# Koopman operator or its dual:

# What matters more?

Alexandre Mauroy *(University of Namur, Belgium)*

UNIVERSITÉ DE NAMUR

naxys
Namur Institute for Complex Systems

# Koopman operator for the practitioner

**Linear lifted dynamics**

$\dot{z} \approx \boldsymbol{A}\, z\,, \qquad z \in \mathbb{R}^N$

**LIFTING**

$$z = \begin{pmatrix} e_1(x) \\ e_2(x) \\ \vdots \\ e_N(x) \end{pmatrix}$$

**Nonlinear dynamics**

$\dot{x} = F(x), \qquad x \in \mathbb{R}^n$

Prediction / MPC *[Korda and Mezic, Automatica 2016]*
Feedback stabilization *[Huang et al., CDC2018]*
State estimation *[Surana, CDC2016]*
Identification *[AM and Goncalves, IEEE TAC2020]*
…

**Message to the theoretician:**
This amounts to considering the evolution of
evaluation functionals in a dual Koopman framework

Rigorous application to
stability analysis and state estimation

# Outline

## Preliminaries

## Stability analysis

*Joint work with C. Mugisho*

## State estimation

*Joint work with J. Mohet and J. Winkin*

# Outline

## Preliminaries

## Stability analysis

*Joint work with C. Mugisho*

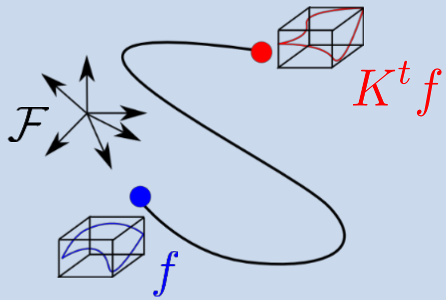
## State estimation

*Joint work with J. Mohet and J. Winkin*

## We consider an operator-theoretic description of dynamical systems

**Koopman operator description**

Operator $f \mapsto K^t f = f \circ \varphi^t$
acting on an observable space

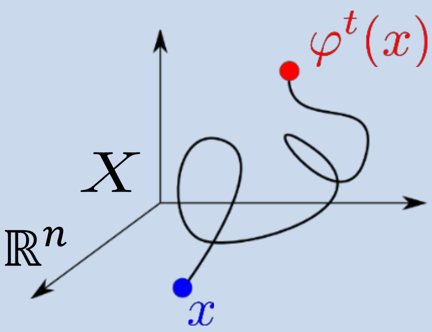



**linear infinite-dimensional operator**

**LIFTING**

**Trajectory-oriented description**

Flow map $x \mapsto \varphi^t(x)$
acting on the state space $X \subset \mathbb{R}^n$



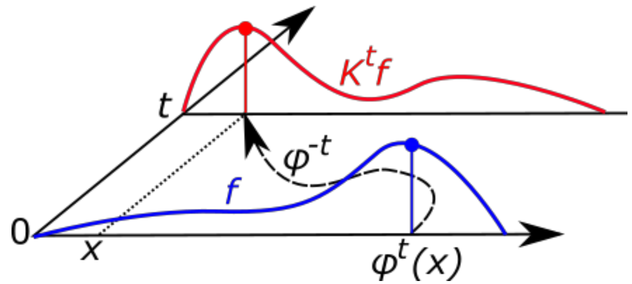

**nonlinear finite-dimensional dynamics**

# The Koopman operator in a nutshell

**Koopman operator semigroup**

$$K^t : \mathcal{F} \to \mathcal{F}, \quad t \geq 0$$

$$K^t f(x) = f \circ \varphi^t(x)$$




**Linearity:** $K^t(a_1 f_1 + a_2 f_2) = a_1 K^t f_1 + a_2 K^t f_2$

**Infinitesimal generator of the Koopman operator**

$$Af = \lim_{t \downarrow 0} \frac{K^t f - f}{t}$$

$$A : \mathcal{D}(A) \to \mathcal{F}$$

$$\dot{x} = F(x) \quad \Rightarrow \quad Af = F \cdot \nabla f$$

Lie derivative

**Koopman dynamical system**

$\dot{f} = Af, \;\; f(0) = f_0 \in \mathcal{F}$ ➔ solution $f(t) = K^t f_0$

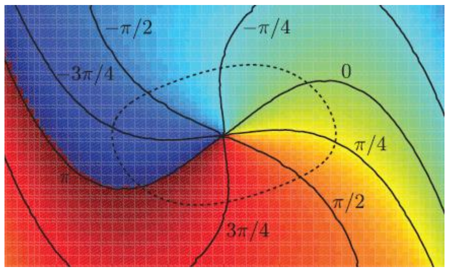
# Since the Koopman operator is linear, it is natural to consider its spectral properties

