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# Time-Varying Semidefinite Programming: Geometry of the Trajectory of Solutions

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# What is Time-Varying Semidefinite Programming?

Time-Varying Semidefinite Programming (TV-SDP) is linear optimization over the cone of positive semidefinite matrices where the problem data (and solutions) depends on time $t \in [t_0, t_f]$:

$$\begin{aligned} \min_{X \in \mathbb{S}^n} \quad & \langle C(t), X \rangle \\ \text{s.t.} \quad & \langle A_i(t), X \rangle = b_i(t), \quad i = 1, \ldots, m \\ & X \succeq 0 \end{aligned} \qquad (\mathrm{SDP}_t)$$

**Previous references**

- *On parametric semidefinite programming*, D. Goldfarb, K. Scheinberg, 1999
- *On computing the nonlinearity interval in parametric semidefinite optimization*, J. Hauenstein et al., 2019
- *Time-varying semidefinite programs*, A.A. Ahmadi, B. El Khadir, 2021

We wanted to geometrically characterize the **trajectory of solutions to TV-SDP** and classify the possible irregular behaviors, as in

- *Parametric Optimization*, J. Guddat, F. Guerra Vasquez, H.T. Jongen, 1990

# Main messages

**What have we done?**

This research resulted in a paper arxiv.org/abs/2104.05445, where our original contributions were:

- definition of **6 types of points** in the trajectory of solutions to TV-SDPs
- a classification theorem: **only** these types can appear
- under generic assumptions, only three types of points can appear

**Why is this useful?**

- we now know what in TV-SDP can go wrong and why
- extra knowledge for algorithms design
- prepares the way for future research on TV-POP

**Summary**

- Crash course on SDP properties and set-valued analysis
- Exposition of the 6 types of points
- The classification theorem
- Sketch of the proof
- An example

# Facial structure of spectrahedra

- A **spectrahedron** $\Sigma$ is an affine section of the PSD cone:

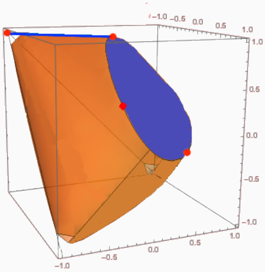
$$\Sigma = \mathbb{S}_+^n \cap \{Ax = b\}$$

- A **face** $F$ of a spectrahedron $\Sigma$ is a set such that

$$F = \Sigma \cap \mathcal{H}$$

  where $\mathcal{H}$ is a supporting hyperplane.
- An **extreme point** is a 0-dimensional face.

- The optimal set of an SDP is always a face.
- The solution is unique if and only if the optimal set is an extreme point
- Generically, the solutions to SDP are unique, hence extreme points



A spectrahedron $\Sigma$

# SDP PROPERTIES

For a primal-dual pair of SDPs $(P, D)$

$$\min_{X \in \mathbb{S}^n} \langle C, X \rangle \quad \text{s.t.} \quad \mathcal{A}[X] = b, \quad X \succeq 0 \tag{P}$$

$$\max_{y \in \mathbb{R}^m, Z \in \mathbb{S}^n} \langle b, y \rangle \quad \text{s.t.} \quad \mathcal{A}^*[y] + Z = C, \quad Z \succeq 0 \tag{D}$$

we have a set of first-order optimality conditions for $(P, D)$:

$$\boxed{\begin{aligned} &\mathcal{A}[X] = b \\ &\mathcal{A}^*[y] + Z = C \\ &X, Z \succeq 0 \\ &\langle X, Z \rangle = 0 \end{aligned}} \tag{KKT}$$

**Strict feasibility:** there exists a strictly feasible primal-dual point:

$$\boxed{\begin{aligned} &\mathcal{A}[X] = b \\ &\mathcal{A}^*[y] + Z = C \\ &X, Z \succ 0 \end{aligned}} \tag{SF}$$

***If strict feasibility holds the (KKT) conditions are necessary & sufficient for optimality.***

# Strict complementarity, non-degeneracy and uniqueness

**QUESTION:** how do we know that a primal-dual solution $(X, Z)$ is unique?

- **Strict complementarity:** $\operatorname{rank}(X) + \operatorname{rank}(Z) = n$
- **Primal non-degeneracy:** $\ker(\mathcal{A}) + T_X\mathbb{S}^n_+ = \mathbb{S}^n$
- **Dual non-degeneracy:** $\operatorname{span}(\mathcal{A}) + T_Z\mathbb{S}^n_+ = \mathbb{S}^n$

Linear-algebraic equivalent conditions for non-degeneracy are available.

