# Polynomial Optimization Techniques for Energy Network Operation and Design

BrainPOP Seminar

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# Overview

1. AC Optimal Power Flow: Variants and Applications
   - AC Optimal Power Flow Problem
   - Optimal Reactive Power Dispatch Problem
   - Application to Network Design
2. Convex Relaxations for ACOPF problems based on Polynomial Optimization
   - Polynomial Optimization and OPF
   - Exploiting sparsity
     - A Relaxation with Moments up to degree 3

# AC Optimal Power Flow: Variants and Applications

# Power Flow Analysis

- Load flow or power flow analysis determines the operating state of a system for a given loading.
- Output of the analysis: voltage and phase angle, real and reactive power, line losses and slack bus power.

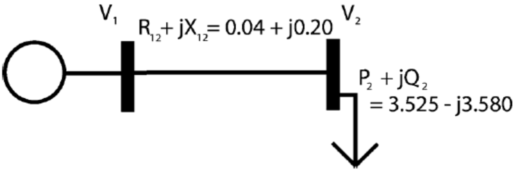
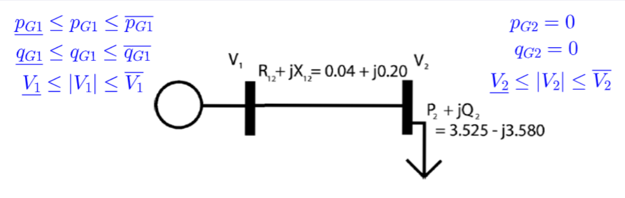


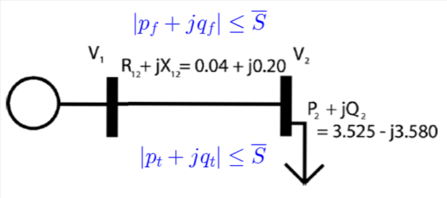

Figure: Two-bus system from [BGMT13]

