \{\sum_{i=1}^r h_i f_i : h_i \in \mathbb{C}[\boldsymbol{x}]\}$
  $\rightarrow$ set of generators $(g_1, \ldots, g_s)$ with special properties.

- Generalize Row Echelon Form for linear systems.

- We can use it to compute
  - Ideal membership: Given h', there exists $h_1, \ldots, h_r$ such $h' = \sum_i h_i f_i$?
  - Certify no common solutions over $\mathbb{C}$: $1 = \sum_i h_i f_i$?
  - Solve: $\{p \in \mathbb{C}^n : f_1(p) = \cdots = f_r(p) = 0\}$.

Gröbner basics

- $\mathbb{C}[\boldsymbol{x}] = \mathbb{C}[x_1, \ldots, x_n]$, polynomial ring in n indeterminates over $\mathbb{C}$.
- Polynomial $\rightarrow \sum_\alpha c_\alpha \boldsymbol{x}^\alpha \in \mathbb{C}[\boldsymbol{x}]$.
- Monomial $\rightarrow \boldsymbol{x}^\alpha$, for $\alpha \in \mathbb{N}^n$.

- Gröbner basis (GB) of $\langle f_1, \ldots, f_r \rangle := \{\sum_{i=1}^r h_i f_i : h_i \in \mathbb{C}[\boldsymbol{x}]\}$
  $\rightarrow$ set of generators $(g_1, \ldots, g_s)$ with special properties.

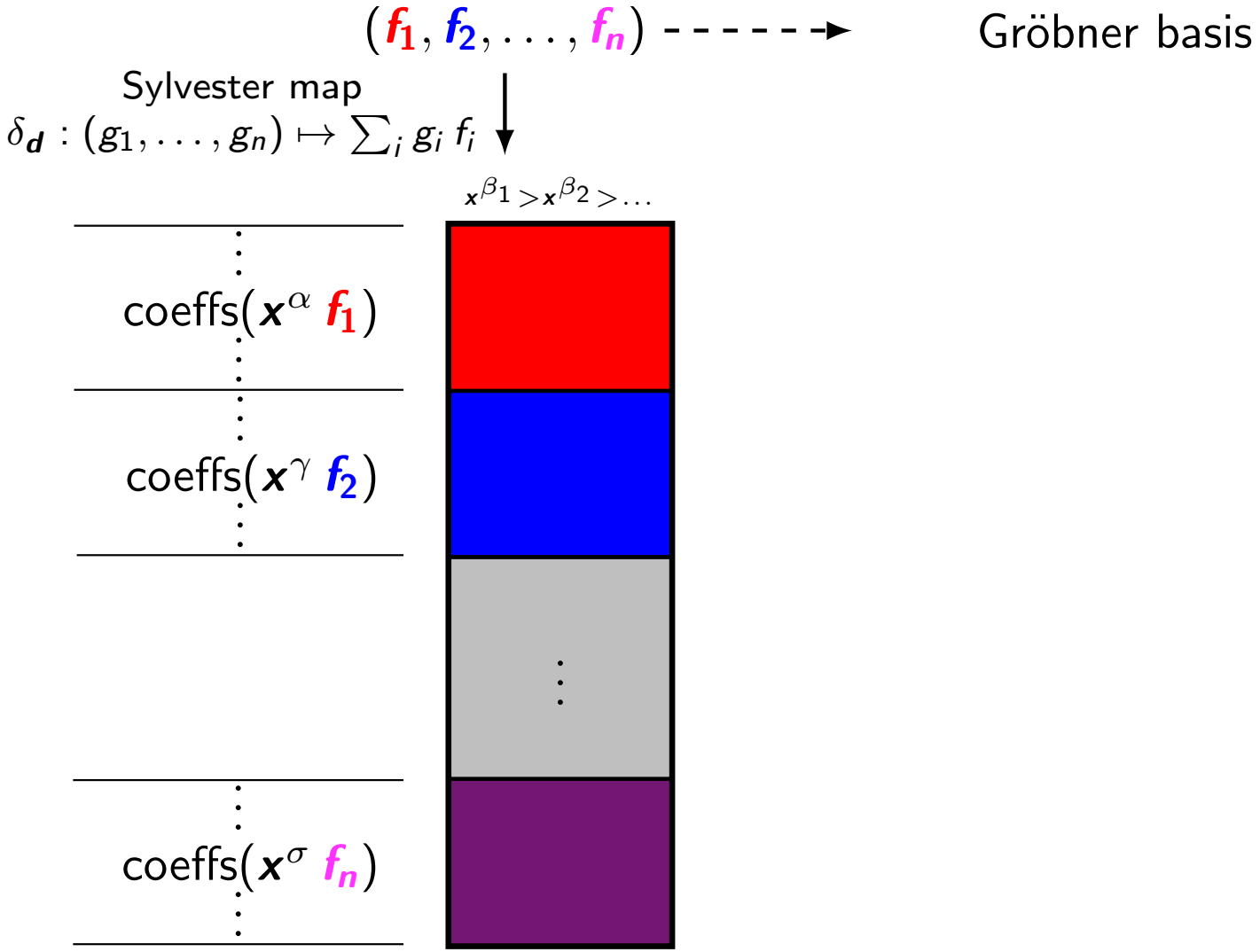
- Generalize Row Echelon Form for linear systems.

- We can use it to compute
  - Ideal membership: Given h', there exists $h_1, \ldots, h_r$ such $h' = \sum_i h_i f_i$?
  - Certify no common solutions over $\mathbb{C}$: $1 = \sum_i h_i f_i$?
  - Solve: $\{p \in \mathbb{C}^n : f_1(p) = \cdots = f_r(p) = 0\}$.
  - Compute other algebraic and geometric invariants.

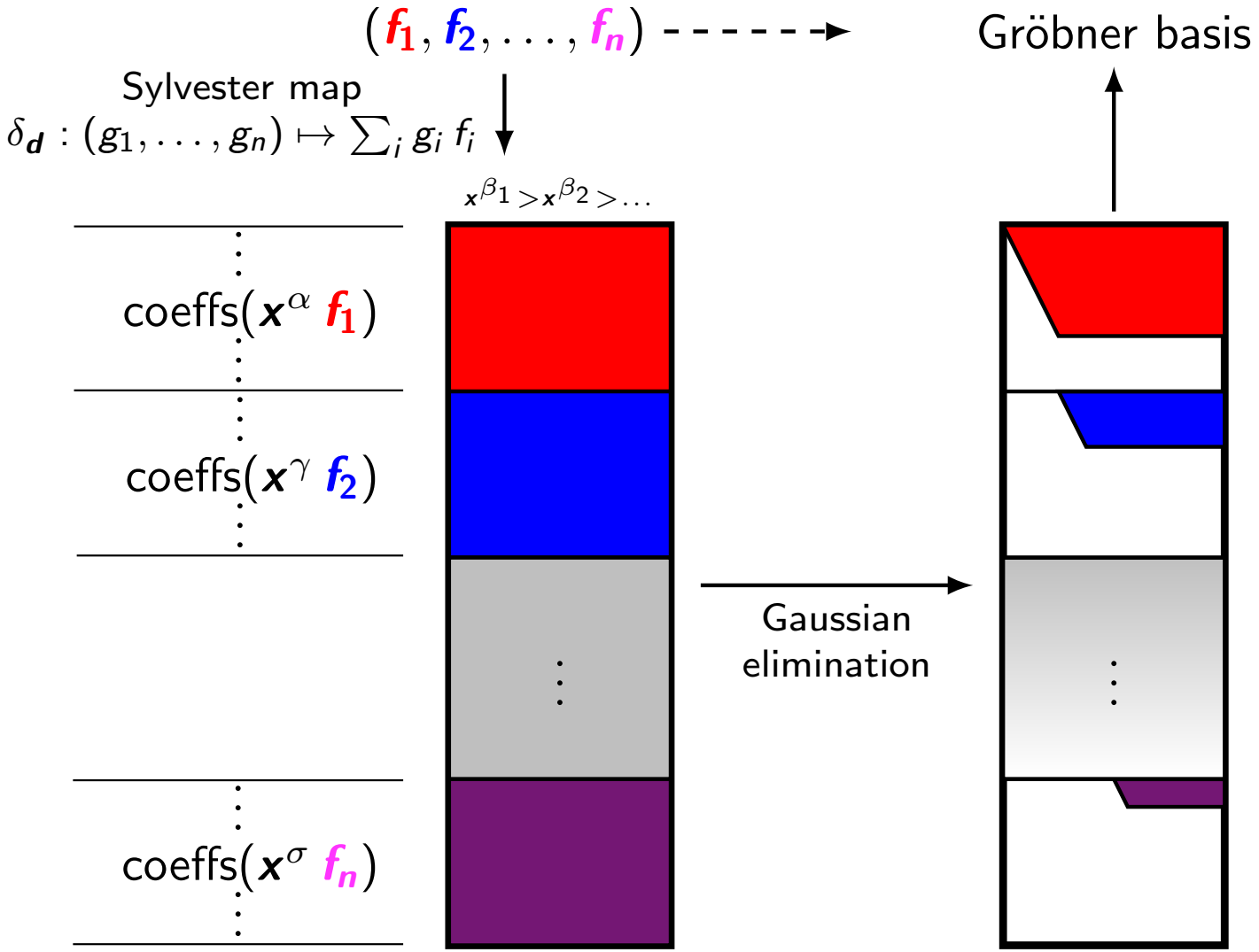
Computing Gröbner bases - [Lazard '83]

$(f_1, f_2, \ldots, f_n)$ - - - - - -► Gröbner basis

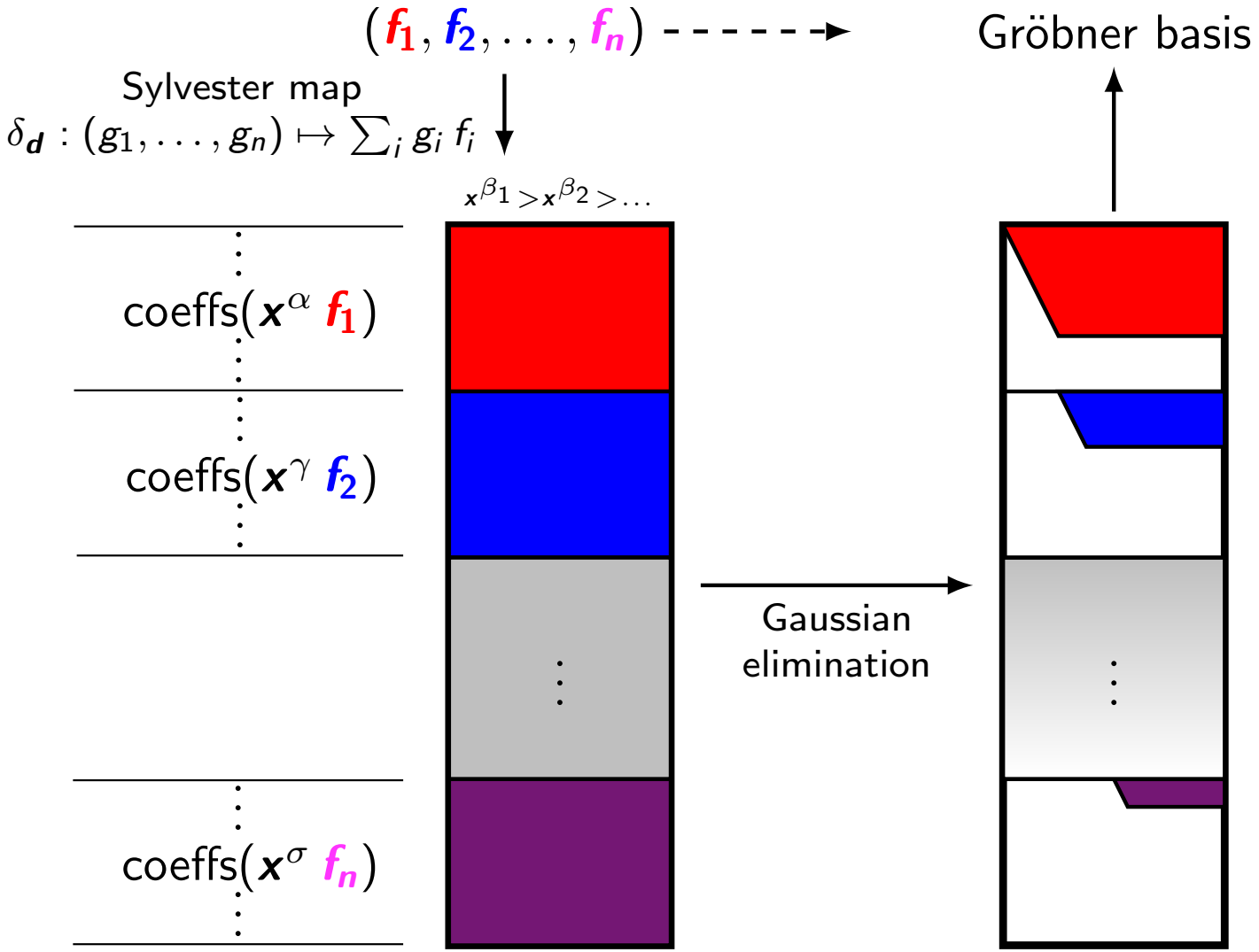
Computing Gröbner bases - [Lazard '83]



Computing Gröbner bases - [Lazard '83]

