Signomial Moment Theory and Upper Bounds for Signomial Optimization

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Key Problem / Motivation

Signomial: A weighted exponential sum supported on $A \subseteq \mathbb{R}^n$:

$$f = \sum_{\boldsymbol{\alpha} \in A} c_{\boldsymbol{\alpha}} \exp\langle \boldsymbol{x}, \boldsymbol{\alpha} \rangle \in \mathbb{R}^A, \quad c_{\boldsymbol{\alpha}} \in \mathbb{R} \text{ for all } \boldsymbol{\alpha} \in A.$$

Signomial Programming:
Given a signomial f and a compact set K, then consider the *optimization problem*

$$\begin{aligned} f_{\mathsf{K}}^{\star} &:= \min\{f(\boldsymbol{x}) \,:\, \boldsymbol{x} \in \mathsf{K}\} \\ &= \sup\{\gamma \in \mathbb{R} \,:\, f(\boldsymbol{x}) - \gamma \geqslant 0 \text{ for all } \boldsymbol{x} \in \mathsf{K}\} \end{aligned}$$

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- Nonconvex optimization problem.
- Problem is NP-hard in general.
- Problem has many applications, e.g., chemical engineering, aircraft design, communications.

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My Talk: Outer approximations & Complete hierarchy of upper bounds.

Outline of the talk:

1. Introduction into the Concept of Signomial Rings.
2. Signomial Moment Theory.
3. Outer approximations of the cone of signomials nonnegative on K.
4. Complete Hierarchy of upper bounds

Signomial Rings

- Let $\mathcal{A} \subseteq \mathbb{R}^n$ be a finite set containing $\mathbf{0}$.
- To every $\boldsymbol{\alpha} \in \mathcal{A}$ associate a "monomial" basis function $\mathrm{e}^{\boldsymbol{\alpha}} \colon \mathbb{R}^n \to \mathbb{R}_{++}$ that takes values $\mathrm{e}^{\boldsymbol{\alpha}}(\boldsymbol{x}) = \exp\langle \boldsymbol{\alpha}, \boldsymbol{x} \rangle$.
- $\mathbb{R}[\mathcal{A}]$ = the $\mathbb{R}$-algebra generated by basis functions $\{\mathrm{e}^{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha} \in \mathcal{A}}$.
- A signomial is a linear combination $f = \sum_{\boldsymbol{\alpha} \in A} c_{\boldsymbol{\alpha}} \mathrm{e}^{\boldsymbol{\alpha}}$.
- A posynomial is a signomial with only *nonnegative* terms.

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Signomials in $\mathbb{R}[\mathcal{A}]$ are "like" polynomials on $\mathbb{R}^{\mathcal{A}}_{++}$.

Note: Polynomial variables $t_{\boldsymbol{\alpha}} = \mathrm{e}^{\boldsymbol{\alpha}}(\boldsymbol{x})$, $t_{\boldsymbol{\beta}} = \mathrm{e}^{\boldsymbol{\beta}}(\boldsymbol{x})$ are *implicitly linked*.

For instance: $t_1 = \exp(x), \; t_2 = \exp(2x) \quad \Rightarrow \quad t_2 = t_1^2$.

Implicit links might be high-degree (even non-algebraic)!

Signomial $\mathcal{A}$-degree

Define the $\mathcal{A}$-degree of a signomial f as

$$\deg_{\mathcal{A}}(f) = \text{min } \# \text{ of products of } \{e^{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha} \in \mathcal{A}} \text{ to express } f.$$

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Using $\mathcal{A}_d = \{\text{sums of } \leqslant d \text{ vectors from } \mathcal{A}\}$, we express

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- Is coordinate-system invariant.
- $\mathbb{R}[\mathcal{A}]_d$ = space of signomials of $\mathcal{A}$-degree at most d.

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Example

Let $\mathcal{A} = \{0, 1, 3\}$. Then,

$$\deg_{\mathcal{A}}(\exp(x)) = 1, \quad \deg_{\mathcal{A}}(\exp(2x)) = 2, \quad \deg_{\mathcal{A}}(\exp(3x)) = 1.$$

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Note, $\deg_{\mathcal{A}}(\exp(3x)) < \deg_{\mathcal{A}}(\exp(x)) + \deg_{\mathcal{A}}(\exp(2x))$.

Signomial Moments

- $\mathcal{A}_\infty$ = smallest subset of $\mathbb{R}^n$ that contains all the $\mathcal{A}_d$.
- $f \in \mathbb{R}[\mathcal{A}] \Leftrightarrow$ sequences $\boldsymbol{f} = (f_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \mathcal{A}_\infty} \in \mathbb{R}^{\mathcal{A}_\infty}$.
- Arbitrary $\boldsymbol{y} \in \mathbb{R}^{\mathcal{A}_\infty} \Leftrightarrow L_{\boldsymbol{y}} : \mathbb{R}[\mathcal{A}] \to \mathbb{R}$ with $L_{\boldsymbol{y}}(\mathrm{e}^{\boldsymbol{\alpha}}) = y_{\boldsymbol{\alpha}}$ - the Riesz functional of $\boldsymbol{y}$.
- Call $\boldsymbol{y} \in \mathbb{R}^{\mathcal{A}_\infty}$ a moment sequence if there is a finite Borel measure μ where

$$y_{\boldsymbol{\alpha}} = \int \mathrm{e}^{\boldsymbol{\alpha}}(\boldsymbol{x}) \mathrm{d}\mu(\boldsymbol{x}) \quad \text{for all} \quad \boldsymbol{\alpha} \in \mathcal{A}_\infty.$$

- $V_d(\,\cdot\,\boldsymbol{y}) : \mathbb{R}[\mathcal{A}] \to \mathbb{R}^{\mathcal{A}_d}$ defined by $f \mapsto (L_{\boldsymbol{y}}(f\mathrm{e}^{\boldsymbol{\alpha}}) \,:\, \boldsymbol{\alpha} \in \mathcal{A}_d)$ is the localizer induced by $\boldsymbol{y}$ and d.

