# Singularities of Spectrahedra

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# What are spectrahedra?

$$\mathbb{S}^d \;=\; \{d \times d \text{ real symmetric matrices}\}$$

$$\mathbb{S}^d_{\mathbb{C}} \;=\; \{d \times d \text{ complex symmetric matrices}\}$$

The convex cone of positive semidefinite matrices :

$$\mathbb{S}^d_{\succcurlyeq} \;=\; \{A \in \mathbb{S}^d \,:\, A \succcurlyeq 0\} \;=\; \{A \in \mathbb{S}^d \,:\, \mathbf{v}^T A \mathbf{v} \geq 0,\ \mathbf{v} \in \mathbb{R}^d\}$$

A spectrahedron is the slice of $\mathbb{S}^d_{\succcurlyeq}$ by an affine-linear subspace

$$L_{\mathbf{A}} \;=\; \{\mathbf{A}(\mathbf{x}) := A_0 + A_1 x_1 + \cdots + A_n x_n \,:\, \mathbf{x} \in \mathbb{R}^n\} \subseteq \mathbb{S}^d :$$

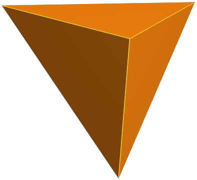
$$S_{\mathbf{A}} \;=\; \mathbb{S}^d_{\succcurlyeq} \cap L_{\mathbf{A}} \;\simeq\; \{\mathbf{x} \in \mathbb{R}^n \,:\, \mathbf{A}(\mathbf{x}) \succcurlyeq 0\}$$

In particular, spectrahedra are convex sets.

# Examples

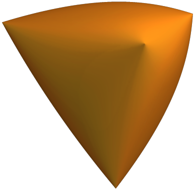
A spectrahedron is a "nonlinear" generalization of a polyhedron:

$$\mathbf{A}(\mathbf{x}) = \begin{pmatrix} a_1^0 & & 0 \\ & \ddots & \\ 0 & & a_d^0 \end{pmatrix} + x_1 \begin{pmatrix} a_1^1 & & 0 \\ & \ddots & \\ 0 & & a_d^1 \end{pmatrix} + x_n \begin{pmatrix} a_1^n & & 0 \\ & \ddots & \\ 0 & & a_d^n \end{pmatrix} = \begin{pmatrix} a_1(\mathbf{x}) & & 0 \\ & \ddots & \\ 0 & & a_d(\mathbf{x}) \end{pmatrix},$$



$$\{\mathbf{x} \in \mathbb{R}^n \,:\, a_1(\mathbf{x}), \ldots, a_d(\mathbf{x}) \geq 0\}$$

$$\mathbf{A}(\mathbf{x}) = \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & x_3 \\ x_2 & x_3 & 1 \end{pmatrix},$$



Elliptope

$$\{\mathbf{x} \in \mathbb{R}^3 \,:\, 1 - x_1^2,\ 1 - x_2^2,\ 1 - x_3^2,\ 1 - x_1^2 - x_2^2 - x_3^2 + 2x_1x_2x_3 \geq 0\}$$

# Spectrahedra, Sums Of Squares and optimization

- **Sum of squares representations of a real polynomial**

$$1+2t+3t^2+5t^4 \;=\; \begin{pmatrix}1 & t & t^2\end{pmatrix} A \begin{pmatrix}1\\ t\\ t^2\end{pmatrix},\; A \;=\; \begin{pmatrix}1 & 1 & a\\ 1 & 3-2a & 0\\ a & 0 & 5\end{pmatrix} \succcurlyeq 0$$

- **Semidefine programming**

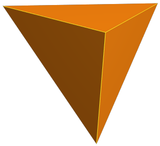
$$\min_{\mathbf{x}\in S_{\mathbf{A}}} \langle \boldsymbol{\ell}, \mathbf{x}\rangle \;=\; \min\{\langle \boldsymbol{\ell}, \mathbf{x}\rangle \;:\; \mathbf{x}\in\mathbb{R}^n,\, A_0 + A_1x_1 + \cdots + A_nx_n \succcurlyeq 0\}$$

| |
|---|
| Spectrahedra are feasible regions of semidefinite programs<br>(polyhedra) (linear) |