In general:

| | | |
|---|---|---|
| **primal non-deg.** | $\Longrightarrow$ | **dual uniqueness** |
| **dual non-deg.** | $\Longrightarrow$ | **primal uniqueness** |

Under strict complementarity:

| | | |
|---|---|---|
| **primal non-deg.** | $\Longleftrightarrow$ | **dual uniqueness** |
| **dual non-deg.** | $\Longleftrightarrow$ | **primal uniqueness** |

Strict complementarity and non-degeneracy hold generically [Alizadeh et al., 1995]
$\Longrightarrow$ solution uniqueness holds generically.

# KKT invertibility

$$\boxed{\begin{aligned}&\mathcal{A}(X) = b\\ &\mathcal{A}^*(y) + Z = C\\ &\langle X, Z\rangle = 0\\ &X, Z \succeq 0\end{aligned}} \quad \longrightarrow \quad \boxed{\begin{gathered} F(X, y, Z, \bar{t}) := \begin{pmatrix} \tilde{\mathcal{A}}\,\mathrm{vec}(X) - b \\ \tilde{\mathcal{A}}^T y + \mathrm{vec}(Z) - \mathrm{vec}(C) \\ \frac{1}{2}\,\mathrm{vec}\,(XZ + ZX) \end{pmatrix} = 0 \\ X, Z \succeq 0 \end{gathered}}$$

where $\tilde{\mathcal{A}} := (\mathrm{svec}(A_1), \ldots, \mathrm{svec}(A_m))^T$.

If strict complementarity and primal-dual non-degeneracy hold

$$J_F(X, y, Z, t) = \begin{pmatrix} \tilde{\mathcal{A}} & 0 & 0 \\ 0 & \tilde{\mathcal{A}}^T & I_{\frac{1}{2}(n+1)n} \\ Z \otimes_s I_n & 0 & I_n \otimes_s X \end{pmatrix}$$

is invertible [Alizadeh et al., 1998].

# Our objects

For a primal-dual pair of TV-SDPs $(P_t, D_t)$ with $t \in T = [t_0, t_f]$

$$\begin{aligned} \min_{X \in \mathbb{S}^n} \ & \langle C(t), X \rangle \\ \text{s.t.} \ & \mathcal{A}(t)[X] = b(t) \\ & X \succeq 0 \end{aligned} \qquad (\mathrm{P}_t)$$

$$\begin{aligned} \max_{y \in \mathbb{R}^m,\, Z \in \mathbb{S}^n} \ & \langle b(t), y \rangle \\ \text{s.t.} \ & \mathcal{A}^*(t)[y] + Z = C(t) \\ & Z \succeq 0 \end{aligned} \qquad (\mathrm{D}_t)$$

we define the primal and dual **feasible set maps**:

$$\mathcal{P}(t) = \{X \in \mathbb{S}^n \mid \mathcal{A}(t)[X] = b(t),\ X \succeq 0\}$$
$$\mathcal{D}(t) = \{(y, Z) \in \mathbb{R}^m \times \mathbb{S}^n \mid \mathcal{A}^*(t)[y] + Z = C(t),\ Z \succeq 0\}$$

the primal and dual **optimal value functions**:

$$p^*(t) = \min_{X \in \mathbb{S}^n} \{\langle C(t), X \rangle \mid \mathcal{A}(t)[X] = b(t),\ X \succeq 0\}$$
$$d^*(t) = \max_{y \in \mathbb{R}^n,\, Z \in \mathbb{S}^n} \{\langle b(t), y \rangle \mid \mathcal{A}^*(t)[y] + Z = C(t),\ Z \succeq 0\}$$

and the primal and dual **optimal set maps**:

$$\mathcal{P}^*(t) = \{X \in \mathcal{P}(t) \mid \langle C(t), X \rangle = p^*(t)\}$$
$$\mathcal{D}^*(t) = \{(y, Z) \in \mathcal{D}(t) \mid \langle b(t), y \rangle = d^*(t)\}$$