# Power Flow Constraints

## Generator capacities and voltage magnitude



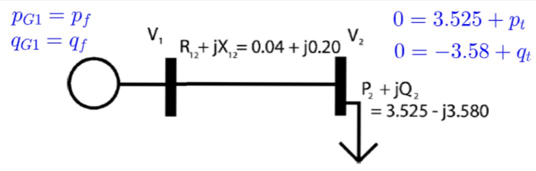

## Line thermal limits



# Power Flow Constraints

## Power balance equations

- Kirchhoff's Current Law: $\sum I_{\text{in}} = \sum I_{\text{out}}$




## Branch Flow Equations

If $y = \frac{1}{0.04+j0.20}$ is the admittance of the line, we have

$$V_1[y(V_1 - V_2)]^* = p_f + jq_f$$

$$V_2[y(V_2 - V_1)]^* = p_t + jq_t$$

# AC Optimal Power Flow Problem

- $\mathcal{P} = (\mathcal{N}, \mathcal{L})$, where $\mathcal{N} = \{1, 2, ..., n\}$ and $\mathcal{L} \subseteq \mathcal{N} \times \mathcal{N}$.
- $\min f(p_G, q_G, p_f, q_f, p_t, q_t, v)$ s.t.
- Gen. cap.: $\underline{p}_{G_g} \leq p_{G_g} \leq \overline{p}_{G_g}, \quad \underline{q}_{G_g} \leq q_{G_g} \leq \overline{q}_{G_g}, \quad \forall g \in \mathcal{G}$
- Line limits: $|p_{fl} + jq_{fl}| \leq \overline{s_l}, \quad |p_{tl} + jq_{tl}| \leq \overline{s_l}, \quad \forall l \in \mathcal{L}$
- Volt. magnitude: $\underline{v_k} \leq |v_k| \leq \overline{v}_k, \quad \forall k \in \mathcal{N}$
- Pow. balance:
  $$\sum_{g \in \mathcal{G}_k} p_{G_g} - p_{Dk} - g'_k |v_k|^2 = \sum_{l=(k,m) \in \mathcal{L}} p_{fl} + \sum_{l=(m,k) \in \mathcal{L}} p_{tl} \quad \forall k \in \mathcal{N},$$
  $$\sum_{g \in \mathcal{G}_k} q_{G_g} - q_{Dk} + b'_k |v_k|^2 = \sum_{l=(k,m) \in \mathcal{L}} q_{fl} + \sum_{l=(m,k) \in \mathcal{L}} q_{tl} \quad \forall k \in \mathcal{N},$$
- Branch flow:
  $$\frac{v_k}{t_l} \left[ \left( j\frac{b'_l}{2} + y_l \right) \frac{v_k}{t_l} - y_l v_m \right]^* = p_{fl} + jq_{fl} \quad \forall l = (k, m) \in \mathcal{L},$$
  $$v_m \left[ -y_l \frac{v_k}{t_l} + \left( j\frac{b'_l}{2} + y_l \right) v_m \right]^* = p_{tl} + jq_{tl} \quad \forall l = (k, m) \in \mathcal{L}.$$
- Angle diff: $|\angle v_k - \angle v_m| \leq \delta_l, \quad \forall l \in \mathcal{L}$
- Ref. bus: $\mathrm{Im}(v_1) = 0$

# Optimal Reactive Power Dispatch Problem

- A subset $\mathcal{U} \subseteq \mathcal{N}$ of the buses can be connected to a shunt element, this yields

$$\sum_{g \in \mathcal{G}_k} p_{G_g} - p_{Dk} - g'_k u_k |v_k|^2 = \sum_{l=(k,m)\in\mathcal{L}} p_{fl} + \sum_{l=(m,k)\in\mathcal{L}} p_{tl} \quad \forall k \in \mathcal{U},$$
$$\sum_{g \in \mathcal{G}_k} q_{G_g} - q_{Dk} + b'_k u_k |v_k|^2 = \sum_{l=(k,m)\in\mathcal{L}} q_{fl} + \sum_{l=(m,k)\in\mathcal{L}} q_{tl} \quad \forall k \in \mathcal{U},$$

$$u_k \in \{0,1\}$$

- Some buses on the network contain transformer and the tap ratio can be selected from a set of discrete values. The set of lines that have a transformer on the *from* end is denoted by $\mathcal{T} \subseteq \mathcal{L}$. We have

$$\frac{v_k}{t_l}\left[\left(j\frac{b'_l}{2} + y_l\right)\frac{v_k}{t_l} - y_l v_m\right]^* = p_{fl} + jq_{fl} \quad \forall l = (k,m) \in \mathcal{T},$$
$$v_m\left[-y_l\frac{v_k}{t_l} + \left(j\frac{b'_l}{2} + y_l\right)v_m\right]^* = p_{tl} + jq_{tl} \quad \forall l = (k,m) \in \mathcal{T}.$$

$$t_l \in \{\underline{t}_l, ..., \overline{t}_l\}$$

- According to safety regulations, DISTRIBUTION power networks may be required to be N-1 secured: should a transformer fail, there exists one network reconfiguration such that the service can be restored for the faulted section of the network using the N-1 remaining power sources.

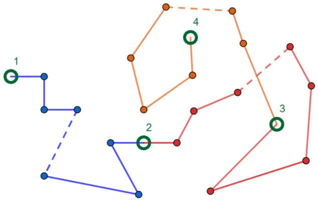



Figure: Initial state

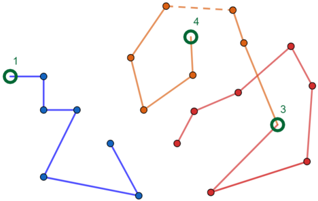



Figure: Failure of transformer T2

# Convex Relaxations for ACOPF problems based on Polynomial Optimization

# OPF as a real POP

- The ACOPF problem can be written as a complex or a real polynomial program on the voltage variables $v \in \mathbb{C}^n$.
- I will consider the formulation on real variables and, therefore, a polynomial program of the form:

$$\rho^* := \inf_x \{f(x) : x \in K\}$$

where $f(x) \in \mathbb{R}[x]$ is a polynomial and $K \subseteq \mathbb{R}^n$ is the basic semialgebraic set

$$K = \{x \in \mathbb{R}^n : g_j(x) \geq 0, j = 1, ..., m\},$$

for some polynomials $g_j(x) \in \mathbb{R}[x], j = 1, ..., m$.

- [Las09] $\rho^* = \rho_{\text{mom}}$ where

$$\rho_{\text{mom}} := \inf_{\mu \in \mathcal{M}(K)_+} \int_K f d\mu \quad \text{s.t.} \quad \int_K d\mu = 1.$$

# Moment Relaxations

- $\mathbb{N}^n_r := \{\alpha \in \mathbb{N}^n : |\alpha| \leq r\}$
- $L_y(f) = \sum_{\alpha \in \mathbb{N}^n} f_\alpha y_\alpha$,
  $y_\alpha := \int x^\alpha d\mu$.
- **Moment Matrix**

  $$M_r(\alpha, \beta) = L_y(x^\alpha x^\beta) = y_{\alpha+\beta},$$

  $\forall \alpha, \beta \in \mathbb{N}^n_r$.
- **Localizing Matrix** For $g \in \mathbb{R}[x]$ with coefficients $\{g_\gamma\}$,

  $$M_r(gy)(\alpha, \beta) = \sum_{\gamma \in \mathbb{N}^n} g_\gamma y_{\gamma+\alpha+\beta}$$

**Moment relaxation of degree $d$:**

$$\inf L_y(f)$$

$$\text{s.t.}$$

$$M_d(y) \succeq 0,$$

$$M_{d-d_j}(g_j y) \succeq 0, \quad j = 1, ..., m,$$

$$y_0 = 1.$$

$$d \geq \max\{\lceil \deg(f)/2 \rceil, \lceil \deg(g_j)/2 \rceil\}$$

# Application of Moment Relaxations to the OPF Problem

- Schor relaxation (first-order moment relaxation) has been well-studied in the context of OPF [Low14, LMBD11, LL11] including sparsity-exploiting approaches [Jab11, MHLD13].
- Schor relaxation is enough to solve a wide class of OPF test problems. However, modifications such as
  - Tightening of voltage bounds
  - Reduction of line flow limits
  - Reduction of active/reactive power demand

  could result on nonzero optimality gap for this relaxation and modified problems [JMPG14, MH14].
- Application of moment-sos hierarchy to OPF [JMPG14]: degree-2 relaxations deliver tight bounds on a broader class of test cases.
- Practical cases are large-scale: developing of sparsity-exploiting techniques for moment relaxations of degree $\geq 2$ and complex hierarchies [WMLM20, JM15, JM18, MH14, WM21].

# Matrices with chordal sparsity graph

- Let $A \in \mathbb{S}^n$. The undirected graph $G(N, E)$ with $N = \{1, ..., n\}$ is called *sparsity graph of A* when $(i,j) \in E$ if and only if $(A)_{ij} \neq 0$ [Koč20].
- For an index set $I \subset \{1, ..., n\}$ let

  $\mathbb{S}^n(I) := \{Y \in \mathbb{S}^n | (Y)_{ij} = 0 \text{ if } (i,j) \notin I \times I\}, \quad \mathbb{S}^n_+(I) := \{Y \in \mathbb{S}^n(I) | Y \succeq 0\}.$
- For an undirected graph with $N = \{1, ..., n\}$ and edge set $E \subseteq N \times N$:

$$\mathbb{S}^n(G) := \{Y \in \mathbb{S}^n | (Y)_{ij} = 0 \text{ if } (i,j) \notin E \cup \{(i,i)\}\}$$

$$\mathbb{S}^n_+(G) := \{Y \in \mathbb{S}^n(G) | Y \succeq 0\}.$$

## Theorem [Kak10]

Let $G(N, E)$ be an undirected graph with maximal cliques $C_1, ..., C_p$. The following two statements are equivalent:

(i) $G(N, E)$ is chordal.