Koopman eigenfunction $\phi_\lambda \in \mathcal{F}$ ⟺ $A\phi_\lambda = \lambda\phi_\lambda$

Koopman eigenvalue $\lambda \in \sigma(A)$ ⟺ $K^t\phi_\lambda = e^{\lambda t}\phi_\lambda$

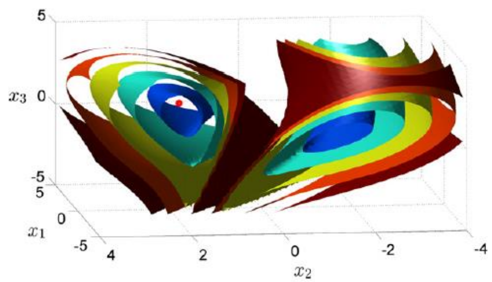
*[Mezic, Nonlinear Dynamics, 2005]*

**Connection to geometric properties**



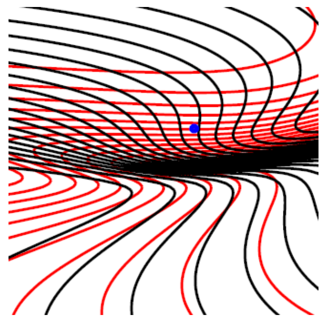

Isochrons

*[AM et Mezic, Chaos 2012]*




Isostables

*[AM, Mezic and Moehlis, Physica D 2013]*



Global linearization

*[Lan et Mezic, Physica D 2013]*

**Stable (hyperbolic) equilibrium** $\mathcal{F}$ *space of analytic functions*

$$\dot{x} = F(x)$$

$F(x^*) = 0$, $F$ analytic

$$\lambda_k \in \sigma\left(\frac{\partial F}{\partial x}(x^*)\right) \subset \sigma(A)$$

$\phi_{\lambda_k}$ : principal eigenfunction

*[AM, Mezic and Moehlis, Physica D 2013] [Mohr and Mezic,* arXiv:1611.01209*]*

# A last ingredient:
# Reproducing kernel Hilbert spaces (RKHS)

A reproducing kernel Hilbert space (RKHS) $\mathcal{H}$ on $X$ over $\mathbb{C}$ is a Hilbert space of functions $f: X \to \mathbb{C}$ such that, for every $x \in X$, the evaluation functional $E_x: \mathcal{H} \to \mathbb{C}$ (defined such that $E_x(f) = f(x)$) is bounded.

**Reproducing kernel**

For all $x \in X$, there exists $k_x \in \mathcal{H}$ such that $E_x(\cdot) = \langle k_x, \cdot \rangle$, i.e. $\langle k_x, f \rangle = f(x)\ \forall f \in \mathcal{H}$

$$\forall x, y \in X, \qquad k(x,y) \overset{\text{def}}{=} \langle k_x, k_y \rangle = k_x(y) = k_y(x)$$

**A special case: the Hardy space on the polydisk** $\mathbb{D}^n = \{w \in \mathbb{C}^n : |w_i| < 1\ \forall i\}$

$$H^2(\mathbb{D}^n) = \{f: \mathbb{D}^n \to \mathbb{C}, \text{analytic}, \|f\| < \infty\}$$

$\|f\|^2 = \sum_{k \in \mathbb{N}} |f_k|^2$, $\langle f, g \rangle = \sum_{k \in \mathbb{N}} f_k \overline{g_k}$ with $f = \sum_{k \in \mathbb{N}} f_k e_k$, $g = \sum_{k \in \mathbb{N}} g_k e_k$

where $\{e_k\}_{k \in \mathbb{N}}$ is an orthonormal basis of monomials

# The lifting dynamics is recovered through the dual Koopman system

**Adjoint operators:** $\langle Af, g\rangle = \langle f, A^* g\rangle \quad \langle K^t f, g\rangle = \langle f, (K^t)^* g\rangle$