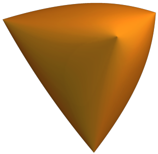
Particular case: **SOS approach to polynomial optimization**

$$\min_{\mathbf{t}\in\mathbb{R}^m} f(\mathbf{t}) \;\geq\; \max\left\{\lambda\in\mathbb{R} \;:\; f(\mathbf{t})-\lambda = \begin{pmatrix}1 & t_1 & \ldots\end{pmatrix} A \begin{pmatrix}1\\ t_1\\ \vdots\end{pmatrix}, A\succcurlyeq 0\right\}$$

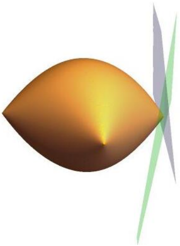
## Smoother than polyhedra but not enough...



Tetrahedron



Elliptope
(a.k.a. Somosa)



Pillow

3D spectrahedra (in general) carry singular points.

In semidefinite programming, there is a positive chance that a linear $\langle \ell, \mathbf{x} \rangle$, $\mathbf{x} \in S$, attains minimum at a singular point of $S$.

**General Problem**

Determine the number of singular points that a general $S$ can have.

# Homogeneous/projective setting

~~Affine~~ Linear space $L_{\mathbf{A}} \subseteq \mathbb{S}^d$:

$$L_{\mathbf{A}} \;=\; \{\mathbf{A}(\mathbf{x}) = A_0 x_0 + A_1 x_1 + \cdots + x_n A_n \;:\; \mathbf{x} \in \mathbb{R}^{n+1}\}.$$

Spectrahedral cone:

$$L_{\mathbf{A}} \cap \mathbb{S}^d_{\succcurlyeq} \;\simeq\; \{\mathbf{x} \in \mathbb{R}^{n+1} \;:\; \mathbf{A}(\mathbf{x}) \succcurlyeq 0\}$$

**Assumption**

$\mathbf{A}(\mathbf{x}) \succ 0$ for some $\mathbf{x} \in \mathbb{R}^{n+1}$ $\Rightarrow$ can take $A_0 = \mathbb{1}$

($L_{\mathbf{A}} \cap \mathbb{S}^d_{\succcurlyeq} \subset \mathbb{R}^{n+1}$ is full-dimensional) (orthogonal congruence applied to $L_{\mathbf{A}}$)

Projective spectrahedron:

$$S_{\mathbf{A}} \;=\; \mathbb{P}(L \cap \mathbb{S}^d_{\succcurlyeq}) \;\simeq\; \{[\mathbf{x}] \in \mathbb{R}\mathrm{P}^n \;:\; \mathbf{A}(\mathbf{x}) \succcurlyeq 0 \text{ or } -\mathbf{A}(\mathbf{x}) \succcurlyeq 0\}$$

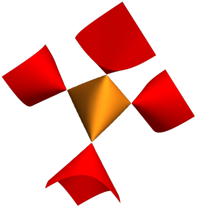
# (Algebraic) Boundary of spectrahedra

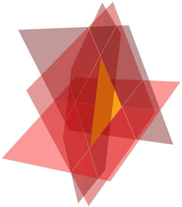
The Euclidean boundary of $S_{\mathbf{A}} \subset \mathbb{R}\mathrm{P}^n$:

$$\partial S_{\mathbf{A}} \;=\; \{[\mathbf{x}] \in S_{\mathbf{A}} \,:\, \det(\mathbf{A}(\mathbf{x})) = 0\}$$

For generic $A_1, \ldots, A_n \in \mathbb{S}^d$ the Zariski closure of $\partial S_{\mathbf{A}} \subset \mathbb{C}\mathrm{P}^n$ is a degree $d$ hypersurface, called *(spectrahedral) symmetroid*:

$$X_{\mathbf{A}} \;=\; \{[\mathbf{x}] \in \mathbb{C}\mathrm{P}^n \,:\, \det(\mathbb{1}x_0 + A_1x_1 + \cdots + A_nx_n) = 0\}$$





$S_{\mathbf{A}} \subset \mathbb{R}\mathrm{P}^n$ is a polyhedron $\Rightarrow$ $X_{\mathbf{A}} \subset \mathbb{C}\mathrm{P}^n$ is a set of hyperplanes.
$(A_1,\ldots,A_n$ are diagonal$)$

