Free spectrahedra and quantum information theory

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Brainpop seminar

Talk outline

Incompatibility in QM

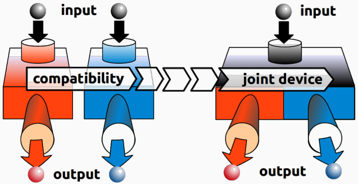
Free spectrahedra

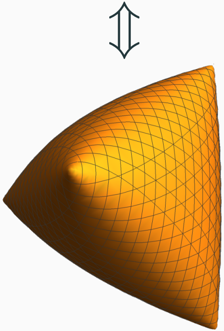
Connecting the two

Proof ideas

Quantum steering

Maximal violation of steering inequalities





Incompatibility in QM

Quantum states and measurements

- Motivation: Classical state $\rightsquigarrow$ probability distributions: $p \in \mathbb{R}^d$, $p \geq 0$, $\sum_i p_i = 1$
- Quantum states $\rightsquigarrow$ density matrices: $\rho \in \mathcal{M}_d(\mathbb{C})$, $\rho \geq 0$, $\operatorname{Tr} \rho = 1$
- Measurement outcomes are labeled $\{1, \ldots, k\}$, need to be assigned probabilities
- Measurements: Tuples of matrices $(E_1, \ldots, E_k)$ such that $(\operatorname{Tr}[E_1\rho], \ldots, \operatorname{Tr}[E_k\rho])$ is a probability distribution for all states ρ
  - $\operatorname{Tr}[E_i\rho] \in \mathbb{R} \rightsquigarrow E_i = E_i^*$
  - $\operatorname{Tr}[E_i\rho] \geq 0 \rightsquigarrow E_i \geq 0$
  - $\sum_i \operatorname{Tr}[E_i\rho] = 1 \rightsquigarrow \sum_i E_i = I_d$
- Tuples of PSD matrices summing to identity are called positive operator-valued measures (POVMs)

Quantum measurements: Compatibility

- Quantum measurements $\rightsquigarrow$ give the probabilities of the classical outcomes when a quantum state enters a measurement apparatus. Mathematically, measurements are modeled by POVMs

Definition

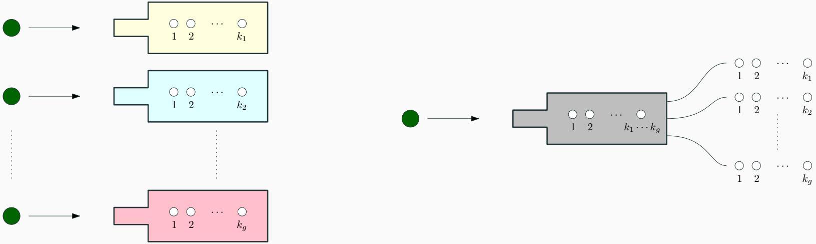
Two POVMs, $A = (A_1, \ldots, A_k)$ and $B = (B_1, \ldots, B_l)$, are called compatible if there exists a third POVM $C = (C_{ij})_{i \in [k], j \in [l]}$ such that

$$\forall i \in [k], \quad A_i = \sum_{j=1}^{l} C_{ij} \qquad \text{and} \qquad \forall j \in [l], \quad B_j = \sum_{i=1}^{k} C_{ij}$$

The definition generalizes to g-tuples of POVMs $A^{(1)}, \ldots, A^{(g)}$, having respectively $k_1, \ldots k_g$ outcomes, where the joint POVM C has outcome set $[k_1] \times \cdots \times [k_g]$.

- Other way to say that: jointly measurable

What does it mean?