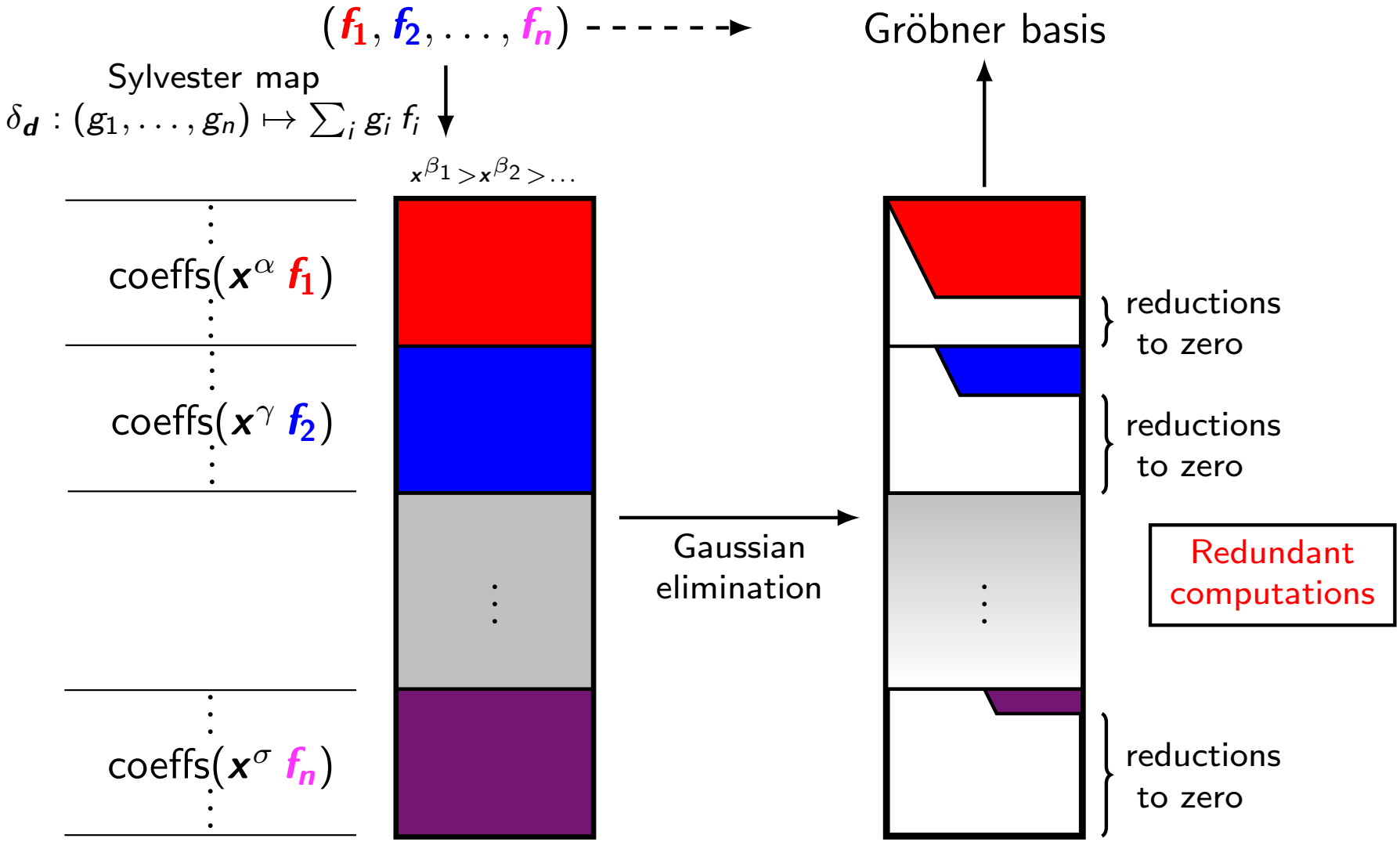


Computing Gröbner bases - [Lazard '83]



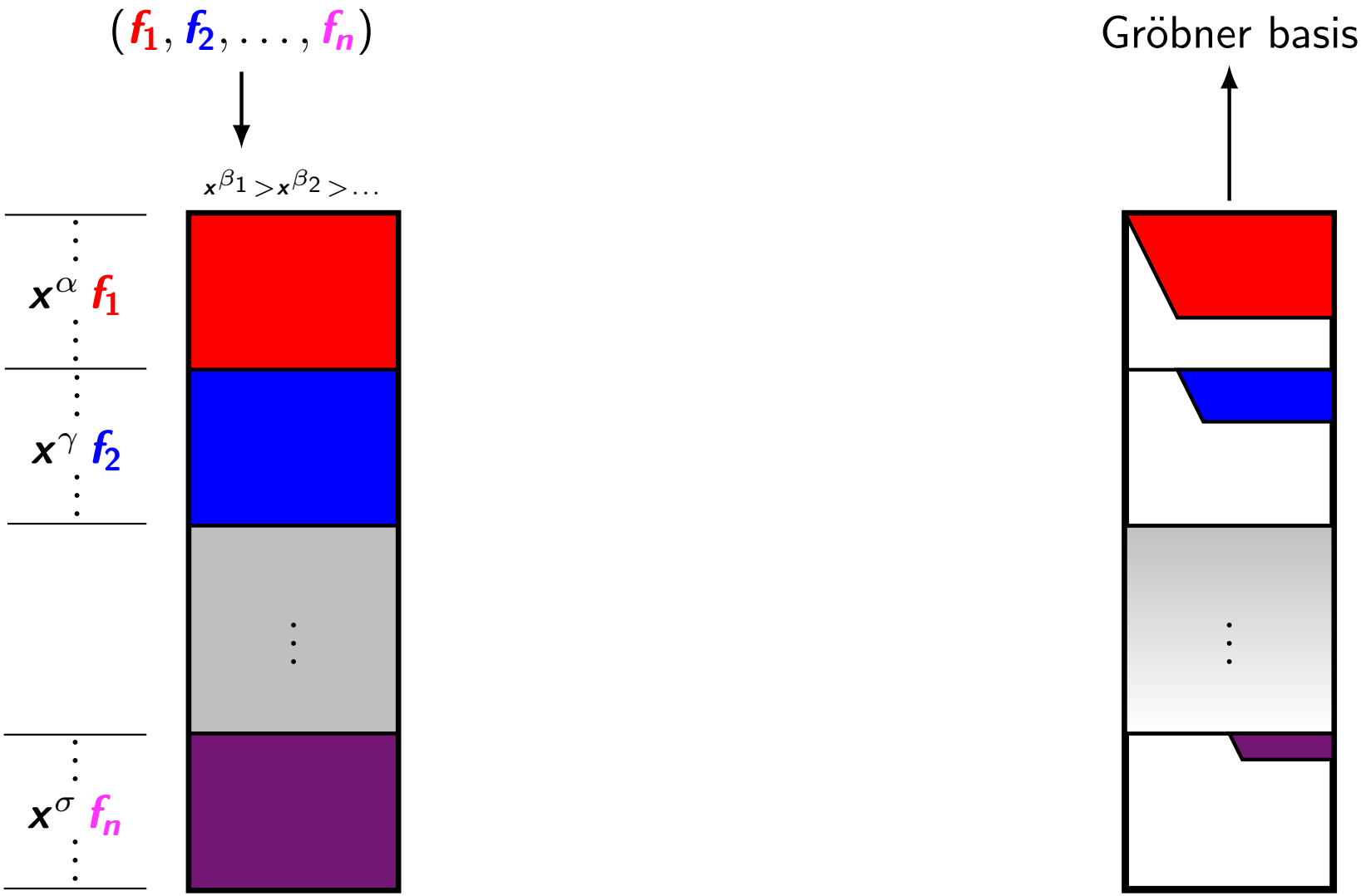
Complexity $\leftrightarrow$ Size of matrix $\rightarrow$ Degree d depends on the system $(f_1, \ldots, f_n)$

Computing Gröbner bases - [Lazard '83]



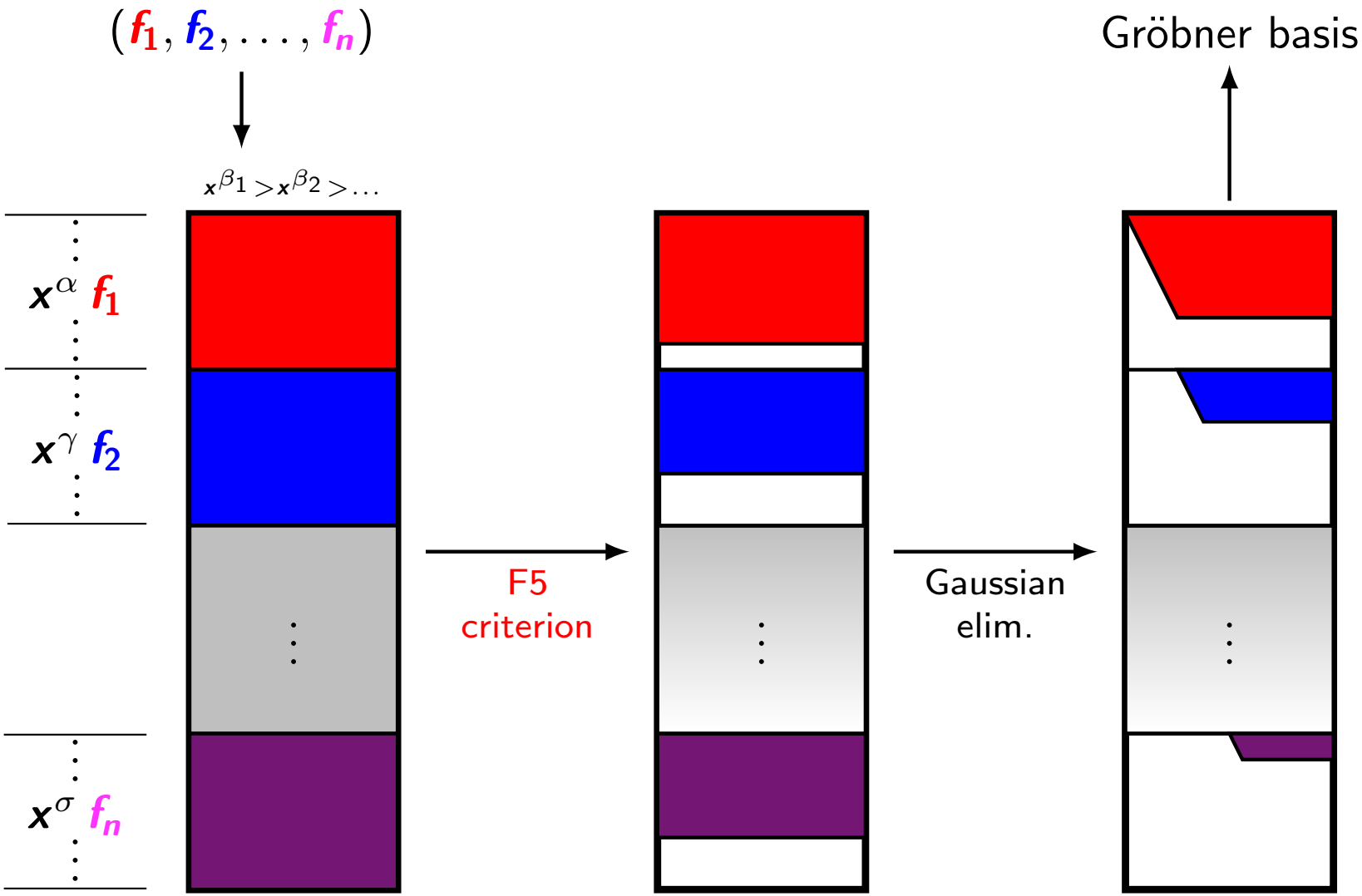
Complexity $\leftrightarrow$ Size of matrix $\rightarrow$ Degree d depends on the system $(f_1, \ldots, f_n)$

Computing Gröbner bases - [Lazard '83] + [Faugère '02]



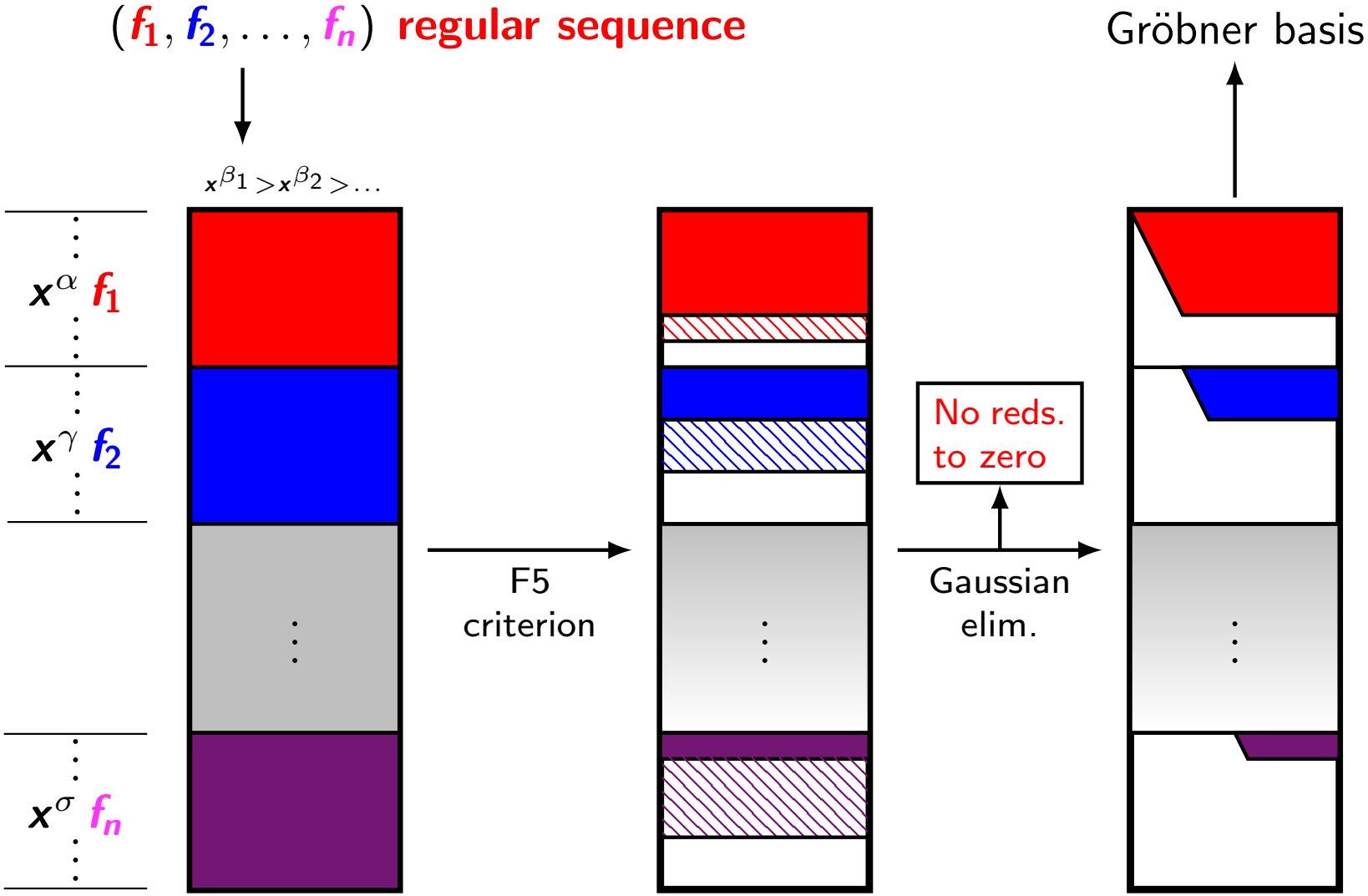
Complexity $\leftrightarrow$ Size of matrix $\rightarrow$ Degree d depends on the system $(f_1, \ldots, f_n)$

Computing Gröbner bases - [Lazard '83] + [Faugère '02]