A **set-valued map** $F$ from a set $T$ to another set $X$ maps a point in $t \in T$ to a non-empty subset $F(t) \subseteq X$:

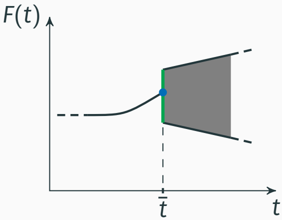
$$F: T \rightrightarrows X$$
$$t \mapsto F(t) \subseteq X$$

The **inner limit** of $F$ for $t$ that goes to $\bar{t}$ is

$$\liminf_{t \to \bar{t}} F(t) := \{\bar{x} \mid \forall \{t_k\}_{k=1}^{\infty} \subseteq T \text{ such that } t_k \to \bar{t},\ \exists \{x_k\}_{k=1}^{\infty} \subseteq X,\ x_k \to \bar{x} \text{ and } x_k \in F(t_k)\}$$

while its **outer limit** for $t$ that goes to $\bar{t}$ is

$$\limsup_{t \to \bar{t}} F(t) := \{\bar{x} \mid \exists \{t_k\}_{k=1}^{\infty} \subseteq T \text{ such that } t_k \to \bar{t},\ \exists \{x_k\}_{k=1}^{\infty} \subseteq X,\ x_k \to \bar{x} \text{ and } x_k \in F(t_k)\}$$



## Painlevé-Kuratowski continuity

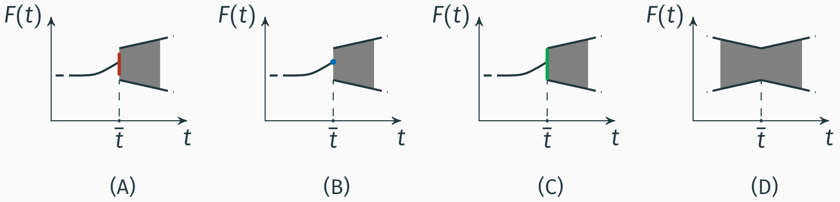
A set-valued map $F : T \rightrightarrows X$ is **inner semi-continuous** at $\bar{t} \in T$ if

$$\liminf_{t \to \bar{t}} F(t) = F(\bar{t})$$

while is **outer semi-continuous** at $\bar{t} \in T$ if

$$\limsup_{t \to \bar{t}} F(t) = F(\bar{t})$$

A set-valued map $F : T \rightrightarrows X$ is **Painlevé-Kuratowski continuous** at $\bar{t}$ if it is both inner and outer semi-continuous at $\bar{t}$.



# Continuity results

**Theorem 1 [e.g. Rockafellar and Wets, 2009]**
If the problem data are continuous functions of time, the primal and dual feasible set maps $\mathcal{P}(t), \mathcal{D}(t)$ are outer semi-continuous.

**Theorem 2 [Hauenstein et al., 2019; Bellon et al., 2021]**
If the problem data are continuous functions of time, strict feasibility holds, the linear constraints are linearly independent, and the operator $\mathcal{A}$ is uniformly bounded in $t$, the primal and dual feasible set maps $\mathcal{P}(t), \mathcal{D}(t)$ are inner semi-continuous.

**Theorem 3 [Hogan, 1974]**
If the problem data are continuous functions of time, the primal and dual optimal set maps $\mathcal{P}^*(t), \mathcal{D}^*(t)$ are outer semi-continuous. In particular, if $\mathcal{P}^*(t), \mathcal{D}^*(t)$ are single valued then they are continuous.

**Theorem 4 [e.g. Rockafeller and Wets, 2009]**
The primal and dual optimal set maps $\mathcal{P}^*(t), \mathcal{D}^*(t)$ can fail to be inner semi-continuous. The subset of points $t \in T$ at which $\mathcal{P}^*(t)$ or $\mathcal{D}^*(t)$ fails to be continuous it is a meager set, i.e. the union of countably many sets that are nowhere dense in T, in particular, it has empty interior.

