# Polynomial system solving in applications

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LAAS BrainPOP Seminar

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- Solutions to these systems often have physical meaning
- This talk will outline techniques to solve these systems
- We will consider three examples
  1. Lagrange systems of polynomial optimization problems
  2. Gaussian mixture moment systems
  3. Power flow equations

# Set up

- Consider the variety defined by polynomials $f_1, \ldots, f_n \in \mathbb{R}[x_1, \ldots, x_n]$

$$\mathcal{V}(F) = \{x \in \mathbb{C}^n : f_1(x) = 0, \ldots, f_n(x) = 0\}$$

where $\dim(\mathcal{V}(F)) = 0$

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  where $\dim(\mathcal{V}(F)) = 0$

- **Goal:** Find all points in $\mathcal{V}(F)$

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## Homotopy continuation

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- Define $H_t := (1-t)H_S + tH_T$ and compute $H_t$ as $t \to 1$
  - Called following *homotopy paths*

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- Define $H_t := (1-t)H_S + tH_T$ and compute $H_t$ as $t \to 1$
    - Called following *homotopy paths*
- Typically use predictor-corrector methods
    - Predict: Take step along tangent direction at a point
    - Correct: Use Newton's method

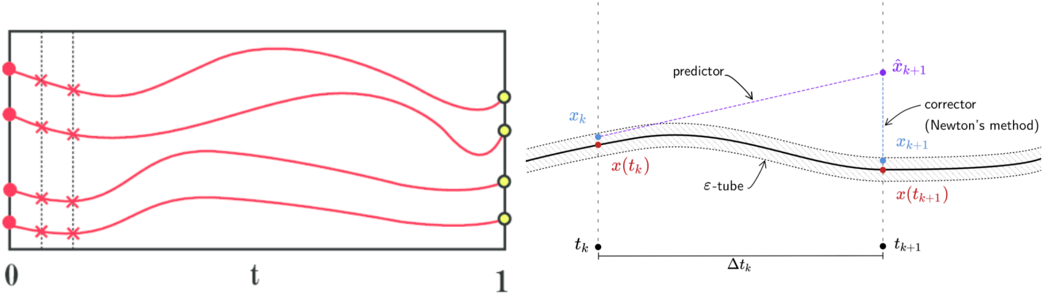


Figure: The homotopy $H_t = (1-t)H_S + tH_T$ (left)[KW14] and a predictor corrector step (right) [BT18]

# Homotopy continuation

## Start system

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  1. The solutions of $H_S$ are easy to find
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- Need to know $|\mathcal{V}(F)|$

**Theorem (Bezout)**

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$$H_S = \langle x_1^{d_1} - 1, \dots, x_n^{d_n} - 1 \rangle$$

- This bound is **generically** tight but can be a strict upper bound when $f_i$ are sparse

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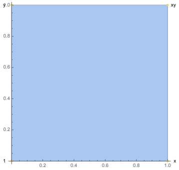
- Ex. If $f(x, y) = 1 + 2x - 3y + xy$, then $\mathrm{Newt}(f) = \mathrm{Conv}\{(0,0),(1,0),(0,1),(1,1)\}$



# Mixed Volume

- Given $k$ polytopes, $P_1, \ldots, P_k$ in $\mathbb{R}^n$ their *mixed volume* is

$$\mathrm{MVol}(P_1, \ldots, P_k) = \sum_{J \subseteq [k]} (-1)^{k-|J|} \cdot \mathrm{Vol}_n(P_J)$$

  where $P_J = P_{i_1} + \ldots + P_{i_{|J|}}$ and $J = (i_1, \ldots, i_{|J|})$

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- Ex. $k = n = 2$,

$$\mathrm{MVol}(P_1, P_2) = \mathrm{Area}(P_1 + P_2) - \mathrm{Area}(P_1) - \mathrm{Area}(P_2)$$

# Start system

## Polyhedral

**Theorem (BKK Bound [Ber75, Kho78, Kou76])**

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- There exists an algorithm that finds this binomial start system [HS95]

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- There exists an algorithm that finds this binomial start system [HS95]
- In general, not easy to compute the mixed volume (#P hard)

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# Example 1

## Critical points of polynomial optimization problems

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- Smooth critical points = solutions to polynomial system

$$\ell_i = \frac{\partial \mathcal{L}}{\partial x_i} = 0$$
$$f_i = \frac{\partial \mathcal{L}}{\partial \lambda_i} = 0$$

**Theorem (L., Nicholson, Rodriguez, Wang [LNRW21])**

*For generic $f_i, g$ the number of smooth critical points to*

$$\min_{x \in \mathbb{R}^n} g(x) \quad \textit{subject to} \quad f_i(x) = 0, \ i \in [m]$$

*is equal to* $\mathrm{MVol}(\ell_1, \ldots, \ell_n, f_1, \ldots, f_m)$. *Moreover, the number of critical points is dictated by* $\mathrm{Newt}(f_1), \ldots, \mathrm{Newt}(f_m), \mathrm{Newt}(g)$.

