LAAS-CNRS
Laboratoire d'analyse et d'architecture des systèmes du CNRS

Hyperbolic plane curves near the non-singular tropical limit

BrainPOP seminar

28 février 2022

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Équipe MAC

[LT21]

Hyperbolic plane curve

Definition

A real algebraic curve $\mathcal{C}$ of degree d in $\mathbb{P}^2$ is hyperbolic with respect to $p \in \mathbb{P}^2(\mathbb{R}) \backslash \mathcal{C}(\mathbb{R})$ if every real line $\mathcal{L} \subset \mathbb{P}^2$ going through p intersect $\mathcal{C}$ in d real points, counted with multiplicity.

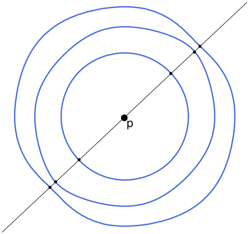

Figure – Hyperbolic curve defined by
$6(x^2+y^2-z^2)(x^2+y^2-2z^2)(x^2+y^2-3z^2)+x^3y^3=0.$

Topology of non-singular hyperbolic curves

Theorem ([Rok78],[HV07])

A non-singular real algebraic curve $\mathcal{C} \subset \mathbb{P}^2$ of degree d is hyperbolic if and only if its real part $\mathcal{C}(\mathbb{R})$ consists of $\lfloor \frac{d}{2} \rfloor$ nested ovals, plus a pseudo-line if d is odd.

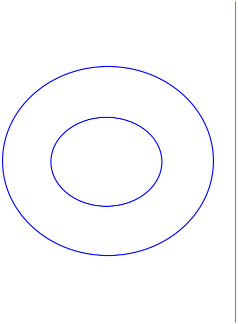

Figure – Topological type in degree 5.

Hyperbolicity locus

Definition

The hyperbolicity locus of a real algebraic curve $\mathcal{C} \subset \mathbb{P}^2$ is the set of points

$$\mathcal{H}_{\mathcal{C}} := \{p \in \mathbb{P}^2(\mathbb{R}) \backslash \mathcal{C}(\mathbb{R}) : \mathcal{C} \text{ hyperbolic with respect to } p\}.$$

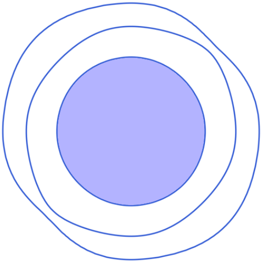

Figure – Hyperbolicity locus of the initial example.

Correspondence with Linear Matrix Inequalities

Definition

A closed convex set $\mathcal{H} \subset \mathbb{P}^2(\mathbb{R})$ of the form

$$\{p \in \mathbb{P}^2(\mathbb{R}) : L_0 p_0 + L_1 p_1 + L_2 p_2 \text{ is positive semi-definite}\},$$

with each L_i a symmetric matrix with real entries, is said to have a Linear Matrix Inequality (LMI) representation.

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Theorem ([HV07])

A closed convex set $\mathcal{H} \subset \mathbb{P}^2(\mathbb{R})$ admits a LMI representation if and only if $\mathcal{H}$ is the hyperbolicity locus of a hyperbolic curve $\mathcal{C} \subset \mathbb{P}^2$.

Fact ([Rok78], [HV07])

Let γ be a path in the space of degree d curves in $\mathbb{P}^2$ such that every curve in γ is non-singular.
If a curve $\mathcal{C}$ in γ is hyperbolic, then every curve in γ is hyperbolic.

Family of non-singular hyperbolic curves

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Let γ be a path in the space of degree *d* curves in $\mathbb{P}^2$ such that every curve in γ is non-singular.
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Goal

Characterise families of non-singular hyperbolic plane curves and their hyperbolicity loci in terms of their logarithmic limit.