**Can we say more?**

# Our assumptions

For a primal-dual pair of TV-SDPs $(P_t, D_t)$ with $t \in T = [t_0, t_f]$

$$\begin{aligned} \min_{X \in \mathbb{S}^n} \ & \langle C(t), X \rangle \\ \text{s.t.} \ & \mathcal{A}(t)[X] = b(t) \qquad (\mathrm{P}_t) \\ & X \succeq 0 \end{aligned}$$

$$\begin{aligned} \max_{y \in \mathbb{R}^m,\, Z \in \mathbb{S}^n} \ & \langle b(t), y \rangle \\ \text{s.t.} \ & \mathcal{A}^*(t)[y] + Z = C(t) \qquad (\mathrm{D}_t) \\ & Z \succeq 0 \end{aligned}$$

we assume that the following assumptions hold:

**LICQ and uniform boundedness of $\mathcal{A}$:** for every $t \in T$, the $m$ matrices $\{A_i(t)\}_{i=1,\dots,m}$ defining $\mathcal{A}(t)$ are linearly independent in $\mathbb{S}^n$, so that $\mathcal{A}(t)$ is surjective. The norm of $\mathcal{A}(t)$ is uniformly bounded.

**Strict feasibility:** for every $t \in T$, problem $(\mathrm{P}_t)$ and its dual $(\mathrm{D}_t)$ are strictly feasible.

**Data differentiability:** data $b(t)$ and $C(t)$ are continuous functions of the time parameter $t$.

# Types of Points summary

**GOAL:** classify points $(X^*, t^*)$ such that $X^* \in \mathcal{P}^*(t^*)$

**OUR APPROACH:** consider a trajectory of solution to TV-SDP in primal (or dual) form given by a smooth branch of a single-valued curve:

$$t \mapsto X^*(t) \in \mathbb{S}^n$$

**QUESTION 1:** how can be sure that a solutions trajectory behave so well?
$\longrightarrow$ strict complementarity and primal-dual non-degeneracy

**QUESTION 2:** how can this good behavior be affected?

(I) regular points

(II) loss of differentiability points

(III) discontinuous isolated multiple points

(IV) discontinuous non-isolated loss of multiple points

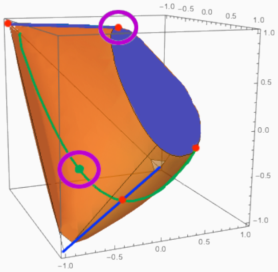
(V) continuous bifurcations points

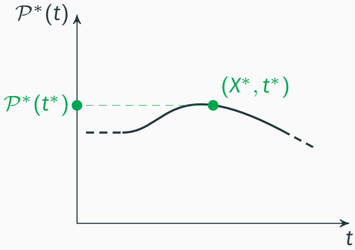
(VI) irregular accumulation points

# Types of points (I)

A **regular point** $(X^*, t^*)$ is such that $\mathcal{P}^*(t^*) = \{X^*\}$ and there exists $\varepsilon > 0$ such that

- $\mathcal{P}^*(t)$ is single-valued and continuous for every $t \in (t^* - \varepsilon, t^* + \varepsilon)$
- $\mathcal{P}^*(t)$ is differentiable at $t^*$

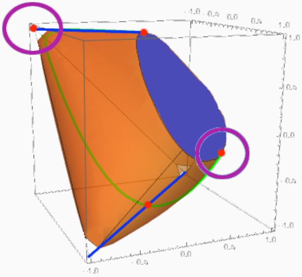


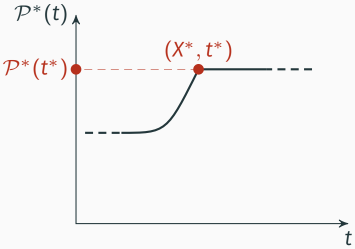



# Types of points (II)

A **non-differentiable point** $(X^*, t^*)$ is such that $\mathcal{P}^*(t^*) = \{X^*\}$ and there exists $\varepsilon > 0$ such that