# Singular points

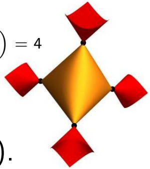
The variety $\Sigma \subset \mathbb{P}(\mathbb{S}^d_{\mathbb{C}})$ of matrices of corank $\geq 2$ has codimension 3.
For $n \geq 3$ any symmetroid

$$X_{\mathbf{A}} \ = \ \{[\mathbf{x}] \in \mathbb{C}\mathrm{P}^n \ : \ \det(\mathbb{1}x_0 + A_1x_1 + \cdots + A_nx_n) = 0\}$$

contains points of corank $\geq 2 \implies X_{\mathbf{A}}$ is singular.
n=3: surface $X_{\mathbf{A}} \subset \mathbb{C}\mathrm{P}^3$ defined by generic $A_1, A_2, A_3 \in \mathbb{S}^d$ has

$$\boxed{\binom{d+1}{3} = \#\mathrm{Sing}(X_{\mathbf{A}}) = \#L_{\mathbf{A}} \cap \Sigma = \deg(\Sigma)}$$

$\binom{3+1}{3} = 4$



singular points, which are all **nodal** (of multiplicity 2).

**Combinatorial type $(\rho, \sigma)$, $\sigma \leq \rho$**

$\rho = \#(\mathrm{Sing}(X_{\mathbf{A}}) \cap \mathbb{R}\mathrm{P}^3)$ is the number of real singularities of $X_{\mathbf{A}}$.
$\sigma = \#(\mathrm{Sing}(X_{\mathbf{A}}) \cap \partial S_{\mathbf{A}})$ is the number of singularities on $\partial S_{\mathbf{A}} \subset X_{\mathbf{A}}$.

## n=3 Reality questions

For generic $A_1, A_2, A_3 \in \mathbb{S}^d$ one has

$$\sigma := \#(\mathrm{Sing}(X_{\mathbf{A}}) \cap \partial S_{\mathbf{A}}) \ \leq \ \rho := \#(\mathrm{Sing}(X_{\mathbf{A}}) \cap \mathbb{R}\mathrm{P}^3) \ \leq \ \binom{d+1}{3},$$

$0 \leq \sigma$ is even, $\rho$ has the same parity as $\binom{d+1}{3}$ and
$0 < \rho$ (for all $X_{\mathbf{A}}$ with fixed $d$) unless $d = -1, 0, 1 \bmod 8$.

d=3: can have either $(\rho, \sigma) = (4, 4)$ or $(\rho, \sigma) = (2, 2)$.

Degtyarev and Itenberg, 2011: $(\rho, \sigma)$ is a combinatorial type of a (generic) quartic d=4 spectrahedron iff $0 \leq \sigma \leq \rho \leq 10 = \binom{4+1}{3}$, both are even and $2 \leq \rho$.

Ottem, Ranestad, Sturmfels and Vinzant, 2015: alternative proof.

**Question 17 in "3264 Questions about Symmetric Matrices" (Sturmfels)**

Classify combinatorial types of quintic spectrahedra in $\mathbb{R}^3$.
Can you realize all $20 = \binom{5+1}{3}$ complex singular points of the quintic symmetroid in the boundary of its spectrahedron?

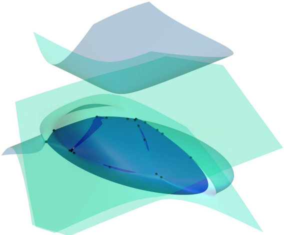
# Main result: classification for $\boxed{d = 5}$

**Brysiewicz, K. and Kummer, 2021**

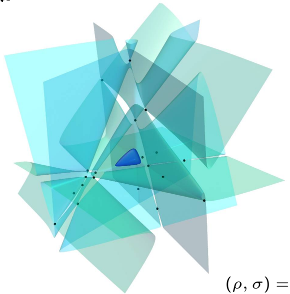
$(\rho, \sigma)$ is a combinatorial type of a generic quintic ($d = 5$) spectrahedron iff $0 \leq \sigma \leq \rho \leq 20$, both are even and $2 \leq \rho$.