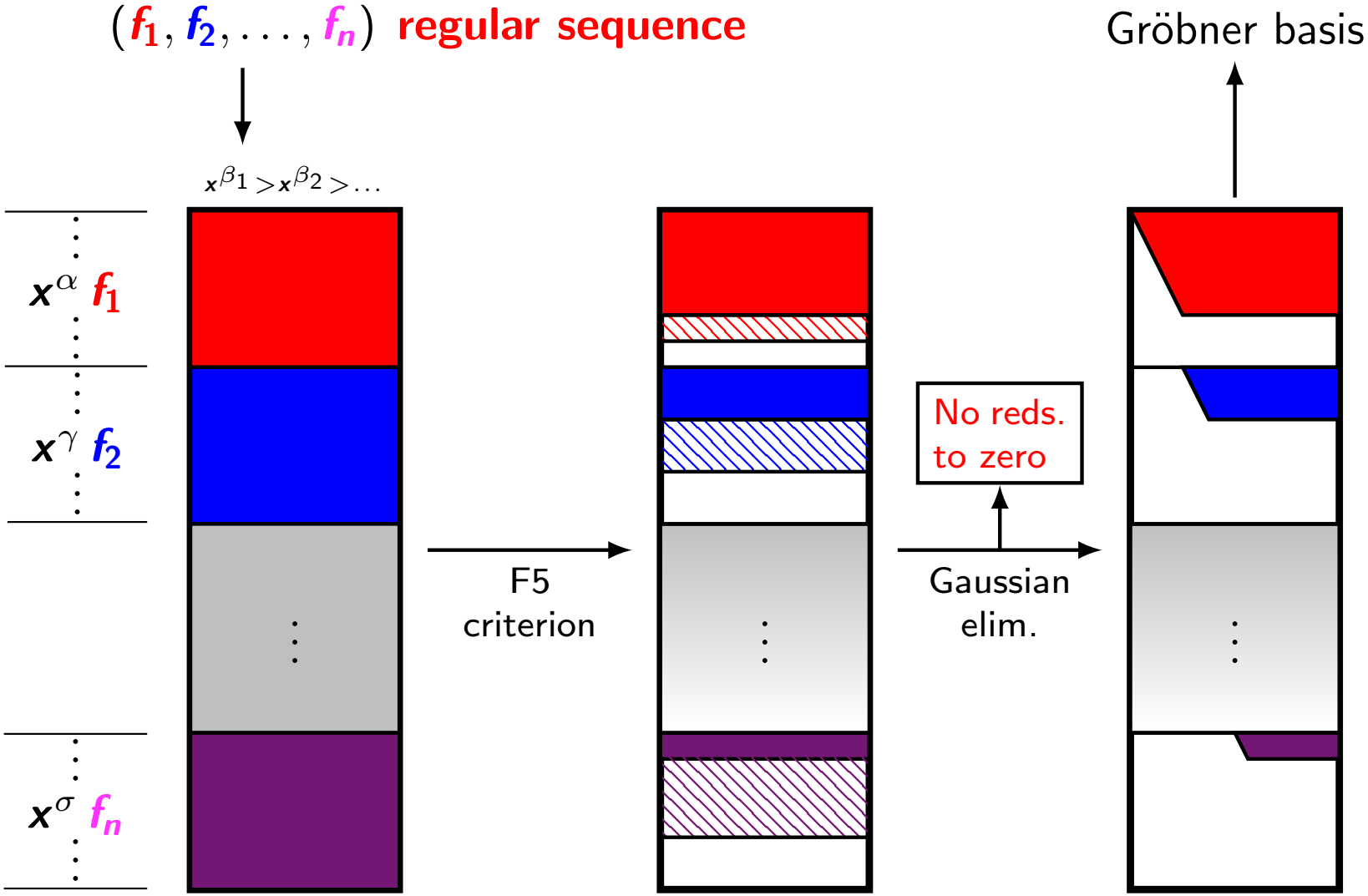
Complexity $\leftrightarrow$ Size of matrix $\rightarrow$ Degree d depends on the system $(f_1, \ldots, f_n)$

Computing Gröbner bases - [Lazard '83] + [Faugère '02]



Complexity $\leftrightarrow$ Size of matrix $\rightarrow$ Degree d depends on the system $(f_1, \ldots, f_n)$

Computing Gröbner bases - [Lazard '83] + [Faugère '02]



Complexity $\leftrightarrow$ Size of matrix $\rightarrow$ Degree d depends on the system $(f_1, \ldots, f_n)$
$\rightarrow$ Macaulay bound: $d \leq \sum_i \deg(f_i) - n + 1$

The resultant and Sylvester-type formulas

Projective resultant

Necessary and sufficient condition for a homogeneous system in $(f_0, \ldots, f_n) \in \mathbb{C}[x_0, \ldots, x_n]^{n+1}$ to have solutions in $\mathbb{P}^n$.

The resultant and Sylvester-type formulas

Projective resultant

Necessary and sufficient condition for a homogeneous system in $(f_0, \ldots, f_n) \in \mathbb{C}[x_0, \ldots, x_n]^{n+1}$ to have solutions in $\mathbb{P}^n$.

Example : Resultant of linear forms = Determinant

$$\begin{cases} a_1\, x + a_2\, y + a_3\, z = 0 \\ b_1\, x + b_2\, y + b_3\, z = 0 \\ c_1\, x + c_2\, y + c_3\, z = 0 \end{cases} \quad \text{has a solution over } \mathbb{P}^2$$

$$\Updownarrow$$

$$\det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = 0.$$

The resultant and Sylvester-type formulas

Projective resultant

Necessary and sufficient condition for a homogeneous system in $(f_0, \ldots, f_n) \in \mathbb{C}[x_0, \ldots, x_n]^{n+1}$ to have solutions in $\mathbb{P}^n$.

Classical way of computing resultant $\rightarrow$ Sylvester-type formula

$$(g_0, \ldots, g_n) \mapsto \sum_{i=0}^{n} g_i \, f_i$$

The resultant and Sylvester-type formulas

Projective resultant

Necessary and sufficient condition for a homogeneous system in $(f_0, \ldots, f_n) \in \mathbb{C}[x_0, \ldots, x_n]^{n+1}$ to have solutions in $\mathbb{P}^n$.

Classical way of computing resultant $\rightarrow$ Sylvester-type formula

$$(g_0, \ldots, g_n) \mapsto \sum g_i \, f_i$$

Macaulay resultant matrix [Macaulay, 1916]

$$\begin{cases} f_1 := & a_1 x^2 + a_2 x y + a_3 x z + \\ & a_4 y^2 + a_5 y z + a_6 z^2 \\ f_2 := & b_1 x + b_2 y + b_3 z \\ f_3 := & c_1 x + c_2 y + c_3 z \end{cases}$$

	x^2	$x\,y$	$x\,z$	y^2	$y\,z$	z^2
f_1	a_1	a_2	a_3	a_4	a_5	a_6
$x\,f_2$	b_1	b_2	b_3			
$y\,f_2$		b_1		b_2	b_3	
$z\,f_2$			b_1		b_2	b_2
$y\,f_3$		c_1		c_2	c_3	
$z\,f_3$			c_1		c_2	c_3

$$\texttt{Determinant} = \texttt{Resultant} \cdot \texttt{ExtraFactor}.$$

The resultant and Sylvester-type formulas

Projective resultant

Necessary and sufficient condition for a homogeneous system in $(f_0, \ldots, f_n) \in \mathbb{C}[x_0, \ldots, x_n]^{n+1}$ to have solutions in $\mathbb{P}^n$.

Classical way of computing resultant $\rightarrow$ Sylvester-type formula

$$(g_0, \ldots, g_n) \mapsto \sum g_i \, f_i$$

Macaulay resultant matrix [Macaulay, 1916]

$$\begin{cases} f_1 := & a_1 x^2 + a_2 x y + a_3 x z + \\ & a_4 y^2 + a_5 y z + a_6 z^2 \\ f_2 := & b_1 x + b_2 y + b_3 z \\ f_3 := & c_1 x + c_2 y + c_3 z \end{cases}$$

	x^2	$x\,y$	$x\,z$	y^2	$y\,z$	z^2
f_1	a_1	a_2	a_3	a_4	a_5	a_6
$x\,f_2$	b_1	b_2	b_3			
$y\,f_2$		b_1		b_2	b_3	
$z\,f_2$			b_1		b_2	b_2
$y\,f_3$		c_1		c_2	c_3	
$z\,f_3$			c_1		c_2	c_3

$$\texttt{Determinant} = \texttt{Resultant} \cdot \texttt{ExtraFactor}.$$

Determinantal formula $\rightarrow$ `ExtraFactor` is a constant.

Solving via Sylvester-type formulas

- We want to compute the two solutions $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{C}^2$ of (f_1, f_2)

$$\begin{cases} f_1 := & \mathbf{1}\, x^2 + \mathbf{-1}\, x\, y + \mathbf{4}\, x + \mathbf{-2}\, y^2 + \mathbf{-5}\, y + \mathbf{3} \\ f_2 := & \mathbf{1}\, x + \mathbf{-1}\, y + \mathbf{-1} \end{cases} .$$

Solving via Sylvester-type formulas

- We want to compute the two solutions $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{C}^2$ of (f_1, f_2)

$$\begin{cases} f_1 := & \mathbf{1}\,x^2 + \mathbf{-1}\,x\,y + \mathbf{4}\,x + \mathbf{-2}\,y^2 + \mathbf{-5}\,y + \mathbf{3} \\ f_2 := & \mathbf{1}\,x + \mathbf{-1}\,y + \mathbf{-1} \end{cases}.$$

- Introduce $f_3 := -1\,x + 2\,y + 1$ and consider a Sylvester-type formula.

$$\left(\begin{array}{c|c} M_{1,1} & M_{1,2} \\ \hline M_{2,1} & M_{2,2} \end{array}\right) =$$

	x^2	$x\,y$	x	y^2	y	1
f_1	**1**	**−1**	**4**	**−2**	**−5**	**3**
$x\,f_2$	**1**	**−1**	**−1**			
$y\,f_2$		**1**		**−1**	**−1**	
f_2			**1**		**−1**	**−1**
$y\,f_3$		−1		2	1	
f_3			−1		2	1

Solving via Sylvester-type formulas

- We want to compute the two solutions $\alpha, \beta \in \mathbb{C}^2$ of (f_1, f_2)

$$\begin{cases} f_1 := & 1\,x^2 + -1\,x\,y + 4\,x + -2\,y^2 + -5\,y + 3 \\ f_2 := & 1\,x + -1\,y + -1 \end{cases}.$$

- Introduce $f_3 := -1\,x + 2\,y + 1$ and consider a Sylvester-type formula.

$$\left(\begin{array}{c|c} M_{1,1} & M_{1,2} \\ \hline M_{2,1} & M_{2,2} \end{array}\right) =$$

	x^2	$x\,y$	x	y^2	y	1
f_1	1	−1	4	−2	−5	3
$x\,f_2$	1	−1	−1			
$y\,f_2$		1		−1	−1	
f_2			1		−1	−1
$y\,f_3$		−1		2	1	
f_3			−1		2	1

- Schur complement of $M_{2,2} \leftrightarrow$ Multiplication map of f_3 in $\mathbb{C}[x,y]/\langle f_1, f_2\rangle$

$$\widetilde{M}_{2,2} = M_{2,2} - M_{2,1}\,M_{1,1}^{-1}\,M_{1,2} = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$$

Solving via Sylvester-type formulas

- We want to compute the two solutions $\alpha, \beta \in \mathbb{C}^2$ of (f_1, f_2)

$$\begin{cases} f_1 := & 1\,x^2 + -1\,x\,y + 4\,x + -2\,y^2 + -5\,y + 3 \\ f_2 := & 1\,x + -1\,y + -1 \end{cases}.$$

- Introduce $f_3 := -1\,x + 2\,y + 1$ and consider a Sylvester-type formula.

$$\left(\begin{array}{c|c} M_{1,1} & M_{1,2} \\ \hline M_{2,1} & M_{2,2} \end{array}\right) = \begin{array}{r} f_1 \\ x\,f_2 \\ y\,f_2 \\ f_2 \\ \hline y\,f_3 \\ f_3 \end{array} \left(\begin{array}{cccc|cc} 1 & -1 & 4 & -2 & -5 & 3 \\ 1 & -1 & -1 & & & \\ & 1 & & -1 & -1 & \\ & & 1 & & -1 & -1 \\ \hline & -1 & & 2 & 1 & \\ & & -1 & & 2 & 1 \end{array}\right)$$

- Schur complement of $M_{2,2} \leftrightarrow$ Multiplication map of f_3 in $\mathbb{C}[x,y]/\langle f_1, f_2\rangle$

$$\widetilde{M}_{2,2} = M_{2,2} - M_{2,1}\,M_{1,1}^{-1}\,M_{1,2} = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$$

Solving via Sylvester-type formulas

- We want to compute the two solutions $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{C}^2$ of (f_1, f_2)

$$\begin{cases} f_1 := & \mathbf{1}\,x^2 + \mathbf{-1}\,x\,y + \mathbf{4}\,x + \mathbf{-2}\,y^2 + \mathbf{-5}\,y + \mathbf{3} \\ f_2 := & \mathbf{1}\,x + \mathbf{-1}\,y + \mathbf{-1} \end{cases}.$$

- Introduce $f_3 := \mathbf{-1}\,x + \mathbf{2}\,y + \mathbf{1}$ and consider a Sylvester-type formula.

$$\left(\begin{array}{c|c} M_{1,1} & M_{1,2} \\ \hline M_{2,1} & M_{2,2} \end{array}\right) = \begin{array}{r} f_1 \\ x\,f_2 \\ y\,f_2 \\ f_2 \\ \hline y\,f_3 \\ f_3 \end{array} \left(\begin{array}{cccc|cc} \mathbf{1} & \mathbf{-1} & \mathbf{4} & \mathbf{-2} & \mathbf{-5} & \mathbf{3} \\ \mathbf{1} & \mathbf{-1} & \mathbf{-1} & & & \\ & \mathbf{1} & & \mathbf{-1} & \mathbf{-1} & \\ & & \mathbf{1} & & \mathbf{-1} & \mathbf{-1} \\ \hline & \mathbf{-1} & & \mathbf{2} & \mathbf{1} & \\ & & \mathbf{-1} & & \mathbf{2} & \mathbf{1} \end{array}\right)$$

- Schur complement of $M_{2,2}$ $\leftrightarrow$ Multiplication map of f_3 in $\mathbb{C}[x,y]/\langle f_1, f_2\rangle$

$$\widetilde{M}_{2,2} = M_{2,2} - M_{2,1}\,M_{1,1}^{-1}\,M_{1,2} = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$$

