Universal equivalence of symplectic groups

Chair of higher algebra
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Main goals

- Formulate a criterion of universal equivalence of symplectic groups $\mathrm{Sp}_{2n}(K)$ and $\mathrm{Sp}_{2n}(M)$ over fields of characteristics not equal to 2.
- Prove the criterion:
  Assume $n, m \geq 1$, K and M — infinite fields of characteristics not equal to 2. Then groups $\mathrm{Sp}_{2n}(K)$ and $\mathrm{Sp}_{2m}(M)$ are universally equivalent, if and only if:
  1) $n = m$;
  2) fields K and M are universally equivalent.

Basic concepts and definitions

Universal formula

A formula φ of signature Σ is called universal, if its normal form is

$$\forall x_1 \ldots \forall x_n \psi(x_1, \ldots, x_n).$$

Universal equivalence

Two algebraic systems $\mathfrak{A}$ and $\mathfrak{B}$ of signature Σ are called universally equivalent, if for every universal proposition φ of signature Σ the following holds true

$$\mathfrak{A} \models \varphi \Longleftrightarrow \mathfrak{B} \models \varphi.$$

Set of universal propositions $\{\varphi \mid \mathfrak{A} \models \varphi\}$ of signature Σ is called universal theory of system $\mathfrak{A}$ and is notated as $Th_\forall(\mathfrak{A})$.
Thus $\mathfrak{A} \equiv_\forall \mathfrak{B} \Leftrightarrow Th_\forall(\mathfrak{A}) = Th_\forall(\mathfrak{B})$.

Basic concepts and definitions

Let us assume that $\mathfrak{A}$ and $\mathfrak{B}$ are algebraic systems of signature Σ with supports A and B

Partial isomorphism

Function $f : A \to B$ is called a partial isomorphism $\mathfrak{A}$ to $\mathfrak{B}$, if following conditions hold true:

1. *$dom(f) \subset A$, $Im(f) \subset B$;*
2. *f is injective;*
3. *f preserves predicates, functions and constants:*

Criterion of universal equivalence of algebraic systems (O. T. O'Meara 1976)

For two systems $\mathfrak{A}$ and $\mathfrak{B}$ to be universally equivalent it is necessary and sufficient that for any finite subsignature $\Sigma_1 \subset \Sigma$ any finite subsystem of system $\mathfrak{A}$ of signature Σ_1 is partially isomorphic to some subsystem $\mathfrak{B}$ of the same signature, and vice versa.

Basic concepts and definitions

Symplectic group $\mathrm{Sp}_{2n}(K)$

Symplectic group $\mathrm{Sp}_{2n}(K)$ — is a group of matrices of $2n \times 2n$ over field K, subject to $A^T QA = Q$, where Q is some fixed antisymetric matrix. In this paper we assume that Q is a fixed matrix:

$$\begin{pmatrix} 0 & 1 & . & . & . \\ -1 & 0 & . & . & . \\ . & . & . & . & . \\ . & . & . & 0 & 1 \\ . & . & . & -1 & 0 \end{pmatrix}$$

It is important for us that $\mathrm{Sp}_{2n}(K)$ is a subgroup of $\mathrm{SL}_{2n}(K)$. Thus all matrices in it have the determinant equal to 1.

Main steps

- Proof of the sufficient condition of universal equivalence of symplectic groups over fields
- The equality of dimensions of universally equivalent groups
- Proof of the necessary condition of universal equivalence of symplectic groups over fields: extraction of some certain matrices, usage of criterion of universal equivalence of algebraic systems (O. T. O'Meara 1976)

Sufficient condition of universal equivalence of symplectic groups over fields. The equality of dimensions.

Theorem (Sufficient condition of universal equivalence)

Let K and M be universally equivalent infinite fields. Then for any natural m: $\mathrm{Sp}_{2m}(K) \equiv_\forall \mathrm{Sp}_{2m}(M)$
Remark: It holds true when K and M are commutative rings.

Lemma about involutions which commute pairwise

Maximum number of involutions which commute pairwise in $\mathrm{Sp}_{2n}(K)$ is equal to $2^n - 1$.

The equality of dimensions

If $Sp_{2n}(K) \equiv_\forall Sp_{2m}(M)$ and infinite fields K and M have characteristics different from 2 then $n = m$.