**Proof strategy:**

- Understand restrictions on $(\rho, \sigma)$ — 65 possible types
- Find explicit representatives for each $(\rho, \sigma)$ numerically
- Certify the numerical answers



$(\rho, \sigma) = (20, 20)$



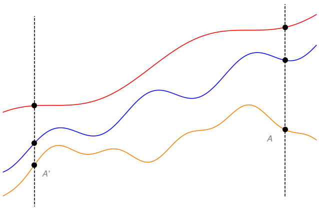
$(\rho, \sigma) = (20, 0)$

# Numerical algebraic geometry

Singular points of $X_{\mathbf{A}} \subset \mathbb{C}\mathrm{P}^3$ are (projective) zeros of the system:

$$F_{\mathbf{A}}: \quad \frac{\partial}{\partial x_i} \det(x_0 \mathbb{1} + x_1 A_1 + x_2 A_2 + x_3 A_3) \;=\; 0, \quad i = 0, \ldots, 3.$$

For generic **complex** matrices $A_1, A_2, A_3 \in \mathbb{S}^5_{\mathbb{C}}$, $F_{\mathbf{A}}$ has 20 zeros.
If we know solutions to a reference system $F_{\mathbf{A}'}$, can solve the desired system $F_{\mathbf{A}}$ using the method of homotopy continuation:



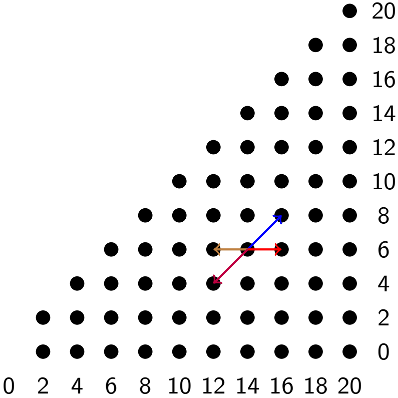
# Neighboring types

How to find $X_{\mathbf{A}} \subset \mathbb{C}\mathrm{P}^3$ with desired combinatorial types?

Generically, the type can change in one of the following 4 ways:

$(+,+)$: $(\rho,\sigma) \to (\rho+2,\sigma+2)$ $(+,0)$: $(\rho,\sigma) \to (\rho+2,\sigma)$

$(-,-)$: $(\rho,\sigma) \to (\rho-2,\sigma-2)$ $(-,0)$: $(\rho,\sigma) \to (\rho-2,\sigma)$




If we are able to find all possible neighboring types for each $(\rho,\sigma)$, then all 65 types can be found.

We achieve this goal using the hill-climbing algorithm

# Hill-climbing algorithm

Given $A_1, A_2, A_3 \in \mathbb{S}^5$, define the sets

- $S_{\mathbb{R}_+}(\mathbf{A})$: definite real nodes $\mathbf{A}(\mathbf{x}) \in \mathbb{S}^5$ of $X_\mathbf{A} \subset \mathbb{C}\mathrm{P}^3$
- $S_{\mathbb{R}_-}(\mathbf{A})$: indefinite real nodes $\mathbf{A}(\mathbf{x}) \in \mathbb{S}^5$ of $X_\mathbf{A} \subset \mathbb{C}\mathrm{P}^3$
- $S_{\mathbb{C}_+}(\mathbf{A})$: non-real nodes $\mathbf{A}(\mathbf{x})$ with definite real part $\mathrm{Re}(\mathbf{A}(\mathbf{x}))$
- $S_{\mathbb{C}_-}(\mathbf{A})$: non-real nodes $\mathbf{A}(\mathbf{x})$ with indefinite $\mathrm{Re}(\mathbf{A}(\mathbf{x}))$

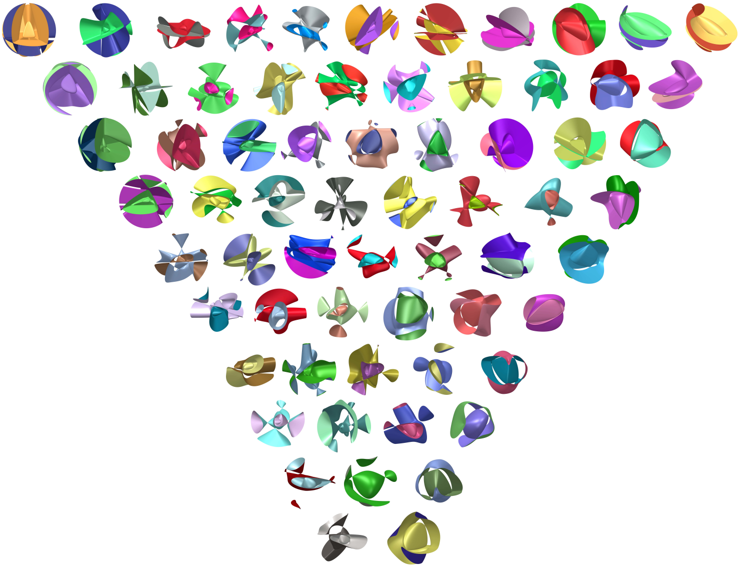
If $X_{\mathbf{A}'}$ has type $(\rho, \sigma)$, go for the 4 neighboring types as follows:

- randomly sample a few $\mathbf{A}$ near $\mathbf{A}'$,
- for each neighboring type that is not found in the sample, as a new $\mathbf{A}'$ choose that sampled $\mathbf{A}$ with the smallest:

$$
\begin{aligned}
(-,-): &\quad \min\{\|\mathbf{A}(\mathbf{x}) - \mathbf{A}(\tilde{\mathbf{x}})\| \,:\, \mathbf{A}(\mathbf{x}),\ \mathbf{A}(\tilde{\mathbf{x}}) \in S_{\mathbb{R}_+}(\mathbf{A})\} \\
(-,0): &\quad \min\{\|\mathbf{A}(\mathbf{x}) - \mathbf{A}(\tilde{\mathbf{x}})\| \,:\, \mathbf{A}(\mathbf{x}),\ \mathbf{A}(\tilde{\mathbf{x}}) \in S_{\mathbb{R}_-}(\mathbf{A})\} \\
(+,+): &\quad \min\{\|\mathrm{Im}(\mathbf{A}(\mathbf{x}))\| \,:\, \mathbf{A}(\mathbf{x}) \in S_{\mathbb{C}_+}(\mathbf{A})\} \\
(+,0): &\quad \min\{\|\mathrm{Im}(\mathbf{A}(\mathbf{x}))\| \,:\, \mathbf{A}(\mathbf{x}) \in S_{\mathbb{C}_-}(\mathbf{A})\}
\end{aligned}
$$

- keep repeating until all neighboring types of $(\rho, \sigma)$ are found...

# 65 combinatorial types



# Random spectrahedra

A random spectrahedron $S_{\mathbf{A}} = \mathbb{P}(L_{\mathbf{A}} \cap \mathbb{S}^d_{\succeq})$ is given by a (random)

$$L_{\mathbf{A}} \;=\; \left\{ \mathbb{1}x_0 + A_1 x_1 + \cdots + x_n A_n \;:\; \mathbf{x} \in \mathbb{R}^{n+1} \right\},$$

where matrices $A_1, \ldots, A_n \sim \mathrm{GOE}(d)$ are sampled independently from the Gaussian Orthogonal Ensemble

$$\mathrm{d}\mu_{\mathrm{GOE}(d)}(A) \;:=\; \frac{1}{Z_d} \exp\left( -\frac{1}{4} \mathrm{Tr}\, A^2 \right) \mathrm{d}A, \quad \mathrm{d}A = \prod_{1 \le i \le j \le d} \mathrm{d}a_{ij}.$$

Remark

The measure $\mu_{\mathrm{GOE}(d)}$ is invariant under $A \mapsto g^{\mathsf{T}} A g$, $g \in O(d)$.

The space $V_{\mathbf{A}} := \mathrm{Span}(A_1, \ldots, A_n)$ is distributed uniformly in the Grassmannian $\mathrm{Gr}_n(\mathbb{S}^d)$ of $n$-planes in $\mathbb{S}^d$.

# Expected number of singular points

Observation: With probability 1 the random spectrahedron $S_{\mathbf{A}}$ has at most $\sigma_{\mathbf{A}} \leq \rho_{\mathbf{A}} \leq \binom{d+1}{3}$ nodes on its (real algebraic) boundary.