- Eigenvalues of $\widetilde{M}_{2,2}$ [Lazard, 1981]

$$f_3(\boldsymbol{\alpha}) = 2 \qquad \text{and} \qquad f_3(\boldsymbol{\beta}) = -2.$$

Solving via Sylvester-type formulas

- We want to compute the two solutions $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{C}^2$ of (f_1, f_2)

$$\begin{cases} f_1 := & \mathbf{1}\, x^2 + \mathbf{-1}\, x\, y + \mathbf{4}\, x + \mathbf{-2}\, y^2 + \mathbf{-5}\, y + \mathbf{3} \\ f_2 := & \mathbf{1}\, x + \mathbf{-1}\, y + \mathbf{-1} \end{cases}.$$

- Introduce $f_3 := \mathbf{-1}\, x + \mathbf{2}\, y + \mathbf{1}$ and consider a Sylvester-type formula.

$$\left(\begin{array}{c|c} M_{1,1} & M_{1,2} \\ \hline M_{2,1} & M_{2,2} \end{array}\right) = \begin{array}{c} f_1 \\ x\, f_2 \\ y\, f_2 \\ f_2 \\ \hline y\, f_3 \\ f_3 \end{array} \left(\begin{array}{cccc|cc} \mathbf{1} & \mathbf{-1} & \mathbf{4} & \mathbf{-2} & \mathbf{-5} & \mathbf{3} \\ \mathbf{1} & \mathbf{-1} & \mathbf{-1} & & & \\ & \mathbf{1} & & \mathbf{-1} & \mathbf{-1} & \\ & & \mathbf{1} & & \mathbf{-1} & \mathbf{-1} \\ \hline & \mathbf{-1} & & \mathbf{2} & \mathbf{1} & \\ & & \mathbf{-1} & & \mathbf{2} & \mathbf{1} \end{array}\right)$$

- Schur complement of $M_{2,2} \leftrightarrow$ Multiplication map of f_3 in $\mathbb{C}[x, y]/\langle f_1, f_2 \rangle$

$$\widetilde{M}_{2,2} = M_{2,2} - M_{2,1}\, M_{1,1}^{-1}\, M_{1,2} = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$$

- Eigenvalues of $\widetilde{M}_{2,2} \leftrightarrow f_3(\boldsymbol{\alpha}) = 2$ and $f_3(\boldsymbol{\beta}) = -2$. [Lazard, 1981]
- Eigenvectors of $\widetilde{M}_{2,2}$ [Auzinger & Stetter, 1988]

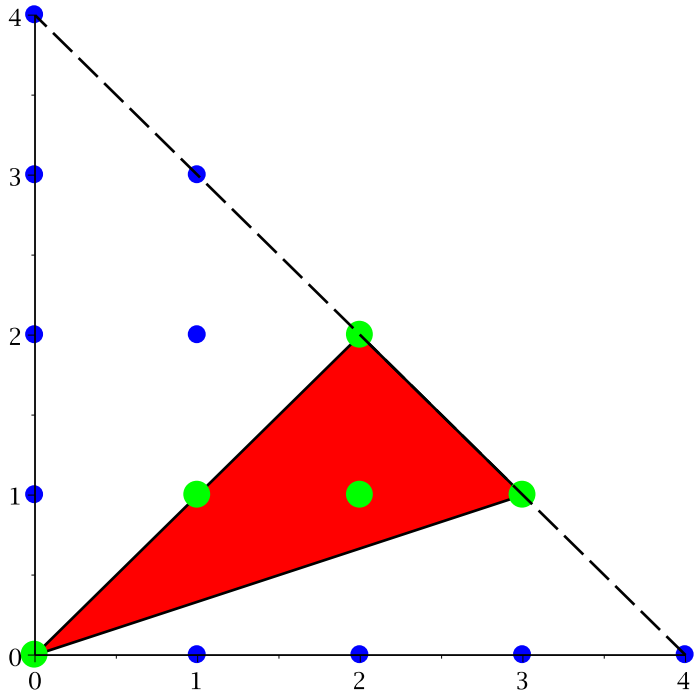
$$\begin{pmatrix} \alpha_y \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \beta_y \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Sparse polynomials / systems

Sparse polynomials / systems

- **Newton polytope** of $f = \sum_\alpha c_\alpha \boldsymbol{x}^\alpha \longrightarrow$ Convex hull of $\{\alpha : c_\alpha \neq 0\}$.
- Sparse polynomial $\longrightarrow$ Its Newton polytope is "small".

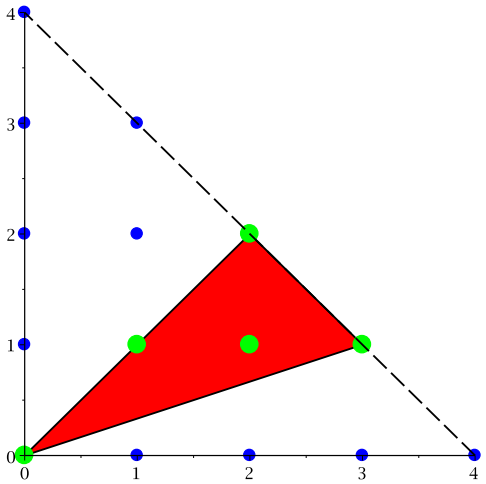
$$1 + xy + x^2y + x^2y^2 + x^3y = \mathbf{1} + \mathbf{0} \cdot x + \mathbf{0} \cdot y + \mathbf{0} \cdot x^2 + \boldsymbol{xy} + \mathbf{0} \cdot y^2 + \mathbf{0} \cdot x^3 + \boldsymbol{x^2y} + \mathbf{0} \cdot xy^2 + \mathbf{0} \cdot y^3 + \mathbf{0} \cdot x^4 + \boldsymbol{x^3y} + \boldsymbol{x^2y^2} + \mathbf{0} \cdot xy^3 + \mathbf{0} \cdot y^4$$



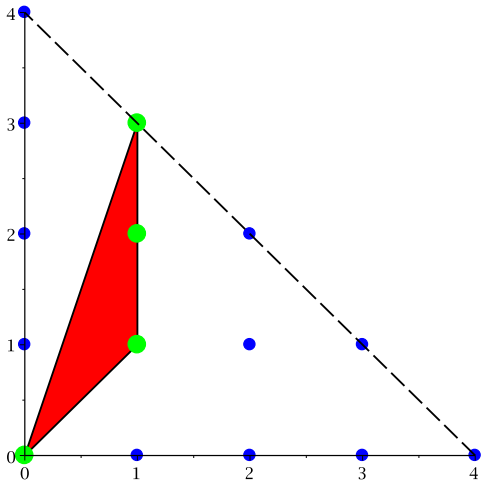
Sparse polynomials / systems

- **Newton polytope** of $f = \sum_\alpha c_\alpha \boldsymbol{x}^\alpha \longrightarrow$ Convex hull of $\{\alpha : c_\alpha \neq 0\}$.
- Sparse polynomial $\longrightarrow$ Its Newton polytope is "small".
- Sparse system $\longrightarrow$ system of sparse polynomials

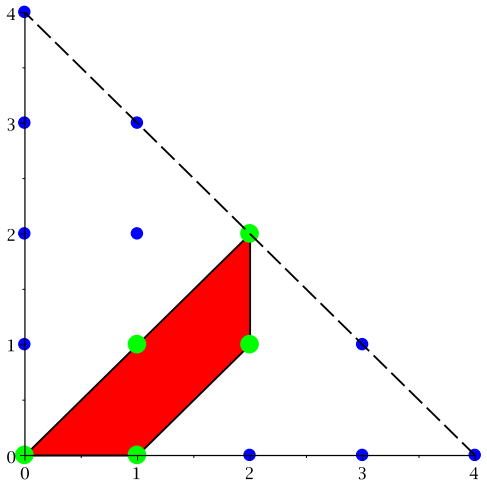
$f_1 := 1 + xy + x^2y + x^2y^2 + x^3y$



$f_2 := 1 + xy + xy^2 + xy^3$



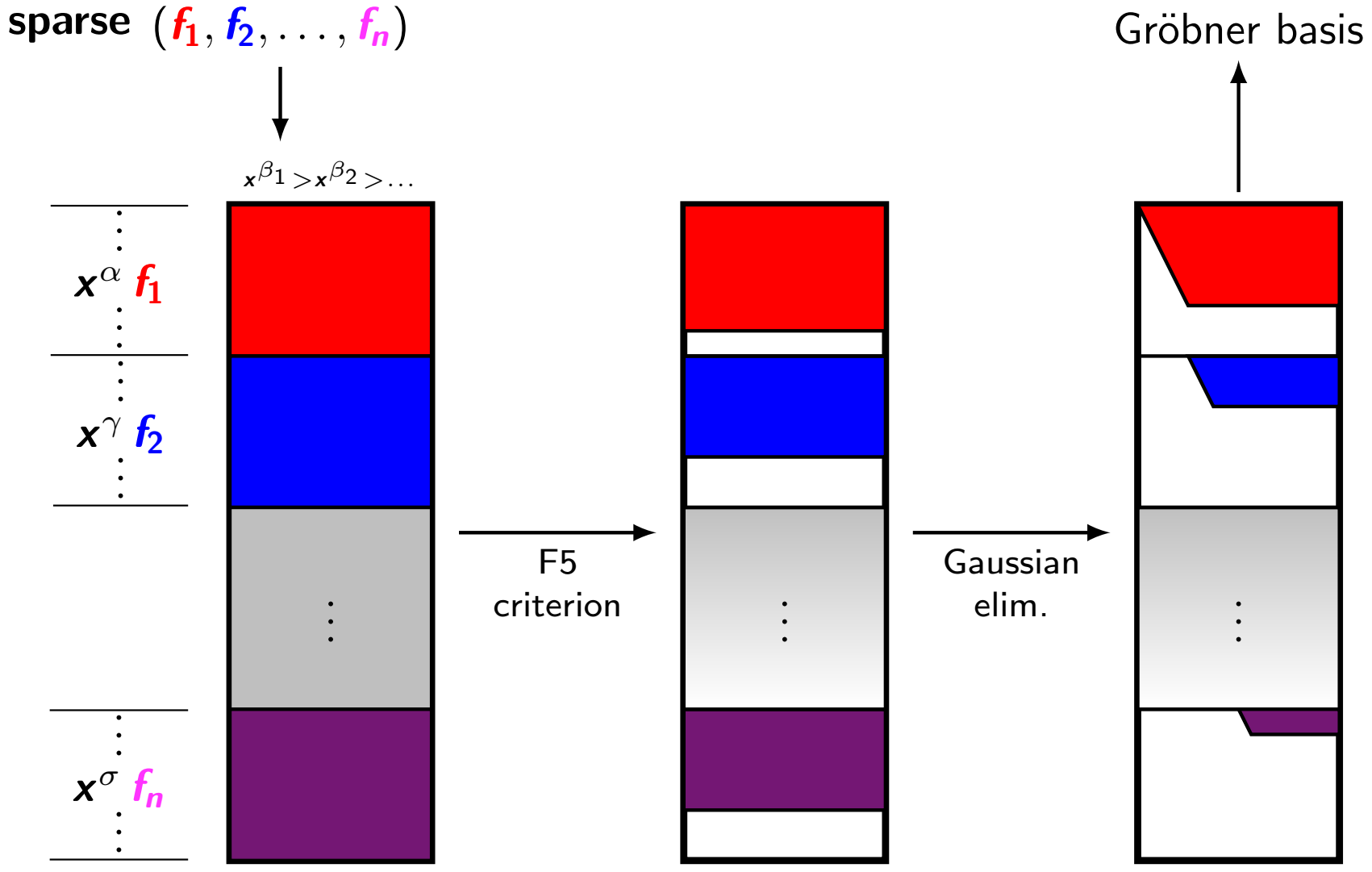
$f_3 := 1 + x + xy + x^2y + x^2y^2$



Gröbner bases for sparse systems

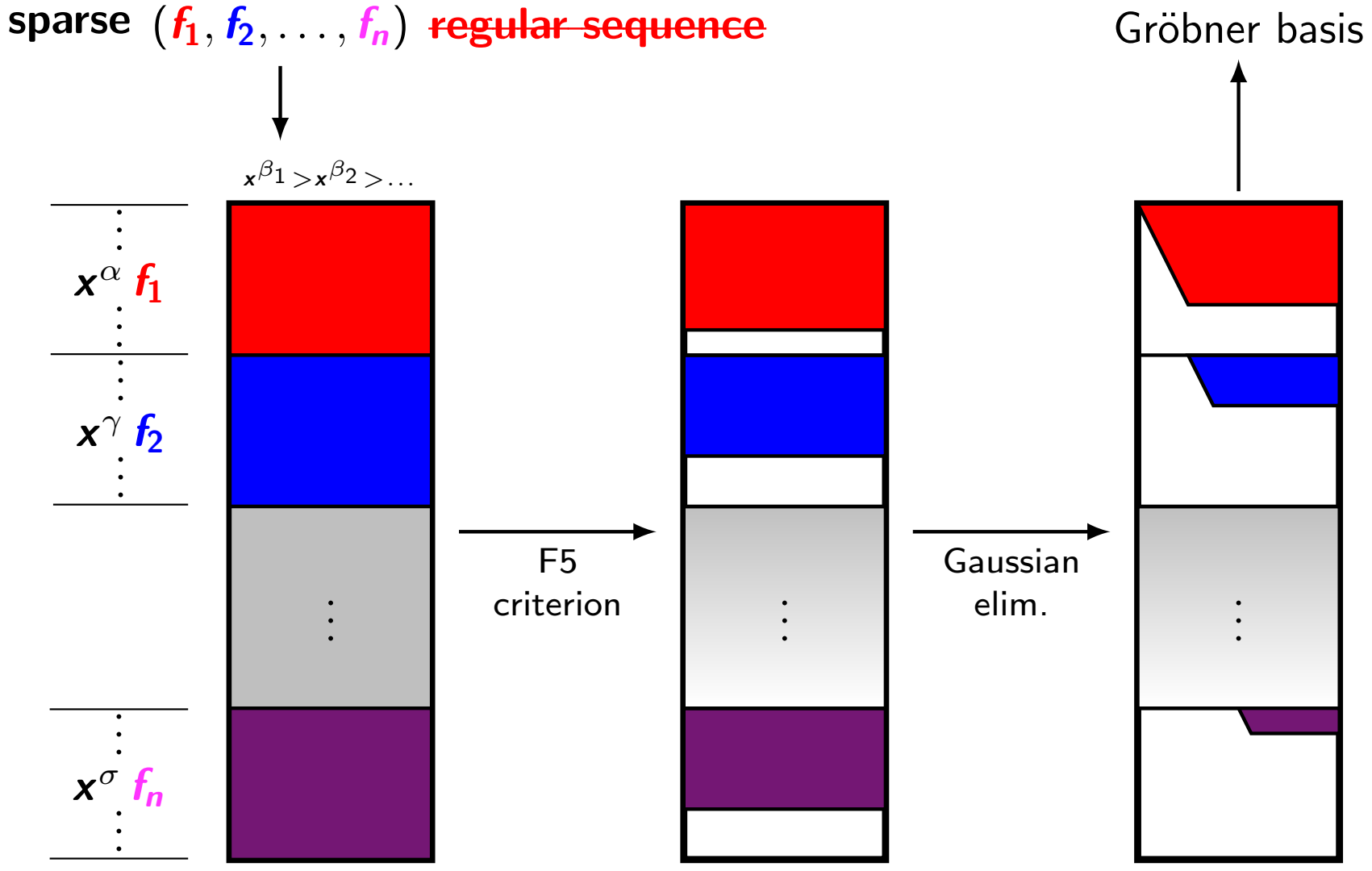
Joint work with Jean-Charles Faugère & Elias Tsigaridas.