$$(K^t)^* k_x = k_{\varphi^t(x)}$$

Assume that $F \in \mathcal{H}$ and $Ae_k \in \mathcal{H}\ \forall k$

$\{e_k\}_{k\in\mathbb{N}}$ is an orthonormal basis

**Dual Koopman system**

$$\dot{f} = A^* f, \quad f \in \mathcal{H}$$

$$\Leftrightarrow$$

$$\dot{z} = \bar{A}^* z, \quad z \in \ell^2$$

$$z_k = \langle f, e_k\rangle$$

**Linear lifted dynamics**

$$\dot{z} \approx \boldsymbol{A}\, z\,, \qquad z \in \mathbb{R}^N$$

$$z = \begin{pmatrix} e_1(x) \\ e_2(x) \\ \vdots \\ e_N(x) \end{pmatrix} = \begin{pmatrix} \langle k_x, e_1\rangle \\ \langle k_x, e_2\rangle \\ \vdots \\ \langle k_x, e_N\rangle \end{pmatrix}$$

➔ lifted states = components of evaluation functional $k_x$ in the basis

$$\bar{A}^* = \begin{pmatrix} \langle A^* e_0, e_0\rangle & \langle A^* e_1, e_0\rangle & \cdots \\ \langle A^* e_0, e_1\rangle & \langle A^* e_1, e_1\rangle & \cdots \\ \langle A^* e_0, e_2\rangle & \langle A^* e_1, e_2\rangle & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

$$e_k\big(\varphi^t(x)\big) = \langle k_{\varphi^t(x)}, e_k\rangle = \langle (K^t)^* k_x, e_k\rangle$$

$$\dot{z} = \frac{d}{dt}\begin{pmatrix} e_1(\varphi^t(x)) \\ e_2(\varphi^t(x)) \\ \vdots \\ e_N(\varphi^t(x)) \end{pmatrix} = \begin{pmatrix} \langle A^* k_x, e_1\rangle \\ \langle A^* k_x, e_2\rangle \\ \vdots \\ \langle A^* k_x, e_N\rangle \end{pmatrix}$$

➔ matrix $\boldsymbol{A}$ = matrix approximation of $\bar{A}^*$

# The lifting dynamics is recovered through the dual Koopman system

**Adjoint operators:** $\langle Af, g\rangle = \langle f, A^* g\rangle \quad \langle K^t f, g\rangle = \langle f, (K^t)^* g\rangle$

$$(K^t)^* k_\mu = k_{\varphi^t_\# \mu}$$

Assume that $F \in \mathcal{H}$ and $Ae_k \in \mathcal{H}\ \forall k$

$\{e_k\}_{k\in\mathbb{N}}$ is an orthonormal basis

$$\langle k_\mu, f\rangle = \int_X f(x)\, \mu(dx)$$

**Dual Koopman system**

$$\dot{f} = A^* f, \quad f \in \mathcal{H}$$

$$\Leftrightarrow$$

$$\dot{z} = \bar{A}^* z, \quad z \in \ell^2$$

$$z_k = \langle f, e_k\rangle$$

**Linear lifted dynamics**

$$\dot{z} \approx \boldsymbol{A}\, z\,, \qquad z \in \mathbb{R}^N$$

$$z = \begin{pmatrix} m_1(\mu) \\ m_2(\mu) \\ \vdots \\ m_N(\mu) \end{pmatrix} = \begin{pmatrix} \langle k_\mu, e_1\rangle \\ \langle k_\mu, e_2\rangle \\ \vdots \\ \langle k_\mu, e_N\rangle \end{pmatrix}$$

➔ lifted states = moments of the measure $\mu$

$$\bar{A}^* = \begin{pmatrix} \langle A^* e_0, e_0\rangle & \langle A^* e_1, e_0\rangle & \cdots \\ \langle A^* e_0, e_1\rangle & \langle A^* e_1, e_1\rangle & \cdots \\ \langle A^* e_0, e_2\rangle & \langle A^* e_1, e_2\rangle & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

$$m_k(\varphi^t_\# \mu) = \langle k_{\varphi^t_\# \mu}, e_k\rangle = \langle (K^t)^* k_\mu, e_k\rangle$$

$$\dot{z} = \frac{d}{dt}\begin{pmatrix} m_1(\varphi^t_\# \mu) \\ m_2(\varphi^t_\# \mu) \\ \vdots \\ m_N(\varphi^t_\# \mu) \end{pmatrix} = \begin{pmatrix} \langle A^* k_\mu, e_1\rangle \\ \langle A^* k_\mu, e_2\rangle \\ \vdots \\ \langle A^* k_\mu, e_N\rangle \end{pmatrix}$$