Definition

Let $(\mathcal{C}_t)_{t\in]0,\varepsilon[}$ be a family of non-singular real algebraic curves in $(\mathbb{R}^\times)^2 \subset \mathbb{P}^2(\mathbb{R})$ defined by a continuous path $\gamma :]0,\varepsilon[\to X_d$, for X_d the space of real algebraic curves of degree d in $(\mathbb{R}^\times)^2$.

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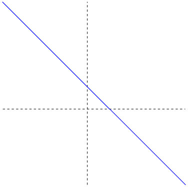
The logarithmic limit of $(\mathcal{C}_t)_t$ is the set $\lim_{t\to 0} \mathrm{Log}_t(\mathcal{C}_t)$, where Log_t is the map

$$\begin{aligned}(\mathbb{R}^\times)^2 &\to \mathbb{R}^2\\ (x,y) &\mapsto (\log_t|x|, \log_t|y|).\end{aligned}$$

Logarithmic limit of a family of lines



(a) Real line $\mathcal{L}_t$ in a family of real lines $(\mathcal{L}_t)_{t\in]0,\varepsilon[}$.

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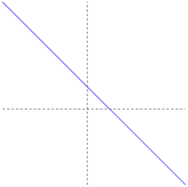

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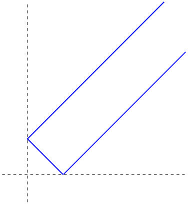
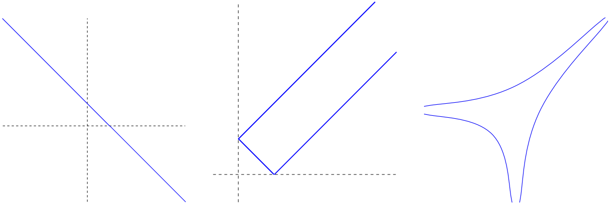


(b) Absolute value $|\mathcal{L}_t|$.



(a) Real line $\mathcal{L}_t$ in a family of real lines $(\mathcal{L}_t)_{t\in]0,\varepsilon[}$.

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(c) Logarithm $\mathrm{Log}_t(\mathcal{L}_t) \subset \mathbb{R}^2$.

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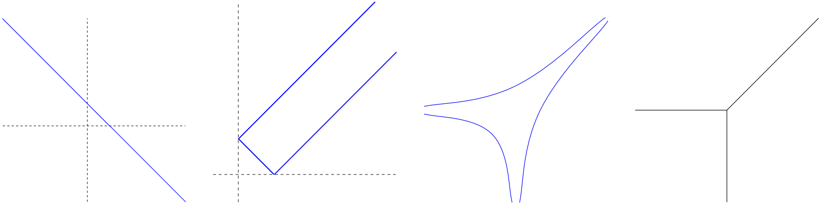


(a) Real line $\mathcal{L}_t$ in a family of real lines $(\mathcal{L}_t)_{t\in]0,\varepsilon[}$.

(b) Absolute value $|\mathcal{L}_t|$.

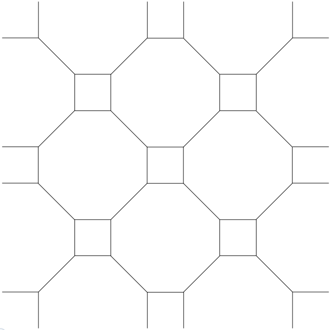
(c) Logarithm $\mathrm{Log}_t(\mathcal{L}_t) \subset \mathbb{R}^2$.

(d) Limit $\lim\limits_{t\to 0} \mathrm{Log}_t(\mathcal{L}_t)$.

Tropical curve

Definition

A non-singular tropical curve $C \subset \mathbb{R}^2$ is a 3-valent graph, with (potentially unbounded) straight edges of rational slope, and satisfying some balancing condition around its vertices.