Computing Gröbner bases for sparse systems



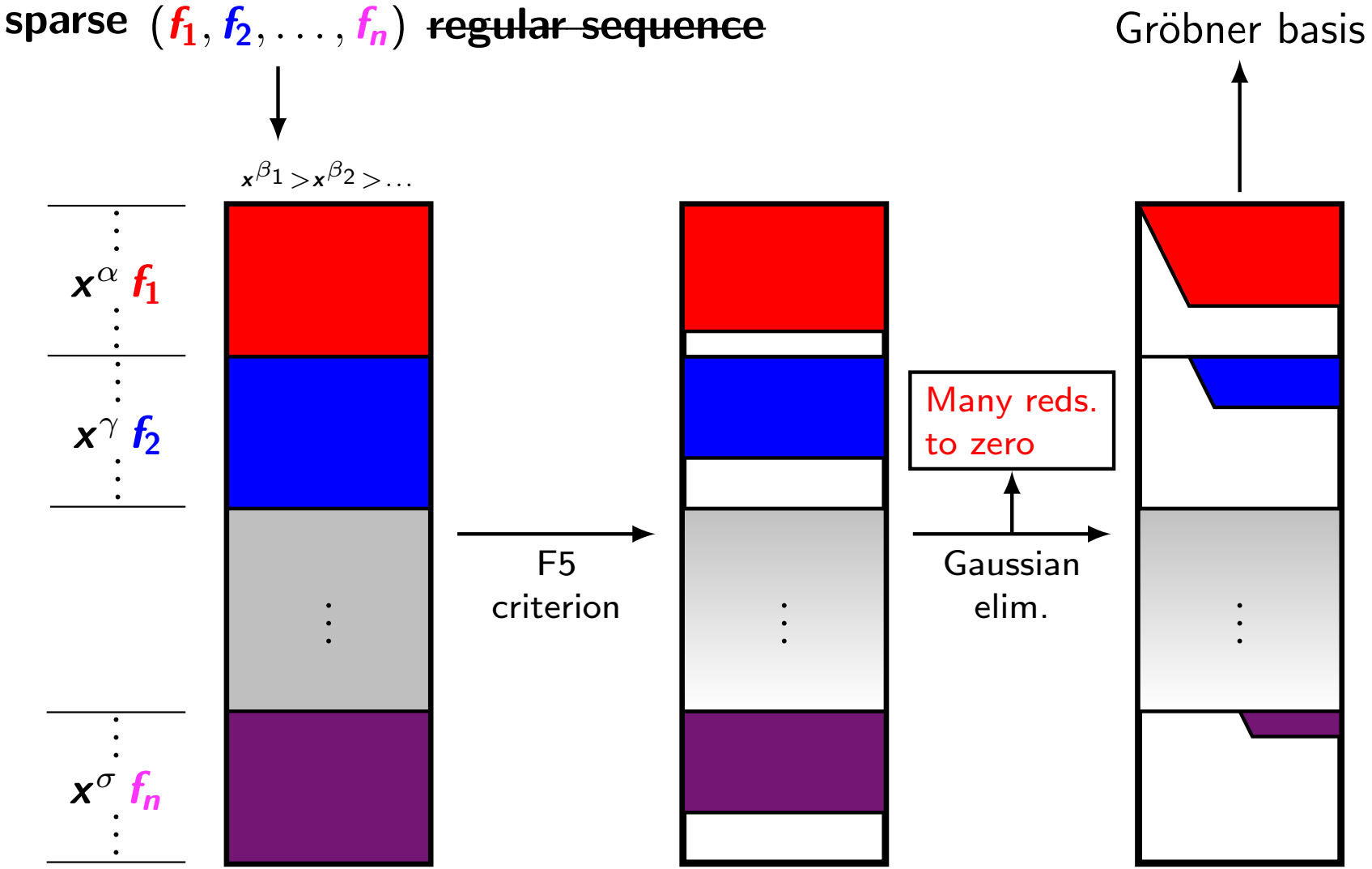
Complexity $\leftrightarrow$ Size of matrix

Computing Gröbner bases for sparse systems



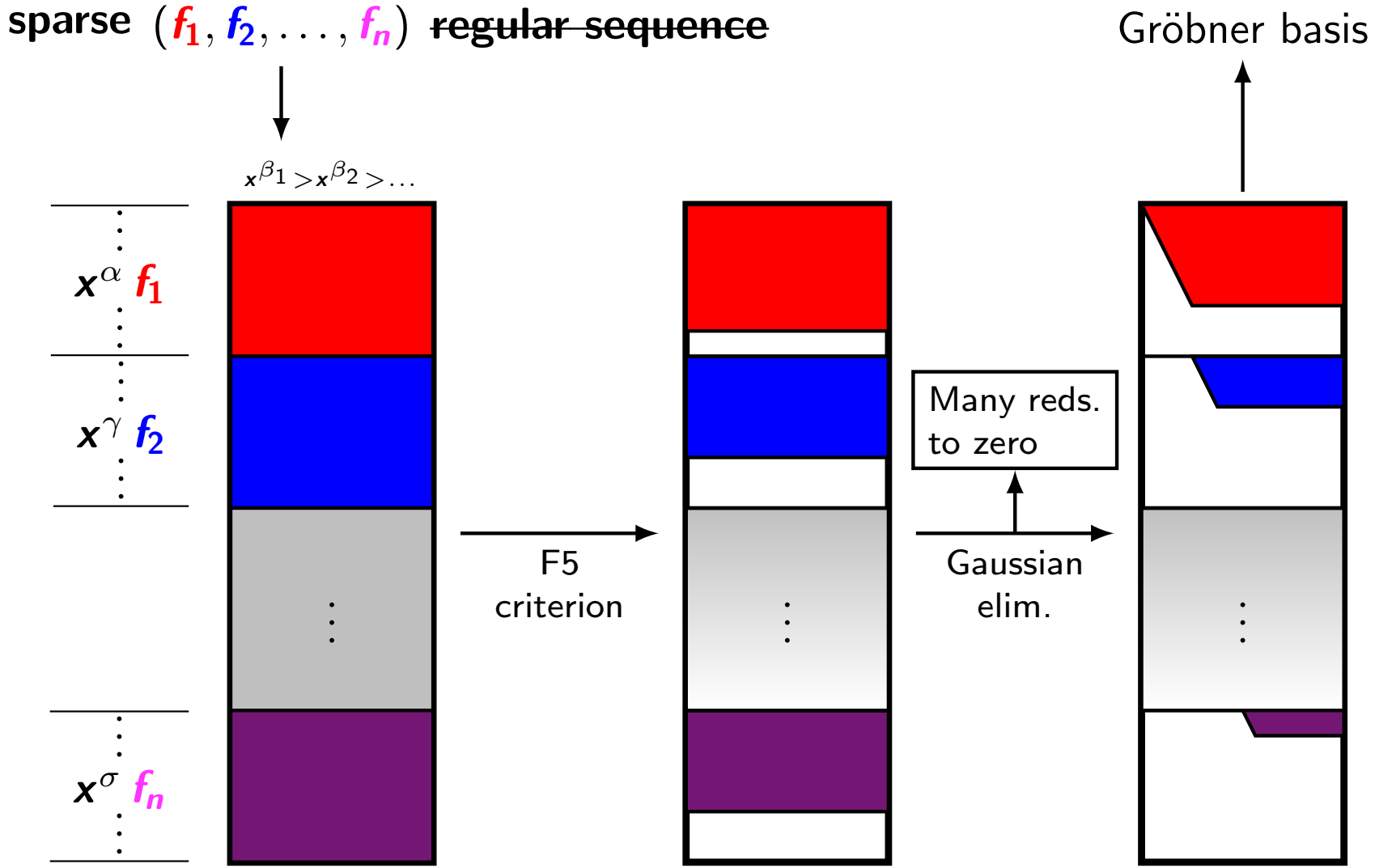
Complexity $\leftrightarrow$ Size of matrix

Computing Gröbner bases for sparse systems



Complexity $\leftrightarrow$ Size of matrix

Computing Gröbner bases for sparse systems



Complexity $\leftrightarrow$ Size of matrix $\rightarrow$ No Macaulay bound for d

Computing Gröbner bases for sparse systems

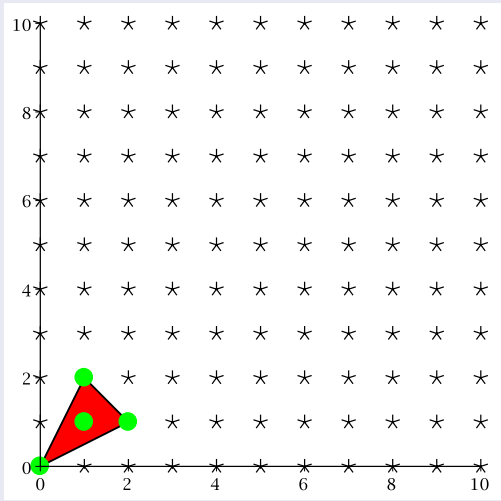
Given $f_1, \ldots, f_n \in \mathbb{C}[\boldsymbol{x}]$ and **different polytopes** $P_i = \texttt{NewtonPolytope}(f_i)$.

Computing Gröbner bases for sparse systems

Given $f_1, \ldots, f_n \in \mathbb{C}[\boldsymbol{x}]$ and **different polytopes** $P_i = \texttt{NewtonPolytope}(f_i)$.

How to exploit sparsity? [Gelfand, Kapranov, & Zelevinsky '90]

- Sparse resultant $\rightarrow$ Generalization of the resultant for sparse systems.



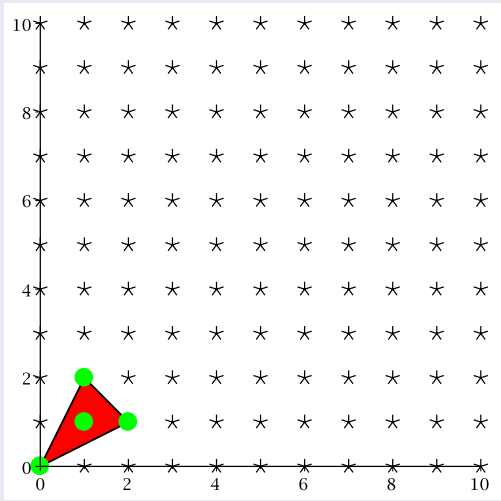
$1 + x\,y + x^2\,y + x\,y^2 \in$ $\mathbb{C}[x, y]$

Computing Gröbner bases for sparse systems

Given $f_1, \ldots, f_n \in \mathbb{C}[\boldsymbol{x}]$ and **different polytopes** $P_i = \texttt{NewtonPolytope}(f_i)$.

How to exploit sparsity? [Gelfand, Kapranov, & Zelevinsky '90]

- Sparse resultant $\rightarrow$ Generalization of the resultant for sparse systems.
- Geometrically $\rightarrow$ Instead of projective space, use proj. toric variety.



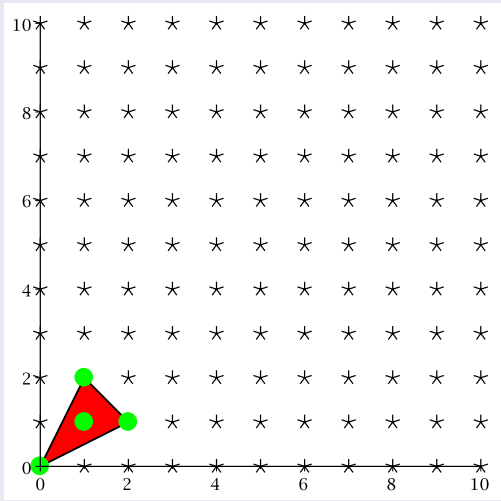
$1 + x\,y + x^2\,y + x\,y^2 \in \mathbb{C}[x, y]$

Computing Gröbner bases for sparse systems

Given $f_1, \ldots, f_n \in \mathbb{C}[\boldsymbol{x}]$ and **different polytopes** $P_i = \texttt{NewtonPolytope}(f_i)$.

How to exploit sparsity? [Gelfand, Kapranov, & Zelevinsky '90]

- Sparse resultant $\rightarrow$ Generalization of the resultant for sparse systems.
- Geometrically $\rightarrow$ Instead of projective space, use proj. toric variety.
- Algebraically $\rightarrow$ Replace classical algebra for a semigroup subalgebra.



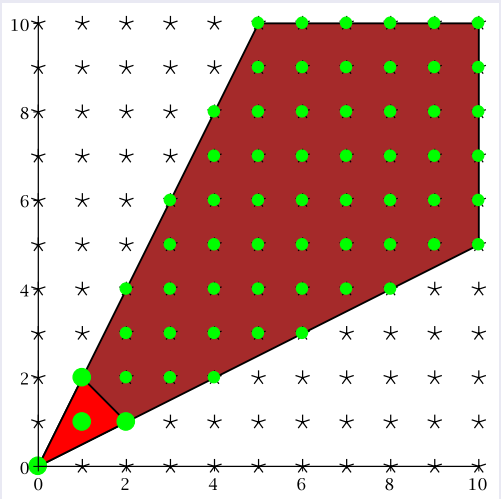
$1 + x\,y + x^2\,y + x\,y^2 \in$ $\mathbb{C}[x, y]$

Computing Gröbner bases for sparse systems

Given $f_1, \ldots, f_n \in \mathbb{C}[\boldsymbol{x}]$ and **different polytopes** $P_i = \texttt{NewtonPolytope}(f_i)$.

