Evaluating regions of attraction of LTI systems with saturation in IQS framework

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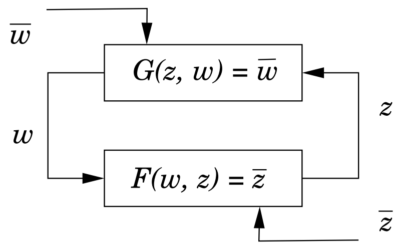
Introduction

- Integral Quadratic Separation framework
- Launcher attitude control: local stability of linear system with saturation
- IQS methodology for the given problem and results

- Well-posedness of a feedback loop



- Uniqueness and boundedness of internal signals for all bounded disturbances

$$\exists\gamma : \forall(\bar{w},\bar{z}) \in L_2 \times L_2 \ , \ \left\| \begin{array}{c} w - w_0 \\ z - z_0 \end{array} \right\| \leq \gamma \left\| \begin{array}{c} \bar{w} \\ \bar{z} \end{array} \right\| , \quad \begin{array}{l} G(z_0, w_0) = 0 \\ F(w_0, z_0) = 0 \end{array}$$

- **iff** exists a topological separator θ
- Negative on the inverse graph of the other component
- Positive definite on the graph of one component of the loop

$$\mathcal{G}^I(\bar{w}) = \{(w,z) \ : \ G(z,w) = \bar{w}\} \subset \{ \ (w,z) \ : \ \theta(w,z) \leq \phi_2(||\bar{w}||)\}$$
$$\mathcal{F}(\bar{z}) = \{(w,z) \ : \ F(w,z) = \bar{z}\} \subset \{ \ (w,z) \ : \ \theta(w,z) > -\phi_1(||\bar{z}||)\}$$

- Issues: How to choose θ ? How to test the separation inequalities ?

Integral Quadratic Separation (IQS)

- Choice of an Integral Quadratic Separator

$$\theta(w,z) = \left\langle \begin{pmatrix} z \\ w \end{pmatrix} \middle| \Theta \begin{pmatrix} z \\ w \end{pmatrix} \right\rangle = \int_0^\infty \begin{pmatrix} z^T(t) & w^T(t) \end{pmatrix} \Theta(t) \begin{pmatrix} z(t) \\ w(t) \end{pmatrix} dt$$

- Identical choice to IQC framework [Megretski, Rantzer, Jönsson]
- IQS is **necessary and sufficient** under assumptions (proof based on [Iwasaki 2001])
- One component is a linear application, can be descriptor form $F(w,z) = \mathcal{A}w - \mathcal{E}z$
- can be time-varying $\mathcal{A}(t)w(t) - \mathcal{E}(t)z(t)$ or frequency dep. $\hat{\mathcal{A}}(\omega)\hat{w}(\omega) - \hat{\mathcal{E}}(\omega)\hat{z}(\omega)$
- $\mathcal{A}(t)$, $\mathcal{E}(t)$ are bounded and $\mathcal{E}(t) = \mathcal{E}_1(t)\mathcal{E}_2$ where $\mathcal{E}_1(t)$ is full column rank
- The other component can be defined in a set

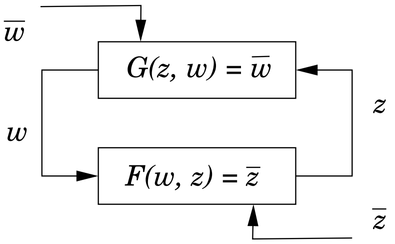
$$G(z,w) = \nabla(z) - w \ , \ \nabla \in \mathbb{W}$$

- $\mathbb{W}$ must have a linear-like property

$$\forall(z_1,z_2) \ , \ \forall\nabla \in \mathbb{W} \ , \ \exists\tilde{\nabla} \in \mathbb{W} \ : \ \nabla(z_1) - \nabla(z_2) = \tilde{\nabla}(z_1 - z_2)$$

- The matrix Θ must satisfy an IQC over $\mathbb{W}$ + an LMI involving $(\mathcal{E}, \mathcal{A})$

■ Global stability of a non-linear system $\dot{x} = f(x,t)$



$$G(z = \dot{x}, w = x) = \int_0^t z(\tau)d\tau - w(t),$$
$$F(w, z, t) = f(w, t) - z(t)$$