- Compatible measurements can be simulated by a single joint measurement, by classically post-processing its outputs
- Examples:
  1. Trivial POVMs $A = (p_i I_d)$ and $B = (q_j I_d)$ are compatible
  2. Commuting POVMs $[A_i, B_j] = 0$ are compatible
  3. If the POVM A is projective, then A and B are compatible iff they commute

Noisy POVMs

- POVMs can be made compatible by adding noise, i.e. mixing in trivial POVMs
- Example: dichotomic POVMs and white noise, $s \in [0, 1]$

$$(E, I - E) \mapsto s(E, I - E) + (1 - s)(\frac{I}{2}, \frac{I}{2}) \qquad \text{or} \qquad E \mapsto sE + (1 - s)\frac{I}{2}$$

- Taking $s = 1/2$ suffices to render any pair of dichotomic POVMs compatible $\rightsquigarrow$ define $C_{ij} := (E_i + F_j)/4$
- From now on, we focus on dichotomic (YES/NO) POVMs

Definition

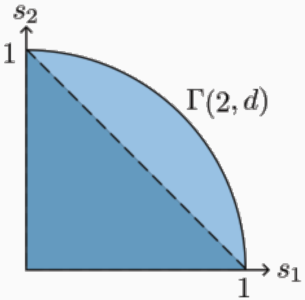
The compatibility region for g measurements on $\mathbb{C}^d$ is the set

$$\Gamma(g, d) := \{s \in [0, 1]^g \; : \; \text{for all quantum effects } E_1, \ldots, E_g \in \mathcal{M}_d(\mathbb{C}),$$
$$\text{the noisy versions } s_i E_i + (1 - s_i) I_d/2 \text{ are compatible}\}$$

Compatibility region

$$\Gamma(g,d) := \{s \in [0,1]^g \ : \text{ for all quantum effects } E_1, \ldots, E_g \in \mathcal{M}_d(\mathbb{C}), \\ \text{the noisy versions } s_i E_i + (1-s_i) I_d/2 \text{ are compatible}\}$$

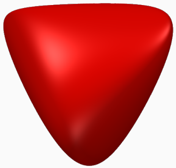
- The set $\Gamma(g,d)$ is convex
- For all $i \in [g]$, $e_i \in \Gamma(g,d)$: every measurement is compatible with $g-1$ trivial measurements
- For $d \geq 2$, $(1,1,\ldots,1) \notin \Gamma(g,d)$: there exist incompatible measurements
- For all $d \geq 2$, $\Gamma(2,d)$ is a quarter-circle



Generally speaking, the set $\Gamma(g,d)$ tells us how robust (to noise) is the incompatibility of g dichotomic measurements on $\mathbb{C}^d$

Free spectrahedra

Free spectrahedra



- A spectrahedron is given by PSD constraints: for $A = (A_1, \dots, A_g) \in (\mathcal{M}_d^{sa})^g$

$$\mathcal{D}_A(1) := \left\{ x \in \mathbb{R}^g \, : \, \sum_{i=1}^{g} x_i A_i \leq I_d \right\}$$

  - $\mathcal{D}_{(\sigma_X,\sigma_Y,\sigma_Z)}(1) = \{(x,y,z) \in \mathbb{R}^3 \, : \, x\sigma_X + y\sigma_Y + z\sigma_Z \leq I_2\} = \text{Bloch ball}$
  - A free spectrahedron is the matricization of a spectrahedron

$$\mathcal{D}_A := \bigsqcup_{n=1}^{\infty} \mathcal{D}_A(n) \quad \text{with} \quad \mathcal{D}_A(n) := \left\{ X \in (\mathcal{M}_n^{sa})^g \, : \, \sum_{i=1}^{g} X_i \otimes A_i \leq I_{nd} \right\}$$

Example: the matrix cube

The matrix cube is the free spectrahedron defined by

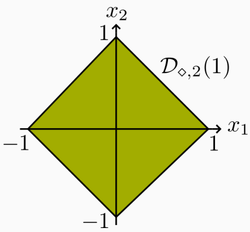
$$\mathcal{D}_{\square,g} := \bigsqcup_{n=1}^{\infty} \{X \in (\mathcal{M}_n^{sa})^g \; : \; \|X_i\|_\infty \leq 1, \quad \forall i \in [g]\}$$

- At level one, $\mathcal{D}_{\square,g}(1)$ is the unit ball of the ℓ^∞ norm on $\mathbb{R}^g$
- As a free spectrahedron, it is defined by $2g \times 2g$ diagonal matrices $\mathcal{D}_{\square,g} = \mathcal{D}_{K_1,\ldots,K_g}$, with $K_i = \operatorname{diag}(e_i) \oplus \operatorname{diag}(-e_i)$