How to exploit sparsity? [Gelfand, Kapranov, & Zelevinsky '90]

- Sparse resultant $\rightarrow$ Generalization of the resultant for sparse systems.
- Geometrically $\rightarrow$ Instead of projective space, use proj. toric variety.
- Algebraically $\rightarrow$ Replace classical algebra for a semigroup subalgebra.



$$1 + x\,y + x^2\,y + x\,y^2 \in \mathbb{C}[x\,y, x^2\,y, x\,y^2] \subset \mathbb{C}[x, y]$$

Computing Gröbner bases for sparse systems

Given $f_1, \ldots, f_n \in \mathbb{C}[\boldsymbol{x}]$ and **different polytopes** $P_i = \texttt{NewtonPolytope}(f_i)$.

How to exploit sparsity?	Compute over semigroup algebras

Computing Gröbner bases for sparse systems

Given $f_1, \ldots, f_n \in \mathbb{C}[\boldsymbol{x}]$ and **different polytopes** $P_i = \texttt{NewtonPolytope}(f_i)$.

How to exploit sparsity? — Compute over semigroup algebras

Our algorithm — [BFT18], [BFT19]

- Algorithm to compute Gröbner basis over multigraded semigroup algebras.

Computing Gröbner bases for sparse systems

Given $f_1, \ldots, f_n \in \mathbb{C}[\boldsymbol{x}]$ and **different polytopes** $P_i = \texttt{NewtonPolytope}(f_i)$.

How to exploit sparsity? — Compute over semigroup algebras

Our algorithm — [BFT18], [BFT19]

- Algorithm to compute Gröbner basis over multigraded semigroup algebras.
- New Koszul-F5 criterion $\rightarrow$ Under mild asumptions, no reds. to zero.

Computing Gröbner bases for sparse systems

Given $f_1, \ldots, f_n \in \mathbb{C}[\boldsymbol{x}]$ and **different polytopes** $P_i = \texttt{NewtonPolytope}(f_i)$.

How to exploit sparsity? — Compute over semigroup algebras

Our algorithm — [BFT18], [BFT19]

- Algorithm to compute Gröbner basis over multigraded semigroup algebras.
- New Koszul-F5 criterion $\rightarrow$ Under mild asumptions, no reds. to zero.
- Complexity bounds for square systems $\rightarrow$ Complexity of solving sparse sys.

Computing Gröbner bases for sparse systems

Given $f_1, \ldots, f_n \in \mathbb{C}[\boldsymbol{x}]$ and **different polytopes** $P_i = \texttt{NewtonPolytope}(f_i)$.

How to exploit sparsity? — Compute over semigroup algebras

Our algorithm — [BFT18], [BFT19]

- Algorithm to compute Gröbner basis over multigraded semigroup algebras.
- New Koszul-F5 criterion $\rightarrow$ Under mild asumptions, no reds. to zero.
- Complexity bounds for square systems $\rightarrow$ Complexity of solving sparse sys.

Biggest matrix has size $\sum_i P_i \cap \mathbb{Z}^n$.

Computing Gröbner bases for sparse systems

Given $f_1, \ldots, f_n \in \mathbb{C}[\boldsymbol{x}]$ and **different polytopes** $P_i = \texttt{NewtonPolytope}(f_i)$.

How to exploit sparsity? — Compute over semigroup algebras

Our algorithm — [BFT18], [BFT19]

- Algorithm to compute Gröbner basis over multigraded semigroup algebras.
- New Koszul-F5 criterion $\rightarrow$ Under mild asumptions, no reds. to zero.
- Complexity bounds for square systems $\rightarrow$ Complexity of solving sparse sys.

Biggest matrix has size $\sum_i P_i \cap \mathbb{Z}^n$. $\sim \sum \deg(f_i)$

Computing Gröbner bases for sparse systems

Given $f_1, \ldots, f_n \in \mathbb{C}[\boldsymbol{x}]$ and **different polytopes** $P_i = \texttt{NewtonPolytope}(f_i)$.

How to exploit sparsity? — Compute over semigroup algebras

Our algorithm — [BFT18], [BFT19]

- Algorithm to compute Gröbner basis over multigraded semigroup algebras.
- New Koszul-F5 criterion $\rightarrow$ Under mild asumptions, no reds. to zero.
- Complexity bounds for square systems $\rightarrow$ Complexity of solving sparse sys.

  Biggest matrix has size $\sum_i P_i \cap \mathbb{Z}^n$. $\sim \sum \deg(f_i)$
- Improvements for special cases $\longrightarrow$ Generalization of the Macaulay bound

Computing Gröbner bases for sparse systems

Given $f_1, \ldots, f_n \in \mathbb{C}[\boldsymbol{x}]$ and **different polytopes** $P_i = \texttt{NewtonPolytope}(f_i)$.

How to exploit sparsity? — Compute over semigroup algebras

Our algorithm [BFT18], [BFT19]

- Algorithm to compute Gröbner basis over multigraded semigroup algebras.
- New Koszul-F5 criterion $\rightarrow$ Under mild asumptions, no reds. to zero.
- Complexity bounds for square systems $\rightarrow$ Complexity of solving sparse sys.

  Biggest matrix has size $\sum_i P_i \cap \mathbb{Z}^n$. $\sim \sum \deg(f_i)$
- Improvements for special cases $\longrightarrow$ Generalization of the Macaulay bound
  - Multihomogeneous systems $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$.

    $\sum \texttt{multideg}(f_i) - (n_1, \ldots, n_r) + (1, \ldots, 1)$

Computing Gröbner bases for sparse systems

Given $f_1, \ldots, f_n \in \mathbb{C}[\boldsymbol{x}]$ and **different polytopes** $P_i = \texttt{NewtonPolytope}(f_i)$.

How to exploit sparsity? — Compute over semigroup algebras

Our algorithm [BFT18], [BFT19]

- Algorithm to compute Gröbner basis over multigraded semigroup algebras.
- New Koszul-F5 criterion $\rightarrow$ Under mild asumptions, no reds. to zero.
- Complexity bounds for square systems $\rightarrow$ Complexity of solving sparse sys.

  Biggest matrix has size $\sum_i P_i \cap \mathbb{Z}^n$. $\sim \sum \deg(f_i)$
- Improvements for special cases $\longrightarrow$ Generalization of the Macaulay bound
  - Multihomogeneous systems $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$.

    $\sum \texttt{multideg}(f_i) - (n_1, \ldots, n_r) + (1, \ldots, 1)$
  - Unmixed systems (all pols same Newton polytope, $P = P_i$).

    $\sim \sum \deg(f_i) - \texttt{codegree}(P) + 1$

Computing Gröbner bases for sparse systems

Given $f_1, \dots, f_n \in \mathbb{C}[\boldsymbol{x}]$ and **different polytopes** $P_i = \texttt{NewtonPolytope}(f_i)$.

How to exploit sparsity? — Compute over semigroup algebras

Our algorithm [BFT18], [BFT19]

- Algorithm to compute Gröbner basis over multigraded semigroup algebras.
- New Koszul-F5 criterion $\rightarrow$ Under mild asumptions, no reds. to zero.
- Complexity bounds for square systems $\rightarrow$ Complexity of solving sparse sys.

  Biggest matrix has size $\sum_i P_i \cap \mathbb{Z}^n$. $\sim \sum \deg(f_i)$
- Improvements for special cases $\longrightarrow$ Generalization of the Macaulay bound
  - Multihomogeneous systems $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$.

    $\sum \texttt{multideg}(f_i) - (n_1, \dots, n_r) + (1, \dots, 1)$
  - Unmixed systems (all pols same Newton polytope, $P = P_i$).

    $\sim \sum \deg(f_i) - \texttt{codegree}(P) + 1$
  - Not all variables in all P_i.

Solving numerically sparse systems

Joint work with Simon Telen.

Symbolic-numerical approach to solve sparse systems

In applications $\rightarrow$ sparse polynomials with noisy coefficients, and
we need to approximate solutions.

Symbolic-numerical approach to solve sparse systems

In applications $\rightarrow$ sparse polynomials with noisy coefficients, and
we need to approximate solutions.

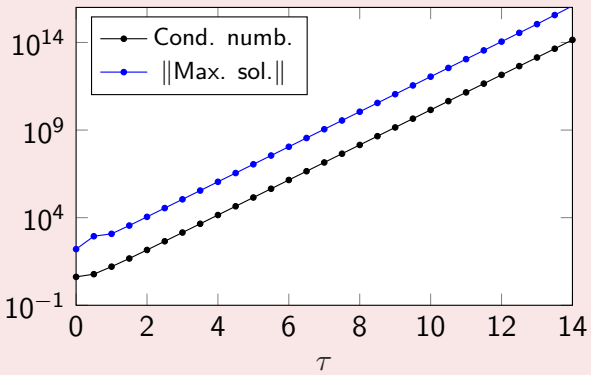
Symbolic-numerical approach to solve sparse systems

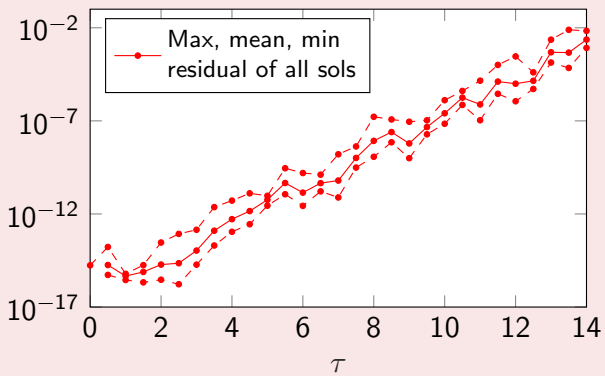
In applications $\rightarrow$ sparse polynomials with noisy coefficients, and we need to approximate solutions.

In degenerate situations (e.g. 'sols. going to ∞') $\rightarrow$ Sparse GBs or resultants $\rightarrow$ large numerical errors (for every solution!).

Example

$$\begin{cases} f_1 := -1 + x + x^2 + y + x\,y, \\ f_2 := -1 + x + (\frac{5}{2} - 10^{-\tau})\,x^2 + 2\,y + \frac{5}{2}\,x\,y. \end{cases}$$





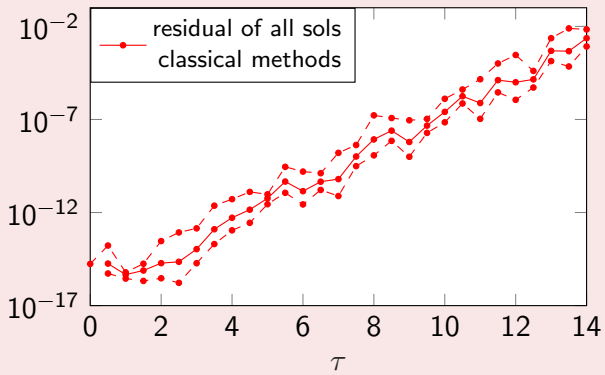
Symbolic-numerical approach to solve sparse systems

In applications $\rightarrow$ sparse polynomials with noisy coefficients, and we need to approximate solutions.

Example

$$f_1 := 1 - x - x^2 - y - x\,y,$$

$$f_2 := 1 - x - (5/2 - 10^{-\tau})\,x^2 - 2\,y - 5/2\,x\,y$$



Symbolic-numerical approach to solve sparse systems

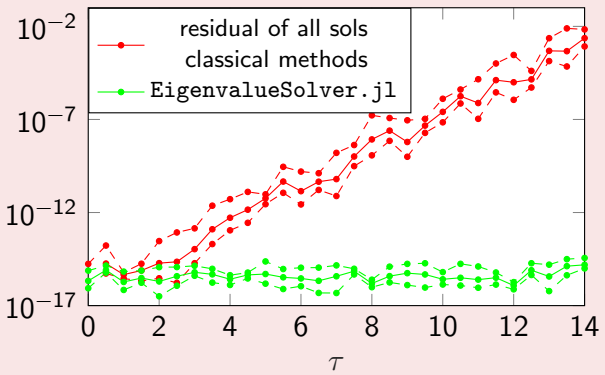
In applications $\rightarrow$ sparse polynomials with noisy coefficients, and we need to approximate solutions.

Our algorithm [BT20],[BT21]

- Homogenize polynomials over Cox ring (homogeneous coord. ring of toric variety).

Example

$$f_1 := 1 - x - x^2 - y - xy,$$
$$f_2 := 1 - x - (5/2 - 10^{-\tau})\,x^2 - 2y - 5/2\,xy$$



Symbolic-numerical approach to solve sparse systems

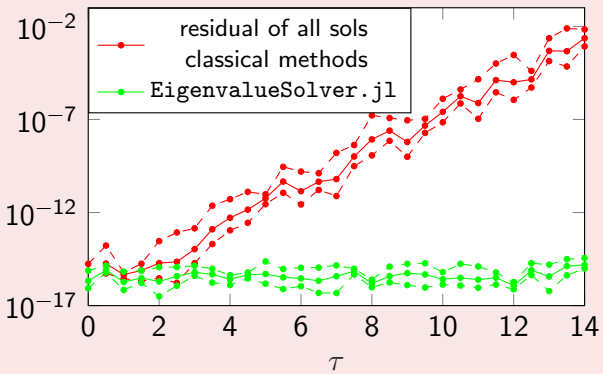
In applications $\rightarrow$ sparse polynomials with noisy coefficients, and we need to approximate solutions.

Our algorithm [BT20],[BT21]

- Homogenize polynomials over Cox ring (homogeneous coord. ring of toric variety).
- Consider Sylvester map ($\boldsymbol{g} \mapsto \sum f_i\, g_i$) at "big enough" (known) degree.

Example

$$f_1 := 1 - x - x^2 - y - x\,y,$$
$$f_2 := 1 - x - (5/2 - 10^{-\tau})\,x^2 - 2\,y - 5/2\,x\,y$$



Symbolic-numerical approach to solve sparse systems

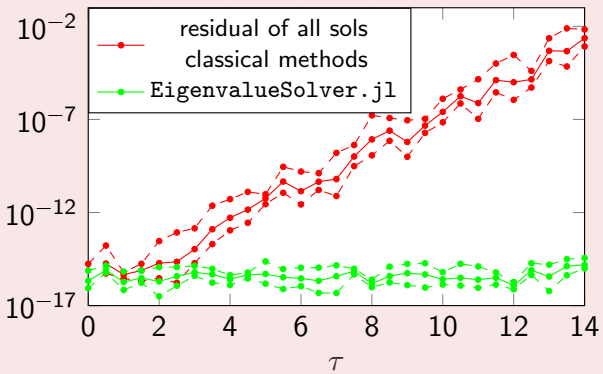
In applications $\rightarrow$ sparse polynomials with noisy coefficients, and we need to approximate solutions.