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- **Implication 1:** Polyhedral start systems will efficiently find all complex critical points for polynomial optimization problems
- **Implication 2:** Only consider monomials corresponding to vertices of Newton polytopes

# Example 1

## Critical points of polynomial optimization problems

- Consider

$$\min_{x\in\mathbb{R}^3} 3x_1 - x_2 + 2x_3 \quad \text{subject to} \quad x_1^2 - x_1x_2 + x_1x_3 - 2x_2^2 + 3x_2x_3 + 4x_3^2 - x_1 + 2x_2 - x_3 - 1 = 0$$

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- The Lagrange system is

$$\begin{aligned}
\ell_1 &= 3 - \lambda(2x_1 - x_2 + x_3 - 1) \\
\ell_2 &= -1 - \lambda(-x_1 - 4x_2 + 3x_3 + 2) \\
\ell_3 &= 2 - \lambda(x_1 + 3x_2 + 8x_3 - 1) \\
f &= x_1^2 - x_1x_2 + x_1x_3 - 2x_2^2 + 3x_2x_3 + 4x_3^2 - x_1 + 2x_2 - x_3 - 1
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- I claim there are only 2 critical points

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- Consider the same optimization problem restricted to the monomials corresponding to vertices of each Newton polytope:

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- Can explicitly solve this system and get 2 complex solutions, then use this as a start system for the general problem
- This generalizes to minimizing a linear function over any hypersurface

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# Example 2

## Gaussian mixture moment systems

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- **Problem:** Given $N$ samples distributed as mixture of $k$ Gaussians, recover mean $\mu_i$, variance, $\sigma_i^2$ and mixing coefficient, $\lambda_i$ of each component
- Many techniques to do this, consider **method of moments**

# Moments

## Definition

Let $f : \mathbb{R} \to \mathbb{R}$ be the probability density function of a random variable $X$. For $i \geq 0$, the $i-th$ *moment of* $X$ is

$$m_i = \mathbb{E}[X^i] = \int_{\mathbb{R}} x^i f(x) dx.$$

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- For parameterized distributions, moments are functions of parameters
- Ex. Suppose $X \sim \text{Unif}[a, b]$ where $-\infty < a < b < \infty$
- The first few moments are:

$$m_1 = \frac{1}{2}(a + b)$$
$$m_2 = \frac{1}{3}(a^2 + 2ab + b^2)$$
$$m_3 = \frac{1}{4}(b^3 + ab^2 + a^2 b + a^3)$$

- Consider a statistical model with $n$ unknown parameters, $\theta = (\theta_1, \dots, \theta_n)$ and the moments up to order $n$ as functions of $\theta$

$$m_1 = g_1(\theta), \dots, m_n = g_n(\theta)$$

and samples $y_1, \dots, y_N$

# Method of Moments

## Procedure

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- **Method of Moments:**
  1. Compute sample moments

$$\overline{m}_i = \frac{1}{N} \sum_{j=1}^{N} y_j^i$$

  2. Solve $g_i(\theta) = \overline{m}_i$ for $i = 1, \ldots, n$ to recover parameters

- The moments of the Gaussian distributions are $M_0(\mu, \sigma^2) = 1$, $M_1(\mu, \sigma^2) = \mu$,

$$M_\ell(\mu, \sigma^2) = \mu M_{\ell-1} + (\ell - 1)\sigma^2 M_{\ell-2}, \qquad \ell \geq 2$$

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## Gaussian Mixture Models

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- There are three special cases of Gaussian mixture models commonly studied in the statistics literature:
  1. The mixing coefficients are known
  2. The mixing coefficients are known and the variances are equal
  3. Only the means are unknown