Example: the matrix diamond

The matrix diamond is the free spectrahedron defined by

$$\mathcal{D}_{\diamondsuit,g} := \bigsqcup_{n=1}^{\infty} \{X \in (\mathcal{M}_n^{sa})^g \ : \ \sum_{i=1}^{g} \epsilon_i X_i \preceq I_n, \quad \forall \epsilon \in \{\pm 1\}^g\}$$



- At level one, $\mathcal{D}_{\diamondsuit,g}(1)$ is the unit ball of the ℓ^1 norm on $\mathbb{R}^g$
- As a free spectrahedron, it is defined by $2^g \times 2^g$ diagonal matrices $\mathcal{D}_{\diamondsuit,g} = \mathcal{D}_{L_1,\ldots,L_g}$, with $L_i = I_2 \otimes \cdots \otimes I_2 \otimes \mathrm{diag}(1,-1) \otimes I_2 \otimes \cdots \otimes I_2$

Spectrahedral inclusion

- Consider two free spectrahedra defined by $(A_1, \ldots, A_g)$ and $(B_1, \ldots, B_g)$
- We write $\mathcal{D}_A \subseteq \mathcal{D}_B$ if, for all $n \geq 1$, $\mathcal{D}_A(n) \subseteq \mathcal{D}_B(n)$
- Clearly, $\mathcal{D}_A \subseteq \mathcal{D}_B \implies \mathcal{D}_A(1) \subseteq \mathcal{D}_B(1)$. For the converse implication to hold, one may need to shrink $\mathcal{D}_A$...

Definition

For a free spectrahedron $\mathcal{D}_A$, we define its set of inclusion constants as

$$\Delta_A(g, d) := \{s \in [0,1]^g \ : \text{for all } g\text{-tuples } B_1, \ldots, B_g \in \mathcal{M}_d(\mathbb{C})^{sa}, \\ \mathcal{D}_A(1) \subseteq \mathcal{D}_B(1) \implies s.\mathcal{D}_A \subseteq \mathcal{D}_B\}$$

- The inclusion constants for the matrix cube play an important role in combinatorial optimization
- We shall be concerned with the inclusion set for the matrix diamond, which we denote by $\Delta(g, d)$; it is the same as the one of the matrix cube

Connecting the two

Compatibility in QM $\iff$ matrix diamond inclusion

To a g-tuple $E \in (\mathcal{M}_d^{\mathrm{sa}})^g$, we associate:

$$\mathcal{D}_{2E-I} := \bigsqcup_{n=1}^{\infty} \{X \in (\mathcal{M}_n^{\mathrm{sa}})^g \ : \ \sum_{i=1}^{g} X_i \otimes (2E_i - I_d) \leq I_{nd}\}$$

Theorem

Let $E \in (\mathcal{M}_d^{sa})^g$ be g-tuple of selfadjoint matrices. Then:

- *The matrices E are quantum effects $\iff \mathcal{D}_{\diamondsuit,g}(1) \subseteq \mathcal{D}_{2E-I}(1)$*
- *The matrices E are compatible quantum effects $\iff \mathcal{D}_{\diamondsuit,g} \subseteq \mathcal{D}_{2E-I}$*

At the intermediate levels $1 \leq n \leq d$, $\mathcal{D}_{\diamondsuit,g}(n) \subseteq \mathcal{D}_{2E-I}(n)$ iff for all isometries $V : \mathbb{C}^n \to \mathbb{C}^d$, the compressed effects $V^* E_i V$ are compatible.

Moreover, the compatibility region is equal to the set of inclusion constants of the matrix diamond: $\forall g, d$, $\Gamma(g, d) = \Delta(g, d)$.

Consequences

Many things are known about the matrix diamond

- For all g, d, $\frac{1}{2d}(1, 1, \ldots, 1) \in \Delta(g, d)$ (Helton *et al.*, 2019)
- For all g, d, $\mathrm{QC}_g := \{s \in [0, 1]^g \ : \ \sum_i s_i^2 \leq 1\} \subseteq \Delta(g, d)$ (Passer *et al.*, 2018)