Our algorithm [BT20],[BT21]

- Homogenize polynomials over Cox ring (homogeneous coord. ring of toric variety).
- Consider Sylvester map ($\boldsymbol{g} \mapsto \sum f_i\, g_i$) at "big enough" (known) degree.
- Using numerical linear algebra tools (QR with optimal pivoting, SVD) $\rightarrow$ reduce to eigenvalue comput. (mult. maps).

Example

$$f_1 := 1 - x - x^2 - y - x\,y,$$
$$f_2 := 1 - x - (5/2 - 10^{-\tau})\,x^2 - 2\,y - 5/2\,x\,y$$



Symbolic-numerical approach to solve sparse systems

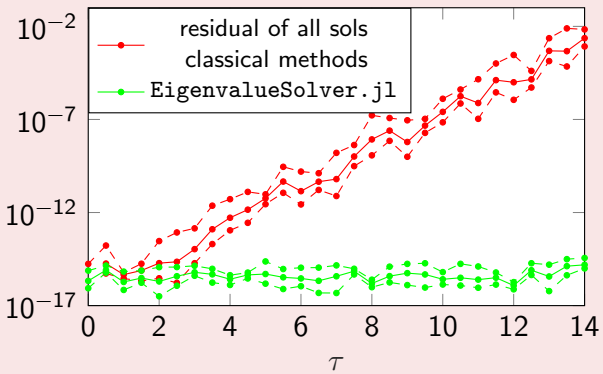
In applications $\rightarrow$ sparse polynomials with noisy coefficients, and we need to approximate solutions.

Our algorithm [BT20],[BT21]

- Homogenize polynomials over Cox ring (homogeneous coord. ring of toric variety).
- Consider Sylvester map ($\boldsymbol{g} \mapsto \sum f_i\, g_i$) at "big enough" (known) degree.
- Using numerical linear algebra tools (QR with optimal pivoting, SVD) $\rightarrow$ reduce to eigenvalue comput. (mult. maps).
- In contrast to homotopy continuation $\rightarrow$ works even better with overdetermined sys.

Example

$$f_1 := 1 - x - x^2 - y - x\,y,$$
$$f_2 := 1 - x - (5/2 - 10^{-\tau})\,x^2 - 2\,y - 5/2\,x\,y$$



Symbolic-numerical approach to solve sparse systems

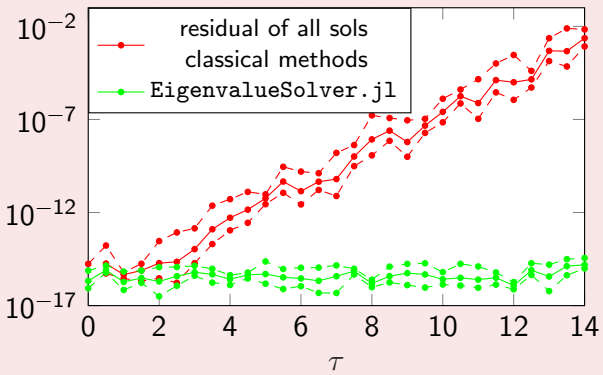
In applications $\rightarrow$ sparse polynomials with noisy coefficients, and we need to approximate solutions.

Our algorithm [BT20],[BT21]

- Homogenize polynomials over Cox ring (homogeneous coord. ring of toric variety).
- Consider Sylvester map ($\boldsymbol{g} \mapsto \sum f_i\, g_i$) at "big enough" (known) degree.
- Using numerical linear algebra tools (QR with optimal pivoting, SVD) $\rightarrow$ reduce to eigenvalue comput. (mult. maps).
- In contrast to homotopy continuation $\rightarrow$ works even better with overdetermined sys.
- Toy implem. in Julia: `EigenvalueSolver.jl`

Example

$$f_1 := 1 - x - x^2 - y - x\,y,$$
$$f_2 := 1 - x - (5/2 - 10^{-\tau})\,x^2 - 2\,y - 5/2\,x\,y$$



Determinantal formulas for mixed multilinear systems

Joint work with J.-C. Faugère, A. Mantzaflaris & E. Tsigaridas.

Solving using resultant formulas

- Until now, solve via Sylvester map ($\boldsymbol{g} \mapsto \sum f_i\, g_i$).

Solving using resultant formulas

- Until now, solve via Sylvester map ($\boldsymbol{g} \mapsto \sum f_i\, g_i$).
- To solve using smaller matrices $\rightarrow$ more general maps.

Solving using resultant formulas

- Until now, solve via Sylvester map ($\boldsymbol{g} \mapsto \sum f_i\, g_i$).
- To solve using smaller matrices $\rightarrow$ more general maps.
- Compute resultant of $\boldsymbol{f} \rightarrow$ formula as factor in det. of matrix $M(\boldsymbol{f})$.

$$\texttt{Det}(M(\boldsymbol{f})) = \texttt{Resultant}(\boldsymbol{f}) \cdot \texttt{ExtraFactor}(\boldsymbol{f}).$$

Solving using resultant formulas

- Until now, solve via Sylvester map ($\boldsymbol{g} \mapsto \sum f_i\, g_i$).
- To solve using smaller matrices $\rightarrow$ more general maps.
- Compute resultant of $\boldsymbol{f} \rightarrow$ formula as factor in det. of matrix $M(\boldsymbol{f})$.

$$\texttt{Det}(M(\boldsymbol{f})) = \texttt{Resultant}(\boldsymbol{f}) \cdot \texttt{ExtraFactor}(\boldsymbol{f}).$$

Generalization of solving strategy for Sylvester maps [**B**FMT18]

- Solve using general class of resultant formulas.

Solving using resultant formulas

- Until now, solve via Sylvester map ($\boldsymbol{g} \mapsto \sum f_i\, g_i$).
- To solve using smaller matrices $\rightarrow$ more general maps.
- Compute resultant of $\boldsymbol{f}$ $\rightarrow$ formula as factor in det. of matrix $M(\boldsymbol{f})$.

$$\texttt{Det}(M(\boldsymbol{f})) = \texttt{Resultant}(\boldsymbol{f}) \cdot \texttt{ExtraFactor}(\boldsymbol{f}).$$

Generalization of solving strategy for Sylvester maps [**B**FMT18]

- Solve using general class of resultant formulas.
- Do not compute the resultant $\rightarrow$ only linear algebra.

Solving using resultant formulas

- Until now, solve via Sylvester map ($\boldsymbol{g} \mapsto \sum f_i\, g_i$).
- To solve using smaller matrices $\rightarrow$ more general maps.
- Compute resultant of $\boldsymbol{f}$ $\rightarrow$ formula as factor in det. of matrix $M(\boldsymbol{f})$.

$$\texttt{Det}(M(\boldsymbol{f})) = \texttt{Resultant}(\boldsymbol{f}) \cdot \texttt{ExtraFactor}(\boldsymbol{f}).$$

Generalization of solving strategy for Sylvester maps [**B**FMT18]

- Solve using general class of resultant formulas.
- Do not compute the resultant $\rightarrow$ only linear algebra.
- Solve by computing eigenvalues. (Matrices are not mult. maps!)

Determinantal formulas for mixed multilinear systems

- We can solve using matrix $M(\boldsymbol{f})$ (in certain class) [BFMT18]

$$\texttt{Det}(M(\boldsymbol{f})) = \texttt{Resultant}(\boldsymbol{f}) \cdot \texttt{ExtraFactor}(\boldsymbol{f}).$$

Determinantal formulas for mixed multilinear systems

- We can solve using matrix $M(\boldsymbol{f})$ (in certain class) [BFMT18]

$$\texttt{Det}(M(\boldsymbol{f})) = \texttt{Resultant}(\boldsymbol{f}) \cdot \texttt{ExtraFactor}(\boldsymbol{f}).$$

- Smallest possible matrix $\leftrightarrow$ no $\texttt{ExtraFactor}(\boldsymbol{f})$ (Determinantal Formula)

$$\texttt{Det}(M(\boldsymbol{f})) = \texttt{Resultant}(\boldsymbol{f})$$

Determinantal formulas for mixed multilinear systems

- We can solve using matrix $M(\boldsymbol{f})$ (in certain class) [BFMT18]
- Smallest possible matrix $\leftrightarrow$ no $\texttt{ExtraFactor}(\boldsymbol{f})$ (Determinantal Formula)

$$\texttt{Det}(M(\boldsymbol{f})) = \texttt{Resultant}(\boldsymbol{f})$$

Determinantal formulas for mixed multilinear systems

- We can solve using matrix $M(\boldsymbol{f})$ (in certain class) [BFMT18]
- Smallest possible matrix $\leftrightarrow$ no $\texttt{ExtraFactor}(\boldsymbol{f})$ (Determinantal Formula)

$$\texttt{Det}(M(\boldsymbol{f})) = \texttt{Resultant}(\boldsymbol{f})$$

- Determinantal formulas do not exist for general systems.

Determinantal formulas for mixed multilinear systems

- We can solve using matrix $M(\boldsymbol{f})$ (in certain class) [**B**FMT18]
- Smallest possible matrix $\leftrightarrow$ no $\texttt{ExtraFactor}(\boldsymbol{f})$ (Determinantal Formula)

$$\texttt{Det}(M(\boldsymbol{f})) = \texttt{Resultant}(\boldsymbol{f})$$

- Determinantal formulas do not exist for general systems.

Determinantal formulas for mixed multilinear sys. [**B**FMT21]

- First det. formulas for mixed sparse systems (different Newton polytopes).

Determinantal formulas for mixed multilinear systems

- We can solve using matrix $M(\boldsymbol{f})$ (in certain class) [**B**FMT18]
- Smallest possible matrix $\leftrightarrow$ no $\texttt{ExtraFactor}(\boldsymbol{f})$ (Determinantal Formula)

$$\texttt{Det}(M(\boldsymbol{f})) = \texttt{Resultant}(\boldsymbol{f})$$

- Determinantal formulas do not exist for general systems.

Determinantal formulas for mixed multilinear sys. [**B**FMT21]

- First det. formulas for mixed sparse systems (different Newton polytopes).
- Special families of mixed multilinear systems.

Determinantal formulas for mixed multilinear systems

- We can solve using matrix $M(\boldsymbol{f})$ (in certain class) [**B**FMT18]
- Smallest possible matrix $\leftrightarrow$ no $\texttt{ExtraFactor}(\boldsymbol{f})$ (Determinantal Formula)

$$\texttt{Det}(M(\boldsymbol{f})) = \texttt{Resultant}(\boldsymbol{f})$$

- Determinantal formulas do not exist for general systems.

Determinantal formulas for mixed multilinear sys. [**B**FMT21]

- First det. formulas for mixed sparse systems (different Newton polytopes).
- Special families of mixed multilinear systems.
- These systems arise in applications $\rightarrow$ Multiparameter eigenvalue problem.

Determinantal formulas for mixed multilinear systems

- We can solve using matrix $M(\boldsymbol{f})$ (in certain class) [**B**FMT18]
- Smallest possible matrix $\leftrightarrow$ no $\texttt{ExtraFactor}(\boldsymbol{f})$ (Determinantal Formula)

$$\texttt{Det}(M(\boldsymbol{f})) = \texttt{Resultant}(\boldsymbol{f})$$

- Determinantal formulas do not exist for general systems.

Determinantal formulas for mixed multilinear sys. [**B**FMT21]

- First det. formulas for mixed sparse systems (different Newton polytopes).
- Special families of mixed multilinear systems.
- These systems arise in applications $\rightarrow$ Multiparameter eigenvalue problem.
  Generalized Eigenvalue Problem

$$\left(\begin{bmatrix} -7 & -3 \\ -8 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 12 & 2 \\ 13 & 1 \end{bmatrix} \right) \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = 0 \quad .$$

Determinantal formulas for mixed multilinear systems

- We can solve using matrix $M(\boldsymbol{f})$ (in certain class) [**B**FMT18]
- Smallest possible matrix $\leftrightarrow$ no $\texttt{ExtraFactor}(\boldsymbol{f})$ (Determinantal Formula)

$$\texttt{Det}(M(\boldsymbol{f})) = \texttt{Resultant}(\boldsymbol{f})$$

- Determinantal formulas do not exist for general systems.

Determinantal formulas for mixed multilinear sys. [**B**FMT21]

- First det. formulas for mixed sparse systems (different Newton polytopes).
- Special families of mixed multilinear systems.
- These systems arise in applications $\rightarrow$ Multiparameter eigenvalue problem.
  - Generalization of the Generalized Eigenvalue Problem

$$\left(\lambda_0 \begin{bmatrix} -7 & -3 \\ -8 & -2 \end{bmatrix} + \lambda_1 \begin{bmatrix} 12 & 2 \\ 13 & 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} -7 & -1 \\ -7 & -1 \end{bmatrix}\right) \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = 0$$

$$\left(\lambda_0 \begin{bmatrix} -11 & -3 \\ 4 & 1 \end{bmatrix} + \lambda_1 \begin{bmatrix} 7 & -1 \\ 1 & 2 \end{bmatrix} + \lambda_2 \begin{bmatrix} -4 & 0 \\ -1 & -1 \end{bmatrix}\right) \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = 0 \,.$$

Determinantal formulas for mixed multilinear systems

- We can solve using matrix $M(\boldsymbol{f})$ (in certain class) [**B**FMT18]
- Smallest possible matrix $\leftrightarrow$ no $\texttt{ExtraFactor}(\boldsymbol{f})$ (Determinantal Formula)

$$\texttt{Det}(M(\boldsymbol{f})) = \texttt{Resultant}(\boldsymbol{f})$$

- Determinantal formulas do not exist for general systems.

Determinantal formulas for mixed multilinear sys. [**B**FMT21]

- First det. formulas for mixed sparse systems (different Newton polytopes).
- Special families of mixed multilinear systems.
- These systems arise in applications $\rightarrow$ Multiparameter eigenvalue problem.
  - Generalization of the Generalized Eigenvalue Problem
  - Applications in physics (Sturm-Liouville theory)

$$\left(\lambda_0 \begin{bmatrix} -7 & -3 \\ -8 & -2 \end{bmatrix} + \lambda_1 \begin{bmatrix} 12 & 2 \\ 13 & 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} -7 & -1 \\ -7 & -1 \end{bmatrix}\right) \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = 0$$

$$\left(\lambda_0 \begin{bmatrix} -11 & -3 \\ 4 & 1 \end{bmatrix} + \lambda_1 \begin{bmatrix} 7 & -1 \\ 1 & 2 \end{bmatrix} + \lambda_2 \begin{bmatrix} -4 & 0 \\ -1 & -1 \end{bmatrix}\right) \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = 0$$

Determinantal formulas for mixed multilinear systems

- We can solve using matrix $M(\boldsymbol{f})$ (in certain class) [**B**FMT18]
- Smallest possible matrix $\leftrightarrow$ no $\texttt{ExtraFactor}(\boldsymbol{f})$ (Determinantal Formula)

$$\texttt{Det}(M(\boldsymbol{f})) = \texttt{Resultant}(\boldsymbol{f})$$

- Determinantal formulas do not exist for general systems.