- $\bar{w}$ plays the rle of the initial conditions, $\bar{z}$ are external disturbances
- Well-posedness: for all bounded initial conditions and all bounded disturbances, the state remains bounded around the equilibrium $\equiv$ global stability

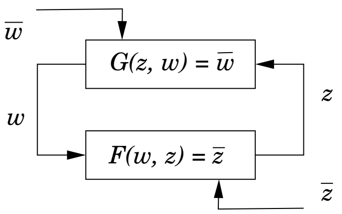
■ For linear systems $\dot{x}(t) = A(t)x(t)$, $\nabla = s^{-1}\mathbf{1}$, $s^{-1} \in \mathbb{C}_+$

- IQS: $\theta(w,z) = \int_0^\infty \begin{pmatrix} z^T(t) & w^T(t) \end{pmatrix} \begin{bmatrix} \mathbf{0} & -P(t) \\ -P(t) & -\dot{P}(t) \end{bmatrix} \begin{pmatrix} z(t) \\ w(t) \end{pmatrix} dt$

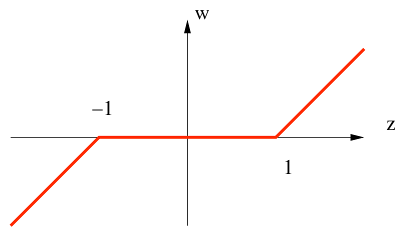
▲ $\theta(w,z) \leq 0$ for all $G(z,w) = 0$ **iff** $P(t) \geq \mathbf{0}$

▲ $\theta(w,z) > 0$ for all $F(w,z) = 0$ **iff** $A^T(t)P(t) + P(t)A(t) + \dot{P}(t) < \mathbf{0}$

■ Global stability of a system with a dead-zone



$$G_1(\dot{x}, x) = \int_0^t \dot{x}(\tau)d\tau - x(t),$$
$$G_2(g, v) = \mathrm{dz}(g(t)) - v(t),$$
$$F_1(x, v, \dot{x}, t) = f_1(x, v, t) - \dot{x}(t),$$
$$F_2(x, v, g, t) = f_2(x, v, t) - g(t)$$



■ IQS applies for linear f_1, f_2

- Dead-zone embedded in a sector uncertainty $\mathbb{W}_\infty = \{\nabla_\infty \ : \ 0 \leq \nabla_\infty(g) \leq g\}$

$$\mathcal{G}_2^I = \{(v,g) \ : \ G_2(g,v) = 0\} \subset \{ (v,g) \ : \ v = \nabla_\infty(g) \ , \ \nabla_\infty \in \mathbb{W}_\infty\}$$

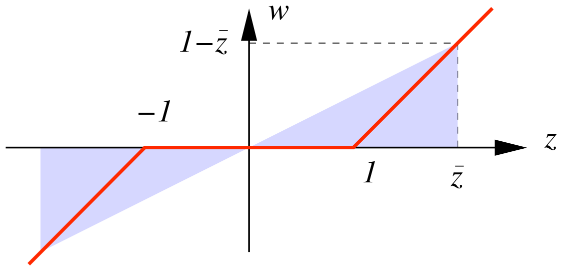
▲ This is the only source of conservatism

- LMI conditions obtained for the IQS defined by

$$\Theta = \left[\begin{array}{cc|cc} \mathbf{0} & \mathbf{0} & -P & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -p_1 \\ \hline -P & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -p_1 & \mathbf{0} & 2p_1 \end{array}\right], \quad \begin{array}{l} P > \mathbf{0}, \\ p_1 > 0. \end{array}$$

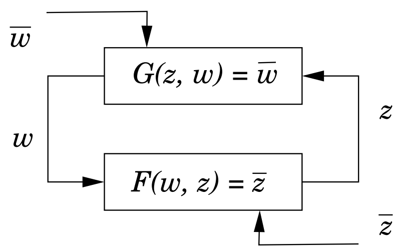
- Launcher in ballistic phase : attitude control
  - neglected atmospheric friction, sloshing modes, ext. perturbation, axes coupling: $I\ddot{\theta} = T$
  - Saturated actuator: $T = \text{sat}_{\bar{T}}(u) = u - \bar{T}\text{dz}(\frac{1}{\bar{T}}u)$
  - PD control $u = -K_P\theta - K_D\dot{\theta}$
- Global stability LMI test fails
  - Sector uncertainty includes $\nabla_\infty = 1$ for which the system is $I\ddot{\theta} = 0$ (unstable)
  - LMIs succeeds (whatever $\bar{g} < \infty$) if dead-zone is restricted to belong to

$$\mathbb{W}_{\bar{g}} = \{\nabla_{\bar{g}} \ : \ 0 \leq \nabla_{\bar{g}}(g) \leq \tfrac{1-\bar{g}}{\bar{g}} g\}$$