# Gaussian Mixture Models

**Theorem (L., Améndola, Rodriguez [LAR21])**

*In all cases, Gaussian mixture models are algebraically identifiable using moment equations of lowest degree. Moreover, the mixed volume of each of set of equations is given below.*

| | Known mixing coefficients | Known mixing coefficients + equal variances | Unknown means |
|---|---|---|---|
| Moment equations | $m_1, \ldots, m_{2k}$ | $m_1, \ldots, m_{k+1}$ | $m_1, \ldots, m_k$ |
| Unknowns | $\mu_i, \sigma_i^2$ | $\mu_i, \sigma^2$ | $\mu_i$ |
| Mixed volume | $(2k-1)!!k!$ | $\frac{(k+1)!}{2}$ | $k!$ |
| Mixed volume tight | Yes for $k \leq 8$ | Yes for $k \leq 8$ | Yes |

| | Mixed Volume | Bezout Bound |
|---|---|---|
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- Our proofs of the mixed volume in the first two cases give a start system that tracks mixed volume number of paths
- In the final case if $\lambda_i = \frac{1}{k}$ and $\sigma_i^2$ are equal, there is a unique solution up to symmetry

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## Monodromy

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- Monodromy works by moving $b$ in a loop while tracking $x$ along that loop
- After one loop, usually find a new solution
- This process then is repeated until some stopping criterion is fulfilled

# Monodromy example

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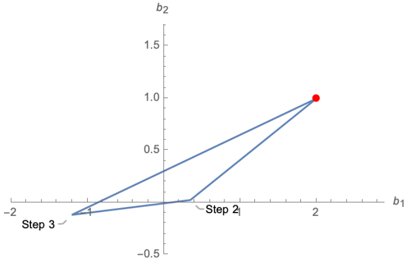
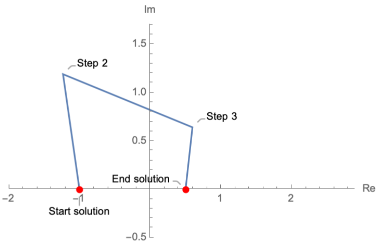
- For almost all $(b_1, b_2) \in \mathbb{C}^2$, $\{x \in \mathbb{C} \ : \ F_b(x) = 0\}$ has two elements
- Consider $(2, 1) \in \mathbb{C}^2$ and $x = -1$, then $F_{(2,1)}(-1) = 0$



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- This action is transitive if and only if the solution variety is irreducible
- **Major benefit:** Using monodromy we can solve up to symmetry [ALR21]

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# Example 3

## Power flow equations

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- These are the **power flow equations**

# Method 2

## Monodromy

### Proposition (**L.**, Boston, Lesieutre 2020 [LBL20])

- The power flow equations with $P_i \neq 0$ has a solution variety that is irreducible.
- The power flow equations with zero active power injections has a solution variety that decomposes into $2^{n-1} + 1$ irreducible components.
  - $2^{n-1}$ consist of a single point of the form $(x_i, y_i) = (\pm 1, 0)$ (trivial).
  - The remaining component consists of all nontrivial solutions to the power flow equations.

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## Lemma (**L.**, Zachariah, Boston, Lesieutre [LZBL20])

- For the power flow equations with zero active power injections, solutions come in pairs.
- For the power flow equations with zero active power injections on bipartite graphs, solutions come in sets of four.
- For the power flow equations with nonzero active power injections on bipartite graphs, solutions come in pairs.

# Comparing Monodromy

| | $K_5$ | $K_6$ | $K_7$ | $K_8$ | $K_9$ | $K_{10}$ |
|---|---|---|---|---|---|---|
| time (s): monodromy | 0.05 | 0.37 | 1.97 | 16.39 | 65.33 | 357.926 |
| time (s): polyhedral | 0.37 | 2.53 | 17.10 | 112.43 | 609.49 | 2637.22 |
| time (s): total degree | 0.21 | 1.45 | 8.17 | 48.78 | 329.60 | 1510.01 |

Table: Numerical results to find all solutions for complete networks

| | $C_5$ | $C_6$ | $C_7$ | $C_8$ | $C_9$ | $C_{10}$ |
|---|---|---|---|---|---|---|
| time (s): monodromy | 0.13 | 0.158 | 1.10 | 1.46 | 2.48 | 2.60 |
| time (s): polyhedral | 2.7 | 3.03 | 5.37 | 14.8 | 56.36 | 211.24 |
| time (s): total degree | 2.11 | 3.40 | 9.76 | 31.91 | 200.41 | 862.50 |

Table: Numerical results to find all solutions for cyclic networks

# Conclusion

- Outlined methods for polynomial system solving
- Considered three applications

**Questions!?**

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