- $\mathcal{P}^*(t)$ is single-valued and continuous for every $t \in (t^* - \varepsilon, t^* + \varepsilon)$
- $\mathcal{P}^*(t)$ is not differentiable at $t^*$

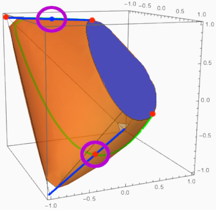


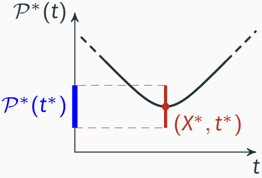


# Types of points (III)

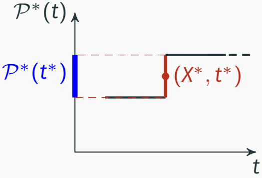
A **discontinuous isolated multiple point** $(X^*, t^*)$ is such that $X^* \in \mathcal{P}^*(t^*)$ and there exists $\varepsilon > 0$ such that

- $\mathcal{P}^*(t)$ is single-valued and continuous for every $t \in (t^* - \varepsilon, t^*) \cup (t^*, t^* + \varepsilon)$
- $\mathcal{P}^*(t)$ is multi-valued at $t^*$





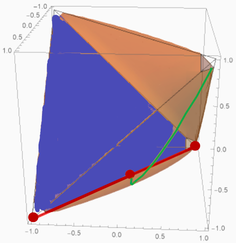

(A)

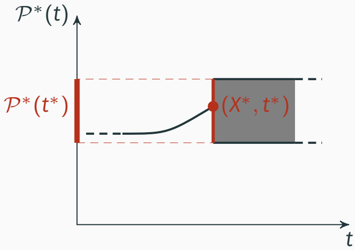



(B)

A **discontinuous non-isolated multiple point** $(X^*, t^*)$ is such that $X^* \in \mathcal{P}^*(t^*)$ and there exists $\varepsilon > 0$ such that

- $\mathcal{P}^*(t)$ is continuous at any $t \in (t^* - \varepsilon, t^*) \cup (t^*, t^* + \varepsilon)$
- $\mathcal{P}^*(t)$ is single-valued for every $t \in (t^* - \varepsilon, t^*)$
- $\mathcal{P}^*(t)$ is multi-valued for every $t \in [t^*, t^* + \varepsilon)$

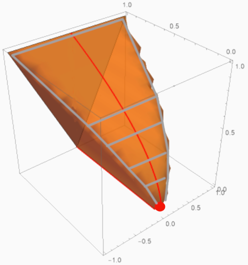


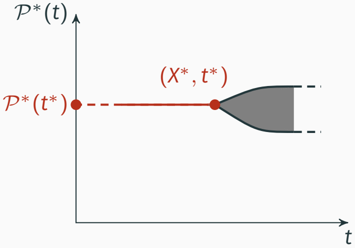


# Types of points (V)

A **continuous bifurcation point** $(X^*, t^*)$ is such that $\mathcal{P}^*(t^*) = \{X^*\}$ and there exists $\varepsilon > 0$ such that

- $\mathcal{P}^*(t)$ is continuous at any $t \in (t^* - \varepsilon, t^* + \varepsilon)$
- $\mathcal{P}^*(t)$ is single-valued for every $t \in (t^* - \varepsilon, t^*]$
- $\mathcal{P}^*(t)$ is multi-valued for every $t \in (t^*, t^* + \varepsilon)$



# Types of points (VI)

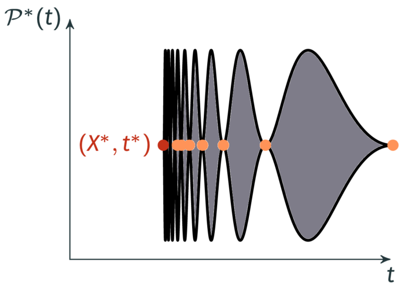
An **irregular accumulation point** $(X^*, t^*)$ is such that $X^* \in \mathcal{P}^*(t^*)$ and there exists $\varepsilon > 0$ such that $\mathcal{P}^*(t)$ is single-valued and continuous for every $t \in (t^* - \varepsilon, t^*)$ and for any $\delta > 0$ there exists $\{t_k\}_{k=1}^{\infty} \subseteq (t^*, t^* + \delta)$ such that