Many things are known about (in-)compatibility

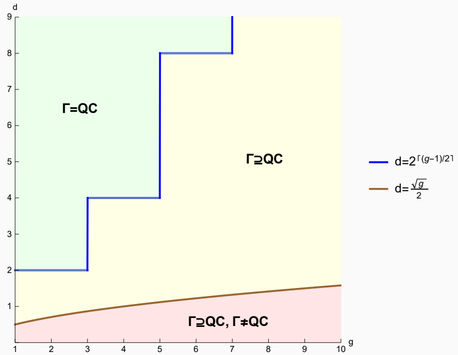
- Some small g, d cases completely solved
- Approximate quantum cloning $\implies$ compatibility

$$\mathrm{Clone}(g, d) := \{s \in [0, 1]^g \ : \exists \text{ quantum channel } \Phi : \mathcal{M}_d \to \mathcal{M}_d^{\otimes g} \text{ s.t.}$$
$$\forall i \in [g], \quad \Phi_i(X) = s_i X + (1 - s_i)\frac{\operatorname{Tr} X}{d} I_d\}$$

Theorem

For all g and $d \geq 2^{\lceil (g-1)/2 \rceil}$, $\Gamma(g, d) = \Delta(g, d) = \mathrm{QC}_g$

Phase diagram



- Connection to free spectrahedra also holds for arbitrary outcomes
- Instead of matrix diamond, consider its generalization, the matrix jewel
- Something similar holds for general probabilistic theories (GPTs)

Proof ideas

Inclusion of spectrahedra and (completely) positive maps

Theorem (Helton et al., 2013)

Let $A \in (\mathcal{M}_D^{sa}(\mathbb{C}))^g$, $B \in (\mathcal{M}_d^{sa}(\mathbb{C}))^g$ such that $\mathcal{D}_A(1)$ is bounded. Then, $\mathcal{D}_A(n) \subseteq \mathcal{D}_B(n)$ iff the unital linear map

$$\Phi : \mathrm{span}\{I, A_1, \ldots, A_g\} \to \mathcal{M}_d^{sa}(\mathbb{C}), \qquad A_i \mapsto B_i$$

is n-positive.

Sketch of the proof of the main theorem:

- Level 1: the extremal points of $\mathcal{D}_{\diamondsuit,g}(1)$ are $\pm e_i$
- The inclusion $\mathcal{D}_{\diamondsuit,g} \subseteq \mathcal{D}_{2E-I}$ holds iff the unital map $\Phi : I_2 \otimes \cdots \otimes I_2 \otimes \mathrm{diag}(1,-1) \otimes I_2 \otimes \cdots \otimes I_2 \mapsto 2E_i - I_d$ is CP
- Arveson's extension theorem: Φ has a (completely) positive extension $\tilde{\Phi}$ to $\mathbb{R}^{2^g}$
- $C_f := \tilde{\Phi}(f)$ is a joint POVM for the E_i's, where $\{f\}$ is a basis of $\mathbb{R}^{2^g}$

Maximally incompatible quantum effects

Lemma (Newman 1932, Hrubeš 2016)

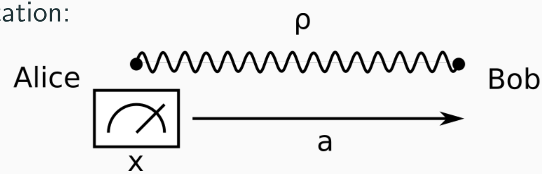
For $d = 2^k$, there exist $2k+1$ anti-commuting, self-adjoint, unitary matrices $F_1, \ldots, F_{2k+1} \in \mathcal{U}_d$. Moreover, 2^k is the smallest dimension where such a $(2k+1)$-tuple exists.

- For $k = 0$, take $F_1^{(0)} := [1]$
- For $k \geq 1$, define $F_i^{(k+1)} = \sigma_X \otimes F_i^{(k)}$ $\forall i \in [2k+1]$ and $F_{2k+2}^{(k+1)} = \sigma_Y \otimes I_{2^k}$, $F_{2k+3}^{(k+1)} = \sigma_Z \otimes I_{2^k}$
- These matrices satisfy, for all $x \in \mathbb{R}_+^g$, $\left\|\sum_{i=1}^g x_i F_i\right\|_\infty = \|x\|_2$, and $\left\|\sum_{i=1}^g x_i \bar{F}_i \otimes F_i\right\|_\infty = \|x\|_1$
- For d large enough, the maximally incompatible g-tuple of quantum effects in $\mathcal{M}_d$ is given by $E_i = (F_i + I_d)/2$