Determinantal formulas for mixed multilinear sys. [**B**FMT21]

- First det. formulas for mixed sparse systems (different Newton polytopes).
- Special families of mixed multilinear systems.
- These systems arise in applications $\rightarrow$ Multiparameter eigenvalue problem.
  - Generalization of the Generalized Eigenvalue Problem
  - Applications in physics (Sturm-Liouville theory)

$$\begin{bmatrix} (-7\,\lambda_0 + 12\,\lambda_1 - 7\,\lambda_2) & (-3\,\lambda_0 + 2\,\lambda_1 - \lambda_2) \\ (-8\,\lambda_0 + 13\,\lambda_1 - 7\,\lambda_2) & (-2\,\lambda_0 + \lambda_1 - \lambda_2) \end{bmatrix} \cdot \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = 0$$

$$\begin{bmatrix} (-11\,\lambda_0 + 7\,\lambda_1 - 4\,\lambda_2) & (-3\,\lambda_0 - \lambda_1) \\ (4\,\lambda_0 + \lambda_1 - \lambda_2) & (\lambda_0 + 2\,\lambda_1 - \lambda_2) \end{bmatrix} \cdot \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = 0 \quad .$$

Determinantal formulas for mixed multilinear systems

- We can solve using matrix $M(\boldsymbol{f})$ (in certain class) [**B**FMT18]
- Smallest possible matrix $\leftrightarrow$ no $\texttt{ExtraFactor}(\boldsymbol{f})$ (Determinantal Formula)

$$\texttt{Det}(M(\boldsymbol{f})) = \texttt{Resultant}(\boldsymbol{f})$$

- Determinantal formulas do not exist for general systems.

Determinantal formulas for mixed multilinear sys. [**B**FMT21]

- First det. formulas for mixed sparse systems (different Newton polytopes).
- Special families of mixed multilinear systems.
- These systems arise in applications $\rightarrow$ Multiparameter eigenvalue problem.
  - Generalization of the Generalized Eigenvalue Problem
  - Applications in physics (Sturm-Liouville theory)

$$\begin{cases} (-7\,\lambda_0 + 12\,\lambda_1 - 7\,\lambda_2)\; v_0 + (-3\,\lambda_0 + 2\,\lambda_1 - \lambda_2)\, v_1 = 0 \\ (-8\,\lambda_0 + 13\,\lambda_1 - 7\,\lambda_2)\; v_0 + (-2\,\lambda_0 + \lambda_1 - \lambda_2)\, v_1 = 0 \\ (-11\,\lambda_0 + 7\,\lambda_1 - 4\,\lambda_2)\; w_0 + (-3\,\lambda_0 - \lambda_1)\; w_1 = 0 \\ (4\,\lambda_0 + \lambda_1 - \lambda_2)\; w_0 + (\lambda_0 + 2\,\lambda_1 - \lambda_2)\; w_1 = 0 \end{cases}.$$

Determinantal formulas for mixed multilinear systems

- We can solve using matrix $M(\boldsymbol{f})$ (in certain class) [**B**FMT18]
- Smallest possible matrix $\leftrightarrow$ no $\texttt{ExtraFactor}(\boldsymbol{f})$ (Determinantal Formula)

$$\texttt{Det}(M(\boldsymbol{f})) = \texttt{Resultant}(\boldsymbol{f})$$

- Determinantal formulas do not exist for general systems.

Determinantal formulas for mixed multilinear sys. [**B**FMT21]

- First det. formulas for mixed sparse systems (different Newton polytopes).
- Special families of mixed multilinear systems.
- These systems arise in applications $\rightarrow$ Multiparameter eigenvalue problem.
  - Generalization of the Generalized Eigenvalue Problem
  - Applications in physics (Sturm-Liouville theory)

$$\begin{cases} f_1 := (-7\,\lambda_0 + 12\,\lambda_1 - 7\,\lambda_2)\; v_0 + (-3\,\lambda_0 + 2\,\lambda_1 - \lambda_2)\; v_1 \\ f_2 := (-8\,\lambda_0 + 13\,\lambda_1 - 7\,\lambda_2)\; v_0 + (-2\,\lambda_0 + \lambda_1 - \lambda_2)\; v_1 \\ f_3 := (-11\,\lambda_0 + 7\,\lambda_1 - 4\,\lambda_2)\; w_0 + (-3\,\lambda_0 - \lambda_1)\; w_1 \\ f_4 := (4\,\lambda_0 + \lambda_1 - \lambda_2)\; w_0 + (\lambda_0 + 2\,\lambda_1 - \lambda_2)\; w_1 \end{cases}.$$

Determinantal formulas for mixed multilinear systems

- We can solve using matrix $M(\boldsymbol{f})$ (in certain class) [**B**FMT18]
- Smallest possible matrix $\leftrightarrow$ no $\texttt{ExtraFactor}(\boldsymbol{f})$ (Determinantal Formula)

$$\texttt{Det}(M(\boldsymbol{f})) = \texttt{Resultant}(\boldsymbol{f})$$

- Determinantal formulas do not exist for general systems.

Determinantal formulas for mixed multilinear sys. [**B**FMT21]

- First det. formulas for mixed sparse systems (different Newton polytopes).
- Special families of mixed multilinear systems.
- These systems arise in applications $\rightarrow$ Multiparameter eigenvalue problem.
  - Generalization of the Generalized Eigenvalue Problem
  - Applications in physics (Sturm-Liouville theory)

$$\begin{cases} f_1 := (-7\,\lambda_0 + 12\,\lambda_1 - 7\,\lambda_2)\; v_0 + (-3\,\lambda_0 + 2\,\lambda_1 - \lambda_2)\; v_1 \\ f_2 := (-8\,\lambda_0 + 13\,\lambda_1 - 7\,\lambda_2)\; v_0 + (-2\,\lambda_0 + \lambda_1 - \lambda_2)\; v_1 \\ f_3 := (-11\,\lambda_0 + 7\,\lambda_1 - 4\,\lambda_2)\; w_0 + (-3\,\lambda_0 - \lambda_1)\; w_1 \\ f_4 := (4\,\lambda_0 + \lambda_1 - \lambda_2)\; w_0 + (\lambda_0 + 2\,\lambda_1 - \lambda_2)\; w_1 \end{cases}.$$

Determinantal formulas for mixed multilinear systems

- We can solve using matrix $M(\boldsymbol{f})$ (in certain class) [**B**FMT18]
- Smallest possible matrix $\leftrightarrow$ no $\texttt{ExtraFactor}(\boldsymbol{f})$ (Determinantal Formula)

$$\texttt{Det}(M(\boldsymbol{f})) = \texttt{Resultant}(\boldsymbol{f})$$

- Determinantal formulas do not exist for general systems.

Determinantal formulas for mixed multilinear sys. [**B**FMT21]

- First det. formulas for mixed sparse systems (different Newton polytopes).
- Special families of mixed multilinear systems.
- These systems arise in applications $\rightarrow$ Multiparameter eigenvalue problem.
  - Generalization of the Generalized Eigenvalue Problem
  - Applications in physics (Sturm-Liouville theory)

$$\begin{cases} f_1 := (-7\,\lambda_0 + 12\,\lambda_1 - 7\,\lambda_2)\; v_0 + (-3\,\lambda_0 + 2\,\lambda_1 - \lambda_2)\; v_1 \\ f_2 := (-8\,\lambda_0 + 13\,\lambda_1 - 7\,\lambda_2)\; v_0 + (-2\,\lambda_0 + \lambda_1 - \lambda_2)\; v_1 \\ f_3 := (-11\,\lambda_0 + 7\,\lambda_1 - 4\,\lambda_2)\; w_0 + (-3\,\lambda_0 - \lambda_1)\; w_1 \\ f_4 := (4\,\lambda_0 + \lambda_1 - \lambda_2)\; w_0 + (\lambda_0 + 2\,\lambda_1 - \lambda_2)\; w_1 \end{cases}.$$

Other applications

Symmetric tensor decomposition [B., Faugère, Perret & Tsigaridas '16 & '21]

(ACM's SIGSAM Distinguished Student Author Award at ISSAC 2016)

- Symmetric tensor decomposition of binary forms:

$$-4704\,x^3 + 3576\,x^2\,y - 912\,x\,y^2 + 78\,y^3 = \frac{1}{6}\,(-26\,x + 7\,y)^3 - \frac{1}{6}\,(22\,x - 5\,y)^3$$

- First quasi-linear algorithm to compute minimal decomposition.

Other applications

Symmetric tensor decomposition [B., Faugère, Perret & Tsigaridas '16 & '21]

(ACM's SIGSAM Distinguished Student Author Award at ISSAC 2016)

- Symmetric tensor decomposition of binary forms:

$$-4704\,x^3 + 3576\,x^2\,y - 912\,x\,y^2 + 78\,y^3 = \frac{1}{6}\,(-26\,x + 7\,y)^3 - \frac{1}{6}\,(22\,x - 5\,y)^3$$

- First quasi-linear algorithm to compute minimal decomposition.

Topological data analysis [B., Gäfvert, Lesnick '21+]

- Optimized GB algorithm to compute Multiparameter Persistence Homology.
- Efficient implementation + Complexity bounds.

Other applications

Symmetric tensor decomposition [B., Faugère, Perret & Tsigaridas '16 & '21]

(ACM's SIGSAM Distinguished Student Author Award at ISSAC 2016)

- Symmetric tensor decomposition of binary forms:

$$-4704\,x^3 + 3576\,x^2\,y - 912\,x\,y^2 + 78\,y^3 = \frac{1}{6}\,(-26\,x + 7\,y)^3 - \frac{1}{6}\,(22\,x - 5\,y)^3$$

- First quasi-linear algorithm to compute minimal decomposition.

Topological data analysis [B., Gäfvert, Lesnick '21+]

- Optimized GB algorithm to compute Multiparameter Persistence Homology.
- Efficient implementation + Complexity bounds.

Computational biology [Schwieger, B., Siebert & Haase '20]

- Gröbner-basis-based approach to construct classifiers for Boolean networks.

Some possible collaborations

- In polynomial optimization $\rightarrow$ There other notions of sparsity.
  $\rightarrow$ Solving techniques over $\mathbb{R}$.
- Our notions of sparsity are orthogonal $\Longrightarrow$ we could merge them.

Some possible collaborations

- In polynomial optimization $\rightarrow$ There other notions of sparsity.
  $\rightarrow$ Solving techniques over $\mathbb{R}$.
- Our notions of sparsity are orthogonal $\Longrightarrow$ we could merge them.

Importing ideas

- Exploit extra sparsity (e.g. correlative) in GB computations.
  (Extension in non-commutative case! $\rightarrow$ non-commutative GB)

Some possible collaborations

- In polynomial optimization $\rightarrow$ There other notions of sparsity.
  $\rightarrow$ Solving techniques over $\mathbb{R}$.
- Our notions of sparsity are orthogonal $\Longrightarrow$ we could merge them.

Importing ideas

- Exploit extra sparsity (e.g. correlative) in GB computations.
  (Extension in non-commutative case! $\rightarrow$ non-commutative GB)
- Use moments to solve sparse polynomial systems over $\mathbb{R}$.

Some possible collaborations

- In polynomial optimization $\rightarrow$ There other notions of sparsity.
  $\rightarrow$ Solving techniques over $\mathbb{R}$.
- Our notions of sparsity are orthogonal $\Longrightarrow$ we could merge them.

Importing ideas

- Exploit extra sparsity (e.g. correlative) in GB computations.
  (Extension in non-commutative case! $\rightarrow$ non-commutative GB)
- Use moments to solve sparse polynomial systems over $\mathbb{R}$.

Some possible collaborations

- In polynomial optimization $\rightarrow$ There other notions of sparsity.
  $\rightarrow$ Solving techniques over $\mathbb{R}$.
- Our notions of sparsity are orthogonal $\Longrightarrow$ we could merge them.

Importing ideas

- Exploit extra sparsity (e.g. correlative) in GB computations.
  (Extension in non-commutative case! $\rightarrow$ non-commutative GB)
- Use moments to solve sparse polynomial systems over $\mathbb{R}$.

Exporting ideas

In practice, input polynomials have different Newton polytopes.

- Can we extend TSSOS hierarchy to exploit this fact?

Some possible collaborations

- In polynomial optimization $\rightarrow$ There other notions of sparsity.
  $\rightarrow$ Solving techniques over $\mathbb{R}$.
- Our notions of sparsity are orthogonal $\Longrightarrow$ we could merge them.

Importing ideas

- Exploit extra sparsity (e.g. correlative) in GB computations.
  (Extension in non-commutative case! $\rightarrow$ non-commutative GB)
- Use moments to solve sparse polynomial systems over $\mathbb{R}$.

Exporting ideas

In practice, input polynomials have different Newton polytopes.

- Can we extend TSSOS hierarchy to exploit this fact?
- How about Positivstellensatz?

Summing-up

Solving structured polynomial systems

- Sparse Gröbner basis.
- Determinantal formulas for sparse resultant.
- Numerical linear algebra approaches.

Summing-up

Solving structured polynomial systems

- Sparse Gröbner basis.
- Determinantal formulas for sparse resultant.
- Numerical linear algebra approaches.

Applications

- Multiparameter Eigenvalue Problem.
- Symmetric Tensor Decomposition (Binary Forms).
- Topological Data Analysis (Multiparameter Persistence Homology).
- Computational Biology (Classifiers on Boolean Networks).

Summing-up

Solving structured polynomial systems

- Sparse Gröbner basis.
- Determinantal formulas for sparse resultant.
- Numerical linear algebra approaches.

Applications

- Multiparameter Eigenvalue Problem.
- Symmetric Tensor Decomposition (Binary Forms).
- Topological Data Analysis (Multiparameter Persistence Homology).
- Computational Biology (Classifiers on Boolean Networks).

Some common interests

- Correlative sparsity in GB? Solve over $\mathbb{R}$?
- Mixed sparsity in polynomial optimization?

Summing-up

Solving structured polynomial systems

- Sparse Gröbner basis.
- Determinantal formulas for sparse resultant.
- Numerical linear algebra approaches.

Applications

- Multiparameter Eigenvalue Problem.
- Symmetric Tensor Decomposition (Binary Forms).
- Topological Data Analysis (Multiparameter Persistence Homology).
- Computational Biology (Classifiers on Boolean Networks).

Some common interests

- Correlative sparsity in GB? Solve over $\mathbb{R}$?
- Mixed sparsity in polynomial optimization?

Thank you!

mbender.github.io