(ii) For any $A \in \mathbb{S}^n(G)$, $A \succeq 0$, there are matrices $Y_k \in \mathbb{S}^n_+(C_k)$, $k = 1, ..., p$, such that $A = Y_1 + Y_2 + \cdots Y_p$.

# Correlative Sparsity

- How do we build the sparsity graph for a polynomial optimization problem?
- For the moment being, I restrict myself to the *correlative sparsity* technique.

**Correlative sparsity pattern (csp) graph [WMLM20]**

is the graph $G^{csp}$ with nodes $V = [n] := \{1, 2, ..., n\}$ and edges $E$ satisfying $\{i, j\} \in E$ is one of the following holds:

(i) there exists $\alpha \in \text{supp}(f)$ s.t. $\alpha_i > 0, \alpha_j > 0$

(ii) there exists $k$, with $1 \leq k \leq m$, s.t. $x_i, x_j \in \text{var}(g_k)$.

# Correlative Sparsity

- Let $\bar{G}^{csp}$ be a chordal extension of $G^{csp}$ and $I_l$,, $l = 1, ..., p$ be the maximal cliques of $\bar{G}^{csp}$.
- Constraints $g_1, ..., g_m$ are partitioned into groups $\{g_j | j \in J_l\}$, $l = 1, ...p$ which satisfy:
  - (i) $J_1, ..., J_p \subseteq [m] := \{1, 2, ..., m\}$ are pairwise disjoint and $\cup_{l=1}^{p} J_l = [m]$
  - (ii) for any $j \in J_l$, $\mathrm{var}(g_j) \subseteq I_l$, $l = 1, ..., p$.

## The moment-SOS hierarchy based on correlative sparsity [WMLM20]

$$
\begin{aligned}
&\inf L_y(f) \\
&\text{s.t.} \\
&M_d(y, I_l) \succeq 0, \quad l = 1, ..., p, \\
&M_{d-d_j}(g_j y, I_l) \succeq 0, \quad j \in J_l, l = 1, ..., p, \\
&y_0 = 1.
\end{aligned}
$$

# Scalability of CS-Hierarchy for Power Networks

- Authors in [MH14] mention computational tractability of a second-order relaxation for OPF problems with approximately 40 buses. However, authors in [WMLM20] report results for a second-order relaxation on cases 73_ieee_rts_api, 73_ieee_rts_sad, 240_pserc and 500_tamu_api of the PGLIB benchmark library [BBC+19] for OPF problems.
- Further reduction of the computational effort:
  - CS-TSSOS Hierarchy [WMLM20]: Application of *term sparsity* separately for each clique $I_l$ of the csp graph.
  - Iterative scheme of [MH14]: selective application of higher-order constraints.
- We next describe the latter approach since our work relies on this idea.

# Heterogeneous-degree Relaxations

- If the optimization problem has already been decomposed into subsystems, using CS for instance, it can be beneficial to define a different degree of the relaxation for each subsystem.
- In the context of OPF, [MH14] exploits the observation that the first-order relaxation is sufficient for large regions of typical OPF problems.
- Let const$(k)$ be the set of power generation and voltage magnitude constraints of node/bus $k \in \mathcal{N}$. Then, all constraints in const$(k)$ are contained in $J_l$ for some $1 \leq l \leq p$.
- Let $p_{Gk}^R$ be the active power injection of node $k$ implied by the solution of the relaxation.
- Let $p_{Gk}^P$ be the active power injection of node $k$ implied by the rank-1 projection of $M_d(y, I_l)$.
- Similarly, for the reactive power, $q_{Gk}^R$ and $q_{Gk}^P$.