Quantum steering

Local hidden state models

Operational interpretation:



- g measurements on Alice's side, k_x outcomes each, quantum systems of dimension d
- Assemblage: tuple $(\sigma_{a|x})_{a,x}$ of PSD matrices such that

$$\sum_{a\in[k_x]} \sigma_{a|x} = \bar{\sigma} \qquad \forall x \in [g]$$

  for some average state $\bar{\sigma}$
- Local hidden state (LHS) model:

$$\forall a \in [k_x],\ x \in [g], \qquad \sigma_{a|x} = \sum_{\lambda\in\Lambda} q_\lambda p(a|x,\lambda)\sigma_\lambda$$

Steering inequalities

- Steering inequality: tuples of self-adjoint matrices $\mathsf{F} := (F_{a|x})_{a,x}$
- LHS value:

$$V_{\mathcal{L}}(\mathsf{F}) := \sup_{\boldsymbol{\sigma} \in \mathcal{L}(g,\mathsf{k},d)} \sum_{a,x} \mathrm{Tr}(\sigma_{a|x} F_{a|x})$$

- Quantum value:

$$V_{\mathcal{Q}}(\mathsf{F}) := \sup_{\boldsymbol{\sigma} \in \mathcal{Q}(g,\mathsf{k},d)} \sum_{a,x} \mathrm{Tr}(\sigma_{a|x} F_{a|x})$$

- Restrict mostly to $k_x = 2$. In this case, F is unbiased if $F_{+|x} = -F_{-|x}$ for all $x \in [g]$

Steering constants

- Set of steering constants is defined as:

  $$\Sigma(g,d) := \{\mathsf{s} \in [0,1]^g \ : \ \forall (F_{\pm|1},\ldots,F_{\pm|g}) \in (\mathcal{M}_d^{\mathrm{sa}})^{2g},\ V_{\mathcal{L}}(\mathsf{F}) \le 1 \implies V_{\mathcal{Q}}(\mathsf{F}^{(\mathsf{s})}) \le 1\}$$

- $\mathsf{F}^{(\mathsf{s})} := \mathsf{s}.\mathsf{F} + (1-\mathsf{s}).\mathsf{F}^{(0)}$ convex mixture with certain trivial steering inequality
- Quantifies how much steerability is available for fixed g, d
- Single number: largest quantum violation

  $$\gamma_{g,d} = \sup_{\mathsf{F}} \frac{V_{\mathcal{Q}}(\mathsf{F})}{V_{\mathcal{L}}(\mathsf{F})}$$

- $\Sigma_0(g,d)$, $\gamma^0_{g,d}$ after restricting to unbiased steering inequalities

Maximal violation of steering inequalities

Connecting steering and the inclusion of the matrix diamond

- To any steering inequality $(F_{a|x})_{a,x}$, can associate (non-monic) free spectrahedron $\hat{\mathcal{D}}_{\tilde{\mathsf{F}}}(n)$

Theorem

For an arbitrary steering inequality F, we have

$$(1) \quad \mathcal{D}_{\square,g}(1) \subseteq \hat{\mathcal{D}}_{\tilde{\mathsf{F}}}(1) \iff V_{\mathcal{L}}(\mathsf{F}) \leq 1.$$

$$(2) \quad \mathcal{D}_{\square,g} \subseteq \hat{\mathcal{D}}_{\tilde{\mathsf{F}}} \iff V_{\mathcal{Q}}(\mathsf{F}) \leq 1.$$

Inclusion constants give maximal violation

Theorem

For all g, $d \in \mathbb{N}$, $\Sigma_0(g, d) = \Sigma(g, d) = \Delta_\square(g, d)$.