Tropical curve as logarithmic limit

Theorem ([Rul01], [Mik04])

The logarithmic limit of a family of $(\mathcal{C}_t)_{t\in]0,\varepsilon[}$ of non-singular real algebraic curves in $(\mathbb{R}^\times)^2$ is a (possibly singular) tropical curve.
Conversely, every non-singular tropical curve $C \subset \mathbb{R}^2$ is the logarithmic limit of a family $(\mathcal{C}_t)_{t\in]0,\varepsilon[}$ of non-singular real algebraic curves in $(\mathbb{R}^\times)^2$.

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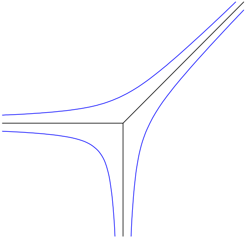
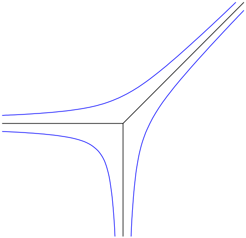
All the families $(\mathcal{C}_t)_{t\in]0,\varepsilon[}$ satisfying Mikhalkin-Rullgaard's theorem can be constructed explicitly.

Near the tropical limit



(a) Near the limit around a vertex.

Near the tropical limit



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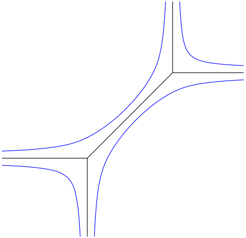
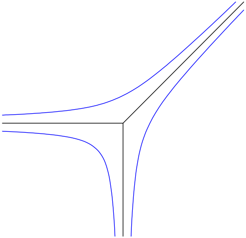


(b) Near the limit around a non-twisted bounded edge.

Near the tropical limit



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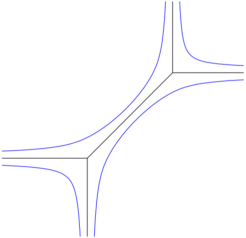
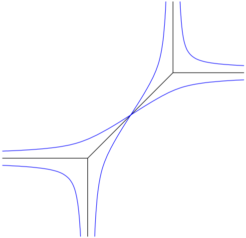
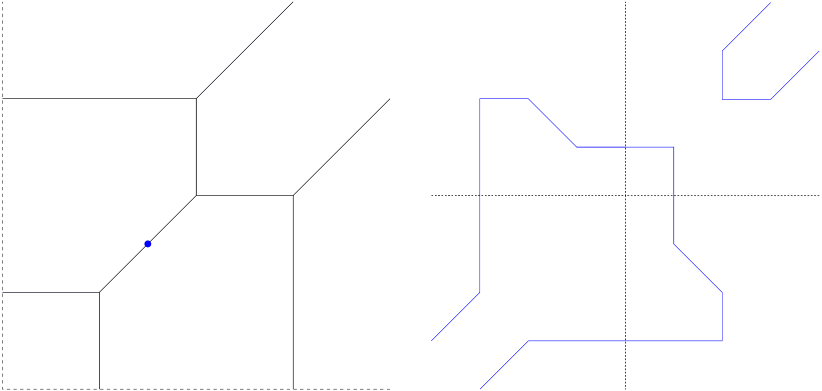


(b) Near the limit around a non-twisted bounded edge.



(c) Near the limit around a twisted bounded edge.

Real part of a tropical curve



(a) Tropical curve C, seen in $(\mathbb{R}_{>0})^2$, equipped with an admissible set of twisted edges T.

(b) Real part $\mathbb{R}C_T \subset (\mathbb{R}^\times)^2$ of C with respect to T, up to symmetry.