# Iterative Solution for Moment Relaxation of [MH14]

- A mismatch is computed by means of

$$S_k^{\text{mis}} = \sqrt{(p_{Gk}^R - p_{Gk}^P)^2 + (q_{Gk}^R - q_{Gk}^P)^2}$$

in order to determine the set $H \subset \mathcal{N}$ such that the order of the relaxation for constraints const$(k)$, $k \in H$, should be increased.

## Iterative Solution for Moment Relaxation

Set $\gamma_k = 1 \quad \forall k \in \mathcal{N}$;
**repeat**
  Solve moment relaxation with $\gamma_k$;
  Calculate power injection mismatches;
  Increase the entries $\gamma$ according to heuristic
**until** *Tolerances are satisfyed*;

# Comments on the Performance of the Iterative Algorithm

| case | Opt. 1st | $f^*$ | Number of Higher Order Buses |
|---|---|---|---|
| 14Q | 3301.34 | 3301.80 | (2nd): 3 |
| 14L | 9353.53 | 9359.20 | (2nd): 4 |
| 39Q | 10804.08 | 11221.00 | (2nd): 31, (3rd): 2 |
| 39L | 41906.85 | 41907.47 | (2nd): 2 |
| 57Q | 7350.68 | 7351.82 | (2nd): 4 |
| 57L | 43907.63 | 43982.18 | (2nd): 2 |
| 118Q | 81427.75 | 81508.48 | (2nd): 4 |
| 118L | 133832.74 | 134903.92 | (2nd): 2 |
| 300 | 719701.55 | 719725.09 | (2nd): 2 |

- Justification to work on such a benchmark is the experiment conducted by randomly perturbing the cost function in the modified 118-bus system from [BGMT13]
- The default interior point solver in MATPOWER [ZMST10] either fails to converge or converges to a local optimum in 6.9% 10000 tested problems (using typical heuristics for the starting solution).

# A Relaxation with Moments up to degree 3

- Since the OPF problem is solely formed of quadratic constraints and objective function, a typical technique for reduction of the computational burden of moment relaxations is the elimination of odd-order moments in the implementation of SDP.
- However, including odd-order moments could be useful to build tight relaxations as we describe next.

# A Relaxation with Moments up to degree-3

- First, note that the reference bus constraint $\mathrm{Im}(v_1) = 0$ and the corresponding voltage magnitude constraint $\underline{v}_1 \leq |v_1| \leq \overline{v}_1$ allow us to break the symmetry of the problem by introduction of the constraint

$$\underline{v}_1 \leq \mathrm{Re}(v_1) \leq \overline{v}_1. \tag{1}$$

- All voltage variables are bounded by means of the voltage constraint $\underline{v_k} \leq |v_k| \leq \overline{v}_k, \quad \forall k \in \mathcal{N}$, and we introduce explicitly the constraints

$$-\overline{v_k} \leq \mathrm{Re}(v_k) \leq \overline{v}_k, \quad k \geq 2, k \in \mathcal{N} \tag{2}$$

$$-\overline{v_k} \leq \mathrm{Im}(v_k) \leq \overline{v}_k, \quad k \geq 2, k \in \mathcal{N} \tag{3}$$

- For simplicity, we will refer to constraints (1)-(3) as constraints $h_m \geq 0$, for $m = 1, ..., 4|\mathcal{N}| - 2$