Scheme of the rest of the proof

For an arbitrary finite submodel $K_1 \subset K$ let us find a finite partially isomorphic submodel $M_1 \subset M$, having constructed a partial isomorphism $\psi\colon K_1 \to M_1$.
This will be done as follows: for each $\alpha \in K_1$ we will choose an addition matrix, a multiplication matrix and several other auxiliary matrices. Let us call this set of matrices $\mathfrak{A}(\alpha)$. Consider $G_K \subset \mathrm{Sp}_{2n}(K)$ such that $\mathfrak{A}(\alpha) \subset G_K$. then according to the criterion of universal equivalence of algebraic systems in $\mathrm{Sp}_{2n}(M)$ there exists a finite submodel G_M, partially isomorphic to G_K. That means that there exists a partial isomorphism $\phi\colon G_K \to G_M$. Later we will show that ϕ maps all matrices from $\mathfrak{A}(\alpha)$ to some system $\mathfrak{B}$ of matrices of the same type, also the images of addition matrices and multiplication matrices correspond to one element β from the field M.
Thus the M_1 that we want to find is the set of all elements of field M, which correspond to the pairs of images of addition matrices and multiplication matrices for all elements in K_1.

Scheme of the rest of the proof

Let us visualize the idea described on the previous page by means of a diagram:

Diagram

$$
\begin{array}{ccc}
\{\mathfrak{A}_\alpha\}_{\alpha \in K_1} & \xrightarrow{\ \phi\ } & \{\mathfrak{B}_\beta\}_{\beta \in M_1} \\
\uparrow & & \downarrow \\
K_1 & \xrightarrow{\ \psi\ } & M_1
\end{array}
$$

K_1 и M_1 are finite subsystems of fields of K and M respectively, connected by means of partial isomorphism ψ.
$\mathfrak{A}(\alpha)$ и $\mathfrak{B}(\beta)$ are sets of matrices, connected by means of partial isomorphism ϕ, corresponding to elements $\alpha \in K$ and $\beta \in M$ respectively.

Extraction of matrices

- Extraction of involutions
- Extraction of block-diagonal matrix with blocks from $\mathrm{SL}_2(K)$ on the main diagonal
- Extraction of partial transvection
- Extraction of $-E$
- Extraction of involutions B_i with one pair of -1 on the main diagonal, and also of matrices X_i
- Extraction of block permutation matrices
- Extraction of transvection
- Extraction of matrix Q
- Extraction of diagonal matrices
- Extraction of multiplication matrices
- Extraction of addition matrices

Involutions and block-diagonal matrix

Extraction of maximal set of involutions which commute pairwise

Maximal set of involutions which commute pairwise can be extracted by means of universal formulas and all of them will have the following shape in some chosen basis: diag$[\alpha_1, \alpha_1, \dots, \alpha_n, \alpha_n]$, where $\alpha_i = \pm 1$.

Further on we will work in the basis where all those involutions are diagonal.

Block-diagonal matrix

If we say that a matrix A commutes with any matrix from the set of involutions extracted in the previous step this matrix A will be block-diagonal diag$[\mathrm{SL}_2(K), \dots, \mathrm{SL}_2(K)]$ meaning that in blocks 2×2 on the diagonal there can be any matrices from $\mathrm{SL}_2(K)$, all other elements are zeros.

Matrix of partial transvection

The condition that matrix A is block-diagonal will be denoted $Block(A)$.

Extraction of matrix of partial transvection

Matrix of partial transvection, which has the following shape (the blocks can be permuted): $\begin{pmatrix} T & \cdot & \cdot & \cdot \\ \cdot & M_2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & M_n \end{pmatrix}$, where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $M_i \in \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix} \right\}$, can be extracted by means of an existential formula:

$$PTransvection(A) := \exists X \Big(Block(A) \& Block(X) \& (X^2 A^2 \neq A^2 X^2) \& \\ \& (XAX^{-1}A = AXAX^{-1}) \& (XAX^{-1} = A^4) \Big)$$

Matrices $-E$ and involutions B_i with one pair of -1 on diagonal also matrices X_i

B_i is an involution with one pair of -1 on diagonal such that those -1 are located in i-th block.
Matrix X_i has shape:

$$\mathrm{diag}\left[\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}\right],$$

where the block $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is on the i-th place.

Lemma

matrices $-E$, B_i and X_i are extracted by means of an existential formulas.