- either a continuous bifurcation or a loss of inner semi-continuity occurs at any $t_k$
- $\lim_{k\to\infty} t_k = t^*$

$$\min z$$

$$\text{s.t.} \quad \begin{pmatrix} 1 & x & y & 0 & 0 \\ x & 1 & z & 0 & 0 \\ y & z & 1 & 0 & 0 \\ 0 & 0 & 0 & 2h(t) & x-y \\ 0 & 0 & 0 & x-y & 2h(t) \end{pmatrix} \succeq 0$$

$$\text{where} \quad h(t) := \begin{cases} t\sin^2 \frac{\pi}{t} & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$$



# Main theorem

**Theorem [Bellon et al. 2021]**
For a primal-dual pair of TV-SDPs $(P_t, D_t)$, $t \in T = [t_0, t_f]$, satisfying LICQ, uniform boundedness of $\mathcal{A}$, strict feasibility, and data are continuous, then:

(i) there can only be points of type (I), (II), (III), (IV), (V), (VI)

(ii) if the data dependence is polynomial and at some $\hat{t} \in T$ all the points $(X, y, Z)$ satisfying

$$\begin{pmatrix} \mathcal{A}(\hat{t})[X] - b(\hat{t}) \\ \mathcal{A}^*(\hat{t})[y] + Z - C(\hat{t}) \\ \langle X, Z \rangle \end{pmatrix} = 0$$

are strictly complementary and primal-dual non-degenerate, only these points can appear:

- regular points (I)
- non-differentiable points (II) *in a finite number*
- discontinuous isolated multiple points (III) *in a finite number*

# PROOF OF MAIN THEOREM (PART II)

We consider a trajectory of solution to TV-SDP form given by a smooth branch of a single-valued curve:

$$(\hat{t}-\varepsilon, \hat{t}) \ni t \mapsto X^*(t) \in \mathbb{S}^n$$

We combine two logical cases partition:

**A** $\mathcal{P}^*(\hat{t})$ is a single-valued
**B** $\mathcal{P}^*(\hat{t})$ is multi-valued

**1** $\exists \varepsilon > 0$ such that $\mathcal{P}^*(t)$ is single-valued for every $t \in (\hat{t}, \hat{t}+\varepsilon)$
**2** $\exists \varepsilon > 0$ such that $\mathcal{P}^*(t)$ is multi-valued for every $t \in (\hat{t}, \hat{t}+\varepsilon)$
**3** $\forall \delta > 0 \; \exists t', t'' \in (\hat{t}, \hat{t}+\delta)$ such that $\begin{cases} \mathcal{P}^*(t') \text{ is single-valued} \\ \mathcal{P}^*(t'') \text{ is multi-valued} \end{cases}$

**A1:** $\begin{cases} \text{regular point} \\ \text{non-differentiable point} \end{cases}$

**A2:** $\begin{cases} \text{continuous bifurcation point} \\ \text{irregular accumulation point} \end{cases}$

**A3:** irregular accumulation point

**B1:** discontinuous isolated multiple point

**B2:** $\begin{cases} \text{discontinuous non-isolated multiple point} \\ \text{irregular accumulation point} \end{cases}$

**B3:** irregular accumulation point

**Lemma A**
For a pair $(P_t, D_t)$ of TV-SDPs with data continuously differentiable in $t \in T$ suppose that $(X^*, y^*, Z^*)$ is a strictly complementary primal-dual non-degenerate optimal solution for $(P_{\hat{t}}, D_{\hat{t}})$ for at $\hat{t} \in T$.