**Breiding, K. and Lerario, 2019**

$$\mathop{\mathbb{E}}_{A_1,A_2,A_3 \sim \mathrm{GOE}(d)} \rho_{\mathbf{A}} = \binom{d}{2} = \frac{d(d-1)}{2}$$

$$\mathop{\mathbb{E}}_{A_i \sim \mathrm{GOE}(d)} \sigma_{\mathbf{A}} = \frac{2^{d-2}}{\sqrt{\pi}(d-2)!} \int_0^\infty \mathop{\mathbb{E}}_{B \sim \mathrm{GOE}(d-2)} \left[ \det(B - t\mathbb{1})^2 \,\middle|\, B - t\mathbb{1} \succcurlyeq 0 \right] e^{-t^2} dt$$

**Expected combinatorial type of a quartic spectrahedron**

$$\left( \mathop{\mathbb{E}}_{A_i \sim \mathrm{GOE}(4)} \sigma_{\mathbf{A}}, \mathop{\mathbb{E}}_{A_i \sim \mathrm{GOE}(4)} \rho_{\mathbf{A}} \right) = \left( 6 - \frac{4}{\sqrt{3}}, 6 \right) \approx (3.69, 6)$$

# Matrices with repeated eigenvalues

Variety of real symmetric matrices with repeated eigenvalues:

$$\Delta \ := \ \{A \in \mathbb{S}^d \ : \ \lambda_i(A) = \lambda_j(A) \text{ for some } i \neq j\}$$

Fact: $\Delta$ is a subvariety of $\mathbb{S}^d$ of codimension 2.

$$\mathrm{rk}(x_0 \mathbb{1} + A_1 x_1 + A_2 x_2 + A_3 x_3) \leq d - 2 \ \Leftrightarrow \ A_1 x_1 + A_2 x_2 + A_3 x_3 \in \Delta$$

With probability 1:

$$\rho_{\mathbf{A}} \ = \ \#\mathrm{Sing}(X_{\mathbf{A}} \cap \mathbb{R}\mathrm{P}^3) \ = \ \{[\mathbf{x}] \in \mathbb{R}\mathrm{P}^3 \ : \ \mathrm{rk}(\overset{x_0 \mathbb{1} + A_1 x_1 + A_2 x_2 + A_3 x_3}{\overset{\shortparallel}{\mathbf{A}(\mathbf{x})}}) \leq d - 2\}$$

$$= \ \{[x_1 : x_2 : x_3] \in \mathbb{R}\mathrm{P}^2 \ : \ A_1 x_1 + A_2 x_2 + A_3 x_3 \in \Delta\} \ = \ \#\mathbb{P}(\Delta \cap V_{\mathbf{A}})$$

Random matrices $A_1, A_2, A_3 \sim \mathrm{GOE}(d)$ are independent

$\Rightarrow$ $V_{\mathbf{A}} = \mathrm{Span}(A_1, A_2, A_3)$ is uniformly distributed in $\mathrm{Gr}_3(\mathbb{S}^d)$.

**Breiding, K. and Lerario, 2018**

$$\mathop{\mathbb{E}}_{V \in \mathrm{Gr}_3(\mathbb{S}^d)} \#\mathbb{P}(\Delta \cap V) \ = \ \frac{\mathrm{Vol}(\mathbb{P}(\Delta))}{\mathrm{Vol}(\mathbb{R}\mathrm{P}^{\dim \mathbb{P}(\Delta)})} \ = \ \binom{d}{2}$$

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$$\mathrm{rk}(x_0 \mathbb{1} + A_1 x_1 + A_2 x_2 + A_3 x_3) \leq d - 2 \iff A_1 x_1 + A_2 x_2 + A_3 x_3 \in \Delta$$

With probability 1:

$$\rho_{\mathbf{A}} = \#\mathrm{Sing}(X_{\mathbf{A}} \cap \mathbb{R}\mathrm{P}^3) = \{[\mathbf{x}] \in \mathbb{R}\mathrm{P}^3 : \mathrm{rk}(\mathbf{A}(\mathbf{x})) \leq d-2\}, \quad \mathbf{A}(\mathbf{x}) = x_0 \mathbb{1} + A_1 x_1 + A_2 x_2 + A_3 x_3$$

$$= \{[x_1 : x_2 : x_3] \in \mathbb{R}\mathrm{P}^2 : A_1 x_1 + A_2 x_2 + A_3 x_3 \in \Delta\} = \#\mathbb{P}(\Delta \cap V_{\mathbf{A}})$$

Random matrices $A_1, A_2, A_3 \sim \mathrm{GOE}(d)$ are independent

$\Rightarrow V_{\mathbf{A}} = \mathrm{Span}(A_1, A_2, A_3)$ is uniformly distributed in $\mathrm{Gr}_3(\mathbb{S}^d)$.

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# Merci pour votre attention!