# A Relaxation with Moments up to degree-3

- If the CS-first-order relaxation is not exact, we increase the order, without increasing the size of the matrices on which we impose SD constraints, as follows:

$$\inf L_y(f)$$

$$\text{s.t.}$$

$$M_1(y, I_l) \succeq 0, \quad l = 1, ..., p,$$

$$M_0(g_j y, I_l) \succeq 0, \quad j \in J_l, l = 1, ..., p,$$

$$y_0 = 1,$$

$$\underline{v}_1 \leq \mathrm{Re}(v_1) \leq \overline{v}_1.$$

$$M_1(h_m y, I_l) \succeq 0, \quad \mathrm{var}(h_m) \in I_l,\ m = 1, ..., 4|\mathcal{N}| - 2,\ l = 1, ..., p$$

$$L_y(h_m \cdot g_j) \geq 0, \quad j \in J_l,\ \mathrm{var}(h_m) \in I_l, m = 1, ..., 4|\mathcal{N}| - 2,\ l = 1, ..., p.$$

# A Relaxation with Moments up to degree-3

- We have a large number of variables which means that even if we do not increase the size of the Semidefinite-matrices on the program, we highly increase the number of semidefinite constraints, namely, because of $M_1(h_m y, I_l) \succeq 0$.

| case | $f^*$ | Opt. Relax. | Time (s) | Max. 2nd Eigval |
|---|---|---|---|---|
| 14Q | 3301.80 | 3301.80 | 9.56 | 5.14e-6 |
| 14L | 9359.20 | 9359.03 | 16.47 | 1.4e-3 |
| 39L | 41907.47 | 41907.47 | 47.73 | 1.99e-7 |
| 57Q | 7351.82 | 7351.82 | 1451 | 5.85e-5 |
| 57L | 43982.18 | 43982.14 | 1745 | 6.54e-6 |

# Can we use an iterative procedure?

- Since our approach considers to increase the order of the relaxation for constraints $h_m$ (as opposed to $g_j$ ), instead of selecting a node $k$ to increase the order of constraints const($k$) we can select one or various cliques $I_l$ for which we will add constraints $M_1(h_m y, I_l) \succeq 0$ and $L_y(h_m \cdot g_j) \geq 0$ for $j \in J_l$ and var($h_m$) $\in I_l$.
- Still in the process of exploring a different heuristic to find the cliques for which the order of the relaxation should be increased.

| case | $f^*$ | Opt. Relax. | Time (s) | Max. 2nd Eigval | Higher Order Cliques |
|---|---|---|---|---|---|
| 14Q | 3301.80 | 3301.80 | 5.48 | 2.4e-5 | 3/6 |
| 14L | 9359.20 | 9359.01 | 11.61 | 1.5e-3 | 4/6 |
| 39L | 41907.47 | 41907.47 | 3.77 | 1.61e-6 | 1/25 |
| 57L | 43982.18 | 43982.08 | 33.47 | 3.12e-6 | 4/36 |

# Work in Progress: using term sparsity

## Branch flow equations

$$\frac{v_k}{t_l}\left[\left(j\frac{b'_l}{2}+y_l\right)\frac{v_k}{t_l}-y_l v_m\right]^* = p_{fl}+jq_{fl} \quad \forall l=(k,m)\in\mathcal{L},$$

$$v_m\left[-y_l\frac{v_k}{t_l}+\left(j\frac{b'_l}{2}+y_l\right)v_m\right]^* = p_{tl}+jq_{tl} \quad \forall l=(k,m)\in\mathcal{L}.$$

**CS:**

| | Re($v_k$) | Re($v_m$) | Im($v_k$) | Im($v_m$) |
|---|---|---|---|---|
| Re($v_k$) | * | * | * | * |
| Re($v_m$) | * | * | * | * |
| Im($v_k$) | * | * | * | * |
| Im($v_m$) | * | * | * | * |

**TS:**

| | Re($v_k$) | Re($v_m$) | Im($v_k$) | Im($v_m$) |
|---|---|---|---|---|
| Re($v_k$) | * | * | | * |
| Re($v_m$) | * | * | * | |
| Im($v_k$) | | * | * | * |
| Im($v_m$) | * | | * | * |

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