(A version of) Viro's patchworking

Theorem ([Vir01])

Let $C \subset \mathbb{R}^2$ be a non-singular tropical curve equipped with an admissible set of twisted edges T.
There exists a family $(\mathcal{C}_t)_{t\in]0,\varepsilon[}$ of non-singular real algebraic curves in $(\mathbb{R}^\times)^2$ such that

- C is the logarithmic limit of $(\mathcal{C}_t)_t$,

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- C is the logarithmic limit of $(\mathcal{C}_t)_t$,
- the set of twisted edges induced by $(\mathcal{C}_t)_t$ is T,
- and, up to symmetry, we have a homeomorphism of pair

$$((\mathbb{R}^\times)^2, \mathcal{C}_t(\mathbb{R})) \simeq ((\mathbb{R}^\times)^2, \mathbb{R}C_T).$$

Terminology

Definition

Let $C \subset \mathbb{R}^2$ be a non-singular tropical curve equipped with an admissible set of twisted edges T.
The couple (C, T) is a hyperbolic tropical curve if for every family $(\mathcal{C}_t)_{t\in]0,\varepsilon[}$ satisfying Viro's Theorem for (C, T), the curves $\mathcal{C}_t$ are all hyperbolic.

Proposition

Let $C \subset \mathbb{R}^2$ be a non-singular tropical curve of degree d, equipped with an admissible set of twisted edges T. The couple (C, T) is hyperbolic if and only if

1. every cycle of C contains an even number of twisted edges, and

Topological criterion for hyperbolicity

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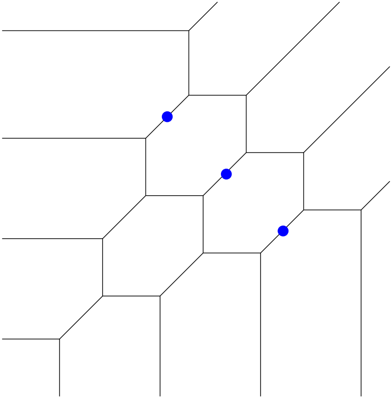
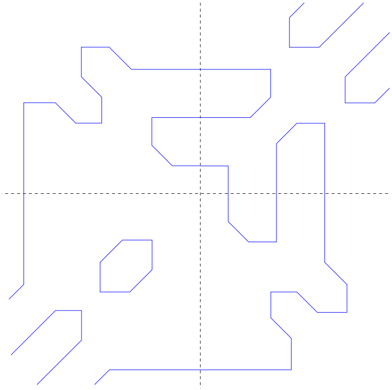
- Item 2 can be easily computed from the data of T (Renaudineau-Shaw 2021).
- No information on the hyperbolicity loci of the curves $\mathcal{C}_t$ obtained this way !

Example of hyperbolic tropical curve



(a) Hyperbolic tropical curve (C, T) of degree 4.



(b) Real part $\mathbb{R}C_T$ (up to symmetry) with 2 projective components.

Terminology, part 2

Definition

Let $C \subset \mathbb{R}^2$ be a non-singular tropical curve equipped with an admissible set of twisted edges T. Let v be a point in $\mathbb{R}^2 \backslash C$.

The couple (C, T) is hyperbolic with respect to v if for every family $(\mathcal{C}_t)_{t\in]0,\varepsilon[}$ satisfying Viro's Theorem for (C, T), there exists a family of points $(v_t)_{t\in]0,\varepsilon[}$ such that

- the curve $\mathcal{C}_t$ is hyperbolic with respect to v_t for all t, and
- the logarithmic limit of $(v_t)_t$ is v.

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(C, T) hyperbolic $\Leftrightarrow$ (C, T) hyperbolic with respect to some point v.

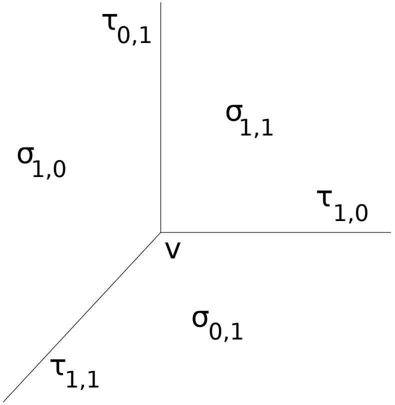

Figure – Subdivision of $\mathbb{R}^2$ with respect to *v*.