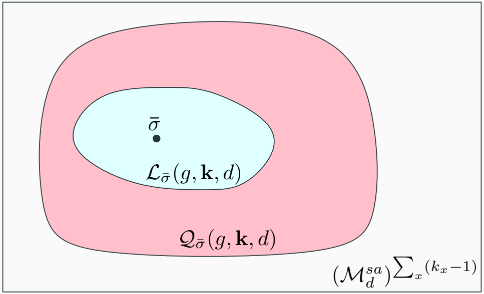
Proposition

For any $s \in [0, 1]$, $s(1, \ldots, 1) \in \Sigma_0(g, d)$ if and only if for all unbiased (g, d)-steering inequalities F it holds that

$$V(\mathrm{F}) := \frac{V_\mathcal{Q}(\mathrm{F})}{V_\mathcal{L}(\mathrm{F})} \leq \frac{1}{s},$$

where we define $V(\mathrm{F}) = 1$ if $V_\mathcal{L}(\mathrm{F}) = V_\mathcal{Q}(\mathrm{F}) = 0$. In particular, the largest such s is equal to the largest unbiased quantum violation $\gamma^0_{g,d}$.

Inclusion constants as set inclusions



Proposition

We have, for all invertible density matrices $\bar{\sigma} \in \mathcal{M}_d^+$,

$$\{s \in [0,1]^g \; : \; s.\mathcal{Q}_{\bar{\sigma}}(g, 2^{\times g}, d) + (1 - s).\bar{\boldsymbol{\sigma}} \subseteq \mathcal{L}_{\bar{\sigma}}(g, 2^{\times g}, d)\} = \Delta_{\square}(g, d).$$

g-independent bounds

Theorem

The largest s such that $s(1,\ldots,1) \in \Delta_\square(g,d)$ for all $g \in \mathbb{N}$ is

$$\tau(d) = 4^{-n}\binom{2n}{n}, \qquad \text{with } n := \lfloor d/2 \rfloor.$$

Asymptotically, this behaves as $\sqrt{2/(\pi d)}$.

- It is possible to construct almost optimal steering inequalities, based on ϵ-nets of $\mathcal{U}(d)$.
- Based on the ideas in Helton *et al.*, 2019

d-independent bounds

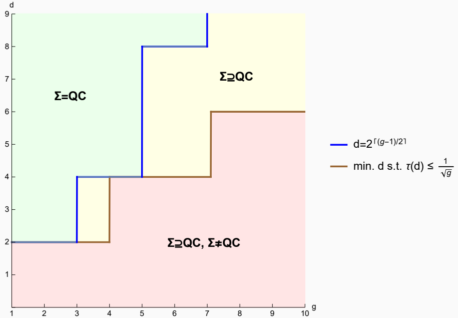
Proposition

Let $d \geq 2^{\lceil (g-1)/2 \rceil}$, $g \geq 2$. Then, $\Sigma(g,d) = \Delta_\Box(g,d) = \left\{ \mathrm{s} \in [0,1]^g \ : \ \sum_{i=1}^g s_i^2 \leq 1 \right\}$.
Generally, for any $d \in \mathbb{N}$, $\Sigma(g,d) = \Delta_\Box(g,d) \supseteq \left\{ \mathrm{s} \in [0,1]^g \ : \ \sum_{i=1}^g s_i^2 \leq 1 \right\}$.

- This shows that the unbiased steering inequalities considered in Marciniak *et al.*, 2015, are optimal
- For $d = 2$ Pauli matrices, in higher dimensions anticommuting self-adjoint unitaries are constructed by taking tensor powers

Phase diagram

$$\Sigma(g,d) := \{\mathsf{s} \in [0,1]^g \ : \ \forall (F_{\pm|1}, \ldots, F_{\pm|g}) \in (\mathcal{M}_d^{\mathrm{sa}})^{2g},\ V_{\mathcal{L}}(\mathsf{F}) \leq 1 \implies V_{\mathcal{Q}}(\mathsf{F}^{(\mathsf{s})}) \leq 1\}$$



$$\mathrm{QC}_g := \left\{ \mathsf{s} \in [0,1]^g \ : \ \sum_{i \in [g]} s_i^2 \leq 1 \right\}$$

Conclusions

- Connection between measurement incompatibility and the inclusion of the matrix diamond
- Yields bounds on the maximal amount of incompatibility for g dichotomic measurements in dimension d
- Connection between quantum steering and the inclusion of the matrix cube
- Allows to place upper bound on the violation of dichotomic steering inequalities for fixed dimension d and fixed number of measurements on Alice's side g
- Shows previously found violations are optimal