Signomial Moments

Definition

A sequence of closed convex cones $(C_d)_{d \geqslant 1}$ is called *$(\mathcal{A}, \mathsf{K})$-complete* if

1. $C_d \subset \mathbb{R}[\mathcal{A}]_d$,
2. every $f \in C_d$ is K-nonnegative, and
3. for every K-positive $f \in \mathbb{R}[\mathcal{A}]$, there exists some d s.t. $f \in C_d$.

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Important:

- Any signomial Positivstellensatz yields such a sequence,
- that is nested (i.e., $C_d \subset C_{d+1}$ for all d.)

Next up...

Now: Take this new framework and develop arbitrarily strong outer approximations of cones of nonnegative signomials.

Goal: Provide complete hierarchy of upper bounds for

$$f^{\star}_{\mathsf{K}} = \min\{f(\boldsymbol{x}) \,:\, \boldsymbol{x} \in \mathsf{K}\}.$$

Note: Adapted from work by Lasserre and inspired by a generalization thereof by de Klerk, Lasserre, Laurent, and Sun.

Outer Approximations

Corollary (D., Murray, 2021)

Let $\mathcal{A}$ be injective, K be compact, and $\boldsymbol{y}$ be the moment sequence of a Borel measure with support K.
Suppose $(C_\ell)_{\ell \geqslant 1}$ is an $(\mathcal{A}, \mathrm{K})$-complete sequence that satisfies $C_\ell \subset C_{\ell+1}$ for all ℓ and denote $C_\infty = \bigcup_{\ell \geqslant 1} C_\ell$.
If P_d denotes the cone of K-nonnegative signomials in $\mathbb{R}[\mathcal{A}]_d$ and

$$Q_\ell := \left\{ f \in \mathbb{R}[\mathcal{A}]_d \,:\, V_\ell(f\boldsymbol{y}) \in C_\ell^\dagger \right\}$$

then $Q_1 \supset Q_2 \supset \cdots \supset Q_\infty = P_d$.

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Follows from our Theorem:
$f \in \mathbb{R}[\mathcal{A}]$ is K-nonnegative $\Leftrightarrow$ $V_\ell(f\boldsymbol{y}) \in C_\ell^\dagger$ for all integers $\ell \geqslant 1$.

Proof ingredients

We establish two basic facts regarding the **existence** and **uniqueness** of representing measures for signomial moment sequences.

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Proposition (D., Murray, 2021)

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Proposition (D., Murray, 2021)

If $\mathcal{A}$ is injective, then Borel measures supported on compact sets are *moment determinate*.

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Consider an $(\mathcal{A}, \mathsf{K})$-complete sequence $(C_d)_{d\geqslant 1}$ where $C_d \subset C_{d+1}$ and each C_d contains all posynomials in $\mathbb{R}[\mathcal{A}]_d$.
For $f \in \mathbb{R}[\mathcal{A}]$ and integers $d \geqslant 1$, define

$$\theta_d := \inf_{\psi} \left\{ \int f \psi \, \mathrm{d}\mu \,:\, \int \psi \, \mathrm{d}\mu = 1,\ \psi \in C_d \right\}.$$

The sequence $(\theta_d)_{d\geqslant 1}$ *monotonically converges* to $f^\star_{\mathsf{K}}$ from *above*.

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The sequence $(\theta_d)_{d \geqslant 1}$ *monotonically converges* to $f_{\mathsf{K}}^\star$ from *above*.

Note: No assumptions on the representation of K and agnostic to precise nature of $(C_d)_{d \geqslant 1}$.

Duality and Computation

We have the following primal-dual pair:

(D) $$\inf \left\{ \langle V_d(f\boldsymbol{y}), \boldsymbol{\psi} \rangle \,:\, \langle V_d(\boldsymbol{y}), \boldsymbol{\psi} \rangle = 1,\ \psi \in C_d \right\},$$

(P) $$\sup \left\{ \theta \,:\, V_d(f\boldsymbol{y}) - \theta V_d(\boldsymbol{y}) \in C_d^{\dagger},\ \theta \in \mathbb{R} \right\}.$$

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Issue: Main obstacle in implementing our hierarchy is computing the moment sequence of the reference measure μ.

Some cases where moments can be derived in closed form or numerically include μ is the uniform measure on a box or over ellipsoids or solid simplices.

Take-home message

1. We introduce new concept of signomial rings and framework for signomial moments.
2. We provide hierarchichal outer-approximations for the signomial nonnegativity cone.
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Open Questions

- Convergence rates for upper bounds.
- Detailed study of signomial moment theory and upper bounds.
- How to efficiently compute moments of reference measure μ.

Thank You

Thank you for your attention!

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