Then there exists $\varepsilon > 0$ and a unique continuously differentiable curve $(X^*(\cdot), y^*(\cdot), Z^*(\cdot))$ defined on $(\hat{t} - \varepsilon, \hat{t} + \varepsilon)$ such that $(X^*(t), y^*(t), Z^*(t))$ is the unique strictly complementary optimal solution for $(P_t, D_t)$ for all $t \in (\hat{t} - \varepsilon, \hat{t} + \varepsilon)$.

*Sketch of proof*

- KKT conds. $\equiv \begin{cases} F(X, y, Z, t) = 0 \\ X, Z \succeq 0 \end{cases}$
- non-degeneracy+strict complementarity $\implies J_F(X^*, y^*, Z^*, \hat{t})$ is invertible
- $F(X(t), y(t), Z(t), t) = 0$ around $\hat{t}$
- if $X(t) \not\succeq 0$ $(P_t)$ would be infeasible, contradicting primal-dual strict feasibility.

# On the finiteness of bad points

**Lemma B [Hauenstein et al., 2019]**
For a pair $(P_t, D_t)$ of TV-SDPs parameterized by $t \in T$ assume that the data dependence is polynomial and at some $\hat{t} \in T$ all the points $(X, y, Z)$ satisfying

$$\begin{pmatrix} \mathcal{A}(t)[X] - b(t) \\ \mathcal{A}^*(t)[y] + Z - C(t) \\ \langle X, Z \rangle \end{pmatrix} = 0$$

are strictly complementary and primal-dual non-degenerate

Then the set of values of the parameter $t$ at which the optimal primal-dual solution to the primal-dual pair of SDPs associated to $t$ is not unique or not strictly complementary is finite.

*Sketch of proof*

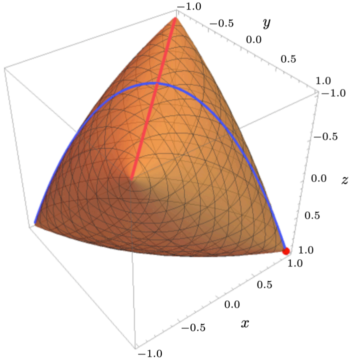
- the set of points where I can't apply the implicit function theorem is a *constructible set*
- a constructible set is either finite or has a finite complement
- the complement of this set is the set where I can apply the implicit function theorem, which cannot be finite

# An example

For $t \in (-2, 3)$ consider the TV-SDP

$$\min \ tx + ty + z$$

$$\text{s.t.} \quad \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} \succeq 0. \qquad (P_t^1)$$



## An example

We have:

$$\mathcal{P}^*(t) = \begin{cases} \begin{pmatrix} 1 & -t/2 & -t/2 \\ -t/2 & 1 & \frac{t^2}{2} - 1 \\ -t/2 & \frac{t^2}{2} - 1 & 1 \end{pmatrix} & \text{for } t \in (-2, 2) \setminus \{0\} \\ \left\{ \begin{pmatrix} 1 & \eta & \theta \\ \eta & 1 & -1 \\ \theta & -1 & 1 \end{pmatrix} \;\middle|\; \begin{matrix} \eta + \theta = 0 \\ \eta, \theta \in [-1, 1] \end{matrix} \right\} & \text{at } t = 0 \\ \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} & \text{for } t \in [2, 3) \end{cases}$$

- regular points for $t \in (-2, 3) \setminus \{0, 2\}$
- discontinuous isolated multiple point at $t = 0$
- loss of differentiability point at $t = 2$

At $t = -1$ there is a strictly complementary and non-degenerate primal-dual solution.
At $t = 0$ and $t = 2$ strict complementarity fails.
At $t = 0$ the dual solution is degenerate.

# Main messages

**What have we done?**

This research resulted in a paper arxiv.org/abs/2104.05445, where our original contributions were:

- definition of **6 types of points** in the trajectory of solutions to TV-SDPs
- a classification theorem: **only** these types can appear
- under generic assumptions, only three types of points can appear

**Why is this useful?**

- we now know what in TV-SDP can go wrong and why
- extra knowledge for algorithms design
- prepares the way for future research on TV-POP

**Summary**

- Crash course on SDP properties and set-valued analysis
- Exposition of the 6 types of points
- The classification theorem
- Sketch of the proof
- An example