➔ matrix $\boldsymbol{A}$ = matrix approximation of $\bar{A}^*$

# Outline

## Preliminaries

## Stability analysis

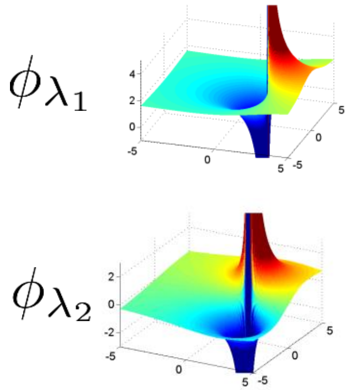
*Joint work with C. Mugisho*

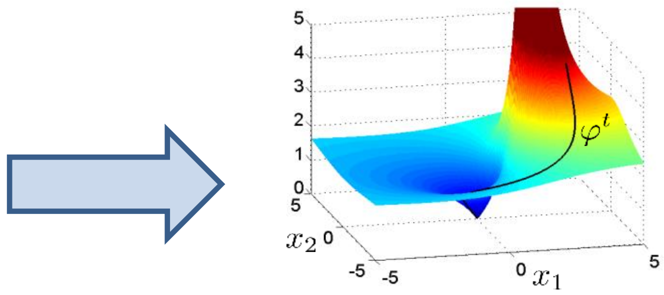
## State estimation

*Joint work with J. Mohet and J. Winkin*

# There are several ways to obtain a Lyapunov function through the Koopman operator

**With the Koopman eigenfunctions**




$$V = \sqrt{\phi_{\lambda_1}^2 + \phi_{\lambda_2}^2}$$

*[AM and Mezic, IEEE Trans. on Aut. Control 2016]*

**From the (finite-dimensional) lifted state dynamics**

**Linear lifted dynamics**
$\dot{z} \approx \boldsymbol{A}\, z, \qquad z \in \mathbb{R}^N$

→ Riccati equation →

**Quadratic Lyapunov function**
$V(x) = z^T \boldsymbol{Q} z$

→ $z = e(x)$ →

**Nonlinear Lyapunov function**
$\bar{V}(x) = e(x)^T \boldsymbol{Q} e(x)$

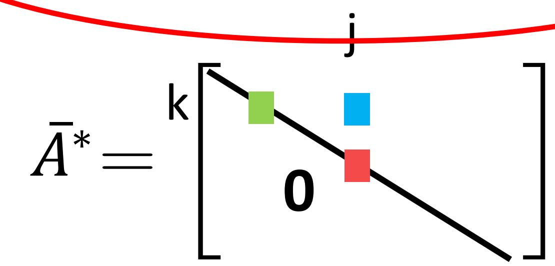
*[AM, Mezic and Sootla, in Koopman operator in Systems and Control (ed. AM, Mezic & Susuki)]*

# A Lyapunov functional for the dual Koopman system yields a Lyapunov function for the nonlinear system

**Dual Koopman system** $\dot{f} = A^* f, \quad f \in \mathcal{H}$

**Hypothesis**

- ➔ triangular form $\langle A^* e_j, e_k \rangle = 0 \ \forall j < k$
- ➔ diagonal entries $\Re(\langle A^* e_k, e_k \rangle) < 0 \ \forall k$




**SATISFIED?**

**Lyapunov functional**

$$\mathcal{V}(f) = \sum_{j \in \mathbb{N}} \epsilon_j \left| \langle f, e_j \rangle \right|^2, \quad \mathcal{V}: \mathcal{H} \to \mathbb{R}^+$$

$$\epsilon_j = \max_{k<j} \frac{|\langle A^* e_j, e_k \rangle|^2}{4\, b_{jk} \left| \Re\left(\langle A^* e_j, e_j \rangle\right)\right| \left|\Re(\langle A^* e_k, e_k \rangle)\right|}$$

$$\mathcal{V}\left((K^t)^* f\right) < \mathcal{V}(f) \quad \forall f \in \mathcal{H} \setminus \{0\}, \ \forall t > 0$$