Matrices of block permutations

Matrix S_{ij} of block permutations is a matrix such that, the i-th and j-th blocks change places when conjugated by it in a block-diagonal matrix. It has shape

$$S_{ij} = \pm \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

(for convenience here we omit all the other blocks of size 2×2 located on the main diagonal of matrix S_{ij} which are $\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; all other elements are zeros).

Matrices of block permutations

matrices of block permutations S_{ij} are characterized by following conditions:

1. S_{ij} commute with all involutions B_k with $k \neq i, j$;
2. $S_{ij} B_i S_{ij}^{-1} = B_j$, where B_i are involutions with one pair of -1 in the i-th block on the main diagonal;
3. S_{ij} preserves the form Q.
4. $S_{ij} X_i = X_j S_{ij}$ and $S_{ij} X_j = X_i S_{ij}$, where X_i is a matrix of shape:
$$\operatorname{diag}\left[\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}\right]$$
with $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in the i-th block on the main diagonal;
5. $S_{ij}^2 = E$;
6. $S_{ij} \pi S_{ij} = \pi$, where π is a matrix of partial transvection which has a block $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ on the i-th place of the main diagonal.

Transvection matrix and matrix Q

Transvection matrix is a block-diagonal matrix which has shape:

$$\begin{pmatrix} T & 0 & \cdot & 0 \\ 0 & T & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & T \end{pmatrix}, \text{ где } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

It is extracted by means of an existential formula:

$$Transvection(A) := \left(PTransvection(A) \& \left(\left(\bigwedge_{i=1}^{n} (S_{1i}A = AS_{1i}) \right) \right) \right).$$

Matrix Q

Matrix Q is extracted by means of an existential formula:

$$\exists Q \exists \sigma \Big(Block(Q) \& Transvection(\sigma) \& \left(Q^2 = -E\right) \&$$

$$\& \left(Q\sigma Q = -\sigma^{-1} Q \sigma^{-1}\right) \Big).$$

Diagonal matrix and multiplication matrix

Diagonal matrix

Diagonal matrix D is extracted by means of an existential formula:

$$Diag(D) := \exists D \exists \sigma \Big(Block(D) \& Transvection(\sigma) \& \\ \& \left(QDQ^{-1} = D^{-1} \right) \& \left(D\sigma D^{-1}\sigma = \sigma D \sigma D^{-1} \right) \Big),$$

where Q is a matrix which has been already extracted.

Multiplication matrix

Matrix $Pr(a)$ consisting of blocks $\begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}$ on the main diagonal will be called a multiplication matrix corresponding to element a. Such matrix can be extracted with the help of diagonal matrix and conjugation by permutation matrices.

Addition matrix and conclusion

Addition matrix

Matrix Sum(a) consisting of blocks $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ *on the main diagonal will be called an addition matrix corresponding to element a.* It is extracted by means of the condition of commuting with all permutation matrices and the formula:

$$SumMatrix(A) := \exists A \exists \sigma \exists D \Big(Block(A) \& Transvection(\sigma) \& \\ \& ProductMatrix(D) \& (A\sigma = \sigma A) \& (DAD^{-1} = A^4) \Big)$$

Connection between addition matrices and multiplication matrices

$$\begin{pmatrix} x & 0 \\ 0 & \frac{1}{x} \end{pmatrix} = -\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{x} & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \qquad (1)$$

Conclusion

Connection between addition matrices and multiplication matrices

Lets note that formula (1) from the previous page can be rewritten as:

$$Pr(x) = -Sum(x)QSum(x^{-1})Q^{-1}Sum(x)Q.$$

Addition matrices, multiplication matrices and all the other auxiliary matrices are extracted by means of existential formulas in group $\mathrm{Sp}_{2n}(K)$. Since $\mathrm{Sp}_{2n}(K) \equiv_\forall \mathrm{Sp}_{2n}(M)$, the same formulas hold true for the images of matrices under the partial isomorphism. Thus all matrices preserve their shape under the isomorphism beause such isomorphism preserves formulas. Finally we can use the criterion of universal equivalence of algebraic systems. Thus we have proved that universal equivalence of groups $\mathrm{Sp}_{2n}(K)$ and $\mathrm{Sp}_{2m}(M)$ implies universal equivalence of fields K and M, which together with $n = m$, terminates the proof of the main theorem.

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Thank you for your attention!