A pencil of tropical lines through a point *v* can be described by the rays τ_η. Each point on a ray τ_η is the vertex of a tropical line going through *v*.

Theorem ([LT21])

Let $C \subset \mathbb{R}^2$ be a non-singular tropical curve of degree d equipped with an admissible set of twisted edges T, and let v be a "generic" point in $\mathbb{R}^2 \backslash C$.
If the couple (C, T) is hyperbolic with respect to v, then

1. Every vertex of C in a face σ_η is incident to an edge of direction η.

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If the couple (C, T) is hyperbolic with respect to v, then

1. Every vertex of C in a face σ_η is incident to an edge of direction η.
2. Every bounded edge of C of direction η and strictly contained in σ_η is twisted.

Characterisation of hyperbolicity

Theorem ([LT21])

Let $C \subset \mathbb{R}^2$ be a non-singular tropical curve of degree d equipped with an admissible set of twisted edges T, and let v be a "generic" point in $\mathbb{R}^2 \backslash C$.
If the couple (C, T) is hyperbolic with respect to v, then

1. Every vertex of C in a face σ_η is incident to an edge of direction η.
2. Every bounded edge of C of direction η and strictly contained in σ_η is twisted.

We need a few more conditions to obtain equivalence, due to the definition "up to symmetry".

Example in degree 4

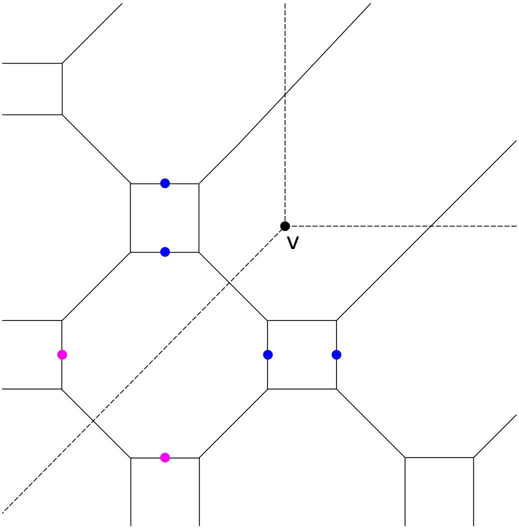

Figure – Couple (C, T) hyperbolic with respect to v.

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A honeycomb is a tropical curve with every edge of direction either $(1, 0)$, $(0, 1)$ or $(1, 1)$.

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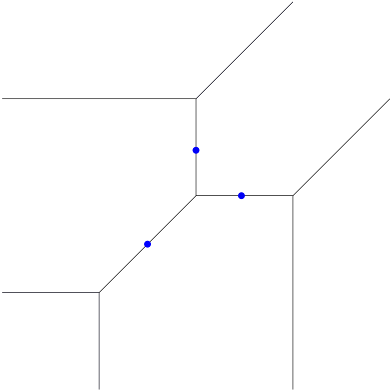
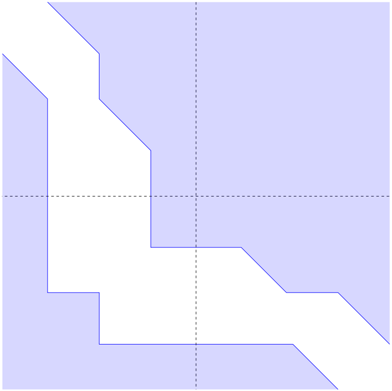
Theorem ([Spe05])

Let $C \subset \mathbb{R}^2$ be a non-singular tropical curve equipped with an admissible set of twisted edges T. The hyperbolicity locus of the real part $\mathbb{R}C_T$ contains an entire orthant of $(\mathbb{R}^\times)^2$ if and only if C is a honeycomb with every bounded edge twisted.



(a) Honeycomb of degree 2 with every bounded edge twisted.



(b) Real part with positive orthant contained in the hyperbolicity locus.