**Nonlinear dynamical system** $\dot{x} = F(x), \quad x \in \mathbb{R}^n$

**Lyapunov function** $V(x) = \mathcal{V}(k_x) = \sum_{j \in \mathbb{N}} \epsilon_j \left| e_j(x) \right|^2$

**CONVERGENT?**

$$V\left(\varphi^t(x)\right) = \mathcal{V}\left(k_{\varphi^t(x)}\right) = \mathcal{V}\left((K^t)^* k_x\right) < \mathcal{V}(k_x) = V(x) \quad \forall t > 0$$

# We obtain systematic "global" stability criteria to estimate basins of attraction

**For analytic vector fields with a (hyperbolic) equilibrium ➔ Hardy space** $\mathcal{F} = H^2(\mathbb{D}^n)$

- ✓ Triangular form (if linearized system has a triangular form)
- ✓ Negative diagonal entries (if locally stable equilibrium)
- ✓ Convergence of the series $V(x) = \sum_{j\in\mathbb{N}} \epsilon_j x^{2\alpha(j)}$ for $|x_i| < \rho$

➔ <u>Systematic criteria</u> in terms of Taylor coefficients of the vector field

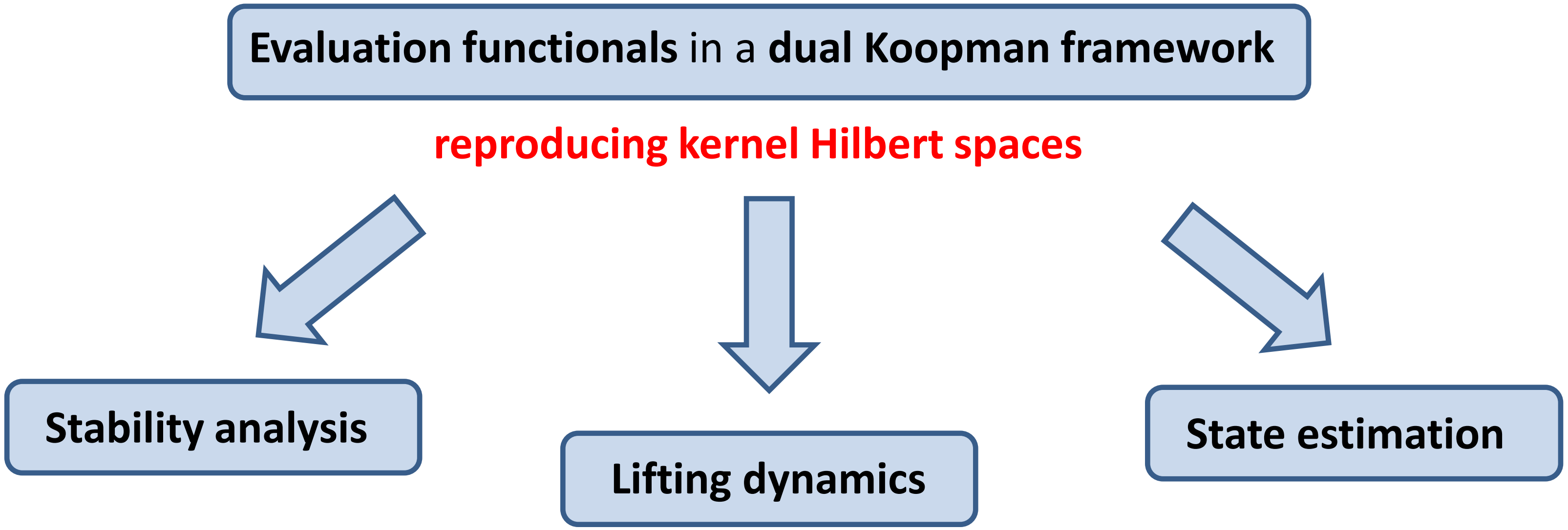
*[Mugisho and AM, CDC 2023]*

Alternatively, truncated series provide <u>Lyapunov function candidates</u> ➔ "Global" results and estimation of basins of attraction

**Extension to uniform exponential stability of switched systems**

Subsystems Jacobian matrices form a solvable Lie algebra (simultaneous triangularization) ➔ Switched system locally uniformly exponentially stable *[Liberzon, Hespanha and Morse, Systems and Control Letters, 1999]*

➔ "Global" results and estimation of basins of attraction *[Mugisho and AM, arXiv:2301.05529v1,2023]*




**A last message to the theoretician:**

**We can do theory!**

# Koopman operator or its dual:

# What matters more?

Alexandre Mauroy *(University of Namur, Belgium)*

UNIVERSITÉ DE NAMUR

naxys
Namur Institute for Complex Systems