  - Useful if one can prove for constrained $x(0)$ that $|g(\theta)| \leq \bar{g}$ holds $\forall \theta \geq 0$.
- How can one prove local properties in IQS framework ?

- Well-posedness of a feedback loop



  - Uniqueness and boundedness of internal signals for all bounded disturbances

$$\exists \gamma : \forall (\bar{w}, \bar{z}) \in L_2 \times L_2 \ , \ \left\| \begin{array}{c} w - w_0 \\ z - z_0 \end{array} \right\| \leq \gamma \left\| \begin{array}{c} \bar{w} \\ \bar{z} \end{array} \right\| , \quad \begin{array}{l} G(z_0, w_0) = 0 \\ F(w_0, z_0) = 0 \end{array}$$

  - How to introduce initial conditions $x(0)$ and “final” conditions $g(\theta)$ in IQS framework?
- Square-root of the Dirac operator: linear operator such that

$$x \mapsto \varphi_\theta x \quad : \quad \begin{array}{l} < \varphi_\theta x | M \varphi_\theta x >= \int_0^\infty \varphi_\theta x^T(t) M \varphi_\theta x(t) dt = x^T(\theta) M x(\theta) \\ < \varphi_{\theta_1} x | M \varphi_{\theta_2} x >= 0 \ \text{ if } \ \theta_1 \neq \theta_2 \end{array}$$

  - Such operator is also used for PDE to describe states on the boundary

Initial conditions dependent IQS

- System with initial and final conditions writes as

$$\left(\begin{array}{c} \varphi_0 x \\ \mathcal{T}_\theta \dot{x} \\ \hline \mathcal{T}_\theta g \\ \hline \varphi_\theta g \end{array}\right) = \left[\begin{array}{cc|c|c} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ A & \mathbf{0} & B & \mathbf{0} \\ \hline C & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & C & \mathbf{0} & \mathbf{0} \end{array}\right] \left(\begin{array}{c} \mathcal{T}_\theta x \\ \varphi_\theta x \\ \hline \mathcal{T}_\theta v \\ \hline \varphi_0 x \end{array}\right)$$

- ⚠ $\mathcal{T}_\theta x$ is the truncated signal such that $\mathcal{T}_\theta x(t) = x(t)$ for $t \leq \theta$ and $= 0$ for $t > \theta$.
- The integration operator is redefined as a mapping of $(\varphi_0 x, \mathcal{T}_\theta \dot{x})$ to $(\mathcal{T}_\theta x, \varphi_\theta x)$.
- Restricted sector constraint assumed to hold up to $t = \theta$ (i.e. $\mathcal{T}_\theta v = \nabla_{\bar{g}} \mathcal{T}_\theta g$)
- Goal is to find sets $1 \geq x^T(0) Q x(0) =< \varphi_0 x | Q \varphi_0 x >$ s.t. $g(\theta) = \|\varphi_\theta g\| < \bar{g}$.

- Problem defined in this way is a well-posedness problem with ∇ composed of 3 blocs
  - IQS framework applies and gives conservative LMI conditions
  - Equivalent to LaSalle invariance principle with $V(x) = x^T Q x$ (ellipsoidal sets of IC)

System augmentation with derivatives

- How to reduce conservatism ?
  - Needed a description of the dead-zone better than sector uncertainty
  - Needed to have dead-zone dependent sets of initial conditions
- Both features derived via descriptor modeling of system augmented with $\dot{v}$ and $\dot{g}$

$$v = \mathrm{dz}(g) \ : \ \left\{ \begin{array}{lll} \text{if } g > 1 & v = g - 1 & \dot{v} = \dot{g} \\ \text{if } |g| \geq 1 & v = 0 & \dot{v} = 0 \\ \text{if } g < -1 & v = g + 1 & \dot{v} = \dot{g} \end{array} \right.$$