Definition

A multi-bridge on a honeycomb C is a set B of parallel edges disconnecting C, such that no proper subset of B disconnects C.

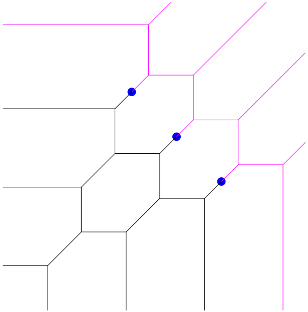
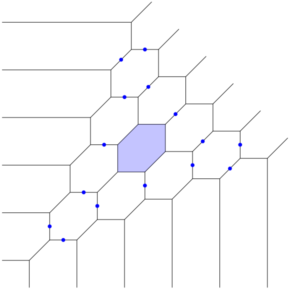


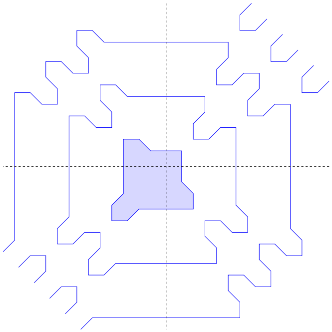
Figure – A multi-bridge on a honeycomb of degree 4.

Theorem ([LT21])

Let $C \subset \mathbb{R}^2$ be a non-singular honeycomb equipped with an admissible set of twisted edges T, such that every cycle of C contains an even number of twisted edges. Then (C, T) is hyperbolic with respect to v if and only if every multi-bridge strictly on the left, below and diagonally above v is twisted.

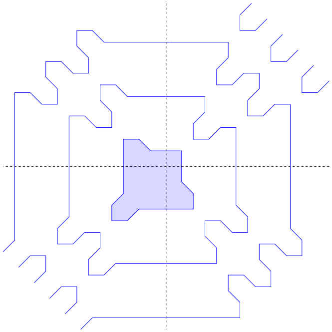


Signed tropical convexity



The hyperbolicity locus $\mathbb{R}H_T$ of a real part $\mathbb{R}C_T$ is TO-convex (in the sense of Loho-Végh, 2019), and its closure $\overline{\mathbb{R}H_T}$ is TC-convex (in the sense of Loho-Skomra, to appear).

Signed tropical convexity



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If C is a honeycomb, then $\overline{\mathbb{R}H_T}$ is given as a finite intersection of closed signed tropical halfspaces.

Tropical spectrahedron

By Helton-Vinnikov, every element of a family

$$\mathcal{H} := (\mathcal{H}_t)_{t\in]0,\varepsilon[}$$

of hyperbolicity loci in $\mathbb{P}^2(\mathbb{R})$ admits a LMI representation.

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Under suitable conditions, the set $\mathcal{H}$ is the hyperbolicity locus of a non-singular algebraic curve over real Puiseux series.

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In that case, the logarithmic limit

$$H := \lim_{t \to 0} \mathrm{Log}_t(\mathcal{H}_t)$$

is a tropical spectrahedron (in the sense of Allamigeon-Gaubert-Skomra 2020).

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Under suitable conditions again, the set H admits a tropical LMI representation.

Future directions

Questions

- Find an algorithm giving a tropical LMI representation of the tropical hyperbolicity locus.

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- For $(\mathcal{H}_t)_{t\in]0,\varepsilon[}$ a family of hyperbolicity loci with logarithmic limit a finite intersection of signed tropical halfspaces, what can we say on each $\mathcal{H}_t$?

Future directions

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- Find an algorithm giving a tropical LMI representation of the tropical hyperbolicity locus.
- For $(\mathcal{H}_t)_{t\in]0,\varepsilon[}$ a family of hyperbolicity loci with logarithmic limit a finite intersection of signed tropical halfspaces, what can we say on each $\mathcal{H}_t$?
- How can we characterise tropical hyperbolicity in higher dimension and codimension ?

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