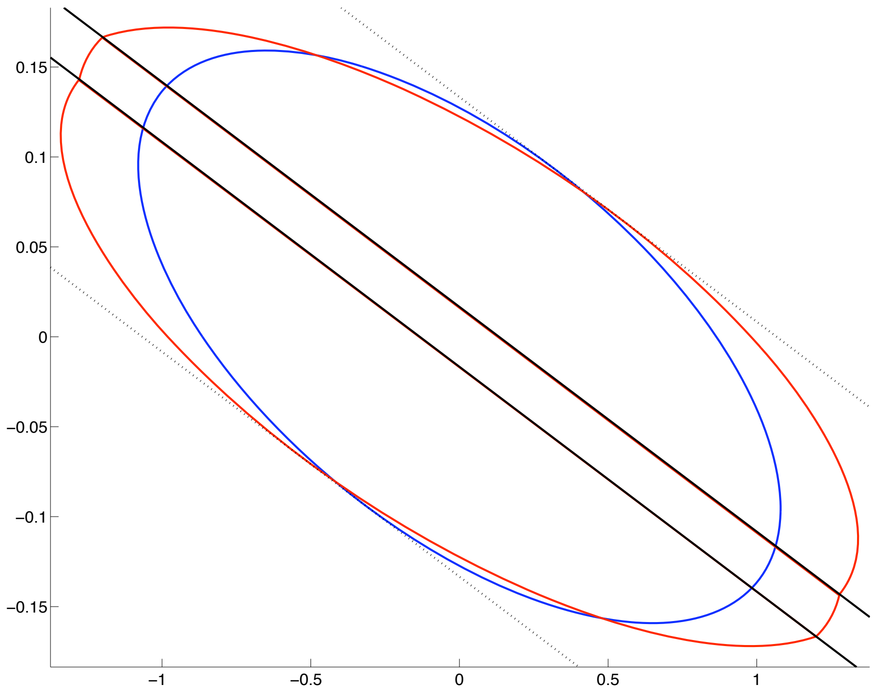
  - For IQS, link between $\dot{v}$ and $\dot{g}$ is embedded in $\dot{v} = \nabla_{\{0,1\}} \dot{g}$, with $\nabla_{\{0,1\}} \in \{0, 1\}$.
  - Also needed to specify that v is the integral of $\dot{v}$ (thus descriptor form)

$$
\left[\begin{array}{cccc|c|c|c|c}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & -C & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\begin{pmatrix}
\varphi_0 x\\ \varphi_0 v\\ \mathcal{T}_\theta \dot{x}\\ \mathcal{T}_\theta \dot{v}\\ \hline \mathcal{T}_\theta g\\ \hline \varphi_\theta g\\ \hline \mathcal{T}_\theta \dot{g}\\ \hline \varphi_\theta g
\end{pmatrix}
=
\left[\begin{array}{cccc|c|c|c|cc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
A & B & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & C & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & C & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0
\end{array}\right]
\begin{pmatrix}
\mathcal{T}_\theta x\\ \mathcal{T}_\theta v\\ \varphi_\theta x\\ \varphi_\theta v\\ \hline \mathcal{T}_\theta v\\ \hline \varphi_\theta v\\ \hline \mathcal{T}_\theta \dot{v}\\ \hline \varphi_0 x\\ \varphi_0 v
\end{pmatrix}
$$

- Problem defined in this way is a well-posedness problem with ∇ composed of 5 blocs
  - IQS framework applies and gives less conservativeLMI conditions
  - Equivalent to LaSalle invariance principle with

$$
V(x) = \begin{pmatrix} x \\ v \end{pmatrix}^T Q_a \begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} x \\ \mathrm{dz}(Cx) \end{pmatrix}^T Q_a \begin{pmatrix} x \\ \mathrm{dz}(Cx) \end{pmatrix}
$$

- LMIs tested on the launcher example



  - Sets of initial conditions for which $|g(\theta)| \leq 8$ is guaranteed
  - Improvement thanks to piecewise quadratic sets of initial conditions

- IQS framework can handle local stability issues
  - Provides LMI tests - conservative
  - System augmentation + descriptor modeling = reduction of conservatism

- Prospectives
  - Improved construction of the IQS $\equiv$ “generalized sector conditions”
  - Further system augmentation with higher derivatives (?)
  - Simultaneous handling of saturation, uncertainties, delays...