Integral Quadratic Separation

applied to polytopic systems

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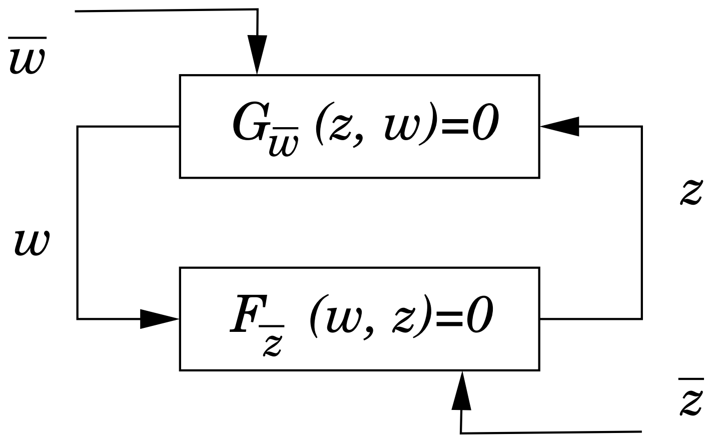
Well-posedness & topological separation



Well-Posedness:

Bounded $(\bar{w}, \bar{z}) \Rightarrow$ unique bounded (w, z)

- [Safonov 80] $\exists\theta$ topological separator:

$$\mathcal{F}(\bar{z}) = \{(w,z)\ :\ F_{\bar{z}}(w,z)=0\} \subset \{\ (w,z)\ :\ \theta(w,z) > -\phi_1(||\bar{z}||)\}$$
$$\mathcal{G}^I(\bar{w}) = \{(w,z)\ :\ G_{\bar{w}}(z,w)=0\} \subset \{\ (w,z)\ :\ \theta(w,z) \leq \phi_2(||\bar{w}||)\}$$

Related results :

- Stability (θ Lyapunov certificate), Passivity (θ storage function), IQC ...
- Robust analysis of Linear uncertain systems [Iwasaki, Scherer]

Integral Quadratic Separation

- For the case of linear application with uncertain operator

$$\mathcal{E}z(t) = \mathcal{A}w(t)\ ,\quad w(t) = [\nabla z](t) \quad \nabla \in \mathbb{\nabla}$$

where $\mathcal{E} = \mathcal{E}_1\mathcal{E}_2$ with $\mathcal{E}_1$ full column rank,

- Integral Quadratic Separator : $\exists \Theta$, matrix, solution of LMI

$$\begin{bmatrix} \mathcal{E}_1 & -\mathcal{A} \end{bmatrix}^{\perp *} \Theta \begin{bmatrix} \mathcal{E}_1 & -\mathcal{A} \end{bmatrix}^{\perp} > 0$$

and Integral Quadratic Constraint (IQC) $\forall \nabla \in \mathbb{\nabla}$

$$\int_0^{\infty} \begin{pmatrix} \mathcal{E}_2 z(t) \\ [\nabla z](t) \end{pmatrix}^* \Theta \begin{pmatrix} \mathcal{E}_2 z(t) \\ [\nabla z](t) \end{pmatrix} dt \leq 0$$

- For some given $\mathbb{\nabla}$, $\exists$ LMI conditions for Θ solution to IQC.
- LMIs are conservative except in few special cases [Meinsma et al., 1997].

- Integral Quadratic Separation Example: impulse-to-norm performance of

$$\left\{ \begin{array}{l} E\dot{x} = Ax + Bv \\ g = Cx + Dv \end{array} \right.$$

- [ECC'09] Equivalent to well-posedness of

$$\begin{bmatrix} E & 0 & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \varphi_0 x \\ \dot{x} \\ \hline \varphi_0 g \\ g \end{pmatrix} = \begin{bmatrix} 0 & B \\ A & 0 \\ 0 & D \\ C & 0 \end{bmatrix} \begin{pmatrix} x \\ v \end{pmatrix}$$

$$x(t) = \left[\mathcal{I} \begin{pmatrix} \varphi_0 x \\ \dot{x} \end{pmatrix} \right](t) = x(0) + \int_0^t \dot{x}(\tau) d\tau$$

$$v = \nabla_{i2n} \begin{pmatrix} \varphi_0 g \\ g \end{pmatrix} \quad : \quad v = \alpha \varphi_0 \mathbf{1}_m \ , \ \ |\alpha| \leq \frac{1}{\gamma} \left\| \begin{matrix} \varphi_0 g \\ g \end{matrix} \right\|$$

Numerical tests for uncertain polytopic applications ?

$$\mathcal{E}(\xi)z(t) = \mathcal{A}(\xi)w(t) \ , \ \ w(t) = [\nabla z](t) \quad \nabla \in \mathbb{W}$$

where $\mathcal{E}(\xi) = \mathcal{E}_1(\xi)\mathcal{E}_2$ with $\mathcal{E}_1(\xi)$ full column rank and

$$\left[\begin{array}{cc} \mathcal{E}_1(\xi) & -\mathcal{A}(\xi) \end{array} \right] = \sum_{i=1}^{\bar{\imath}} \xi_i \left[\begin{array}{cc} \mathcal{E}_1^{[i]} & -\mathcal{A}^{[i]} \end{array} \right]$$

is a modeling of parametric (constant) uncertainties constrained by

$$\xi \in \Xi = \left\{ \xi_i \geq 0 \ , \ \ \sum_{i=1}^{\bar{\imath}} \xi_i = 1 \right\} .$$

- Give LMI tests
- Control the numerical complexity / conservatism trade-off

General Slack Variables result

- If $\Theta^{[i]}$ satisfy the IQC conditions w.r.t. $\mathbb{W}$ and $\exists H$ s.t. for all vertices

$$\Theta^{[i]} > H \left[\begin{array}{cc} \mathcal{E}_1^{[i]} & -\mathcal{A}^{[i]} \end{array} \right] + \left[\begin{array}{cc} \mathcal{E}_1^{[i]} & -\mathcal{A}^{[i]} \end{array} \right]^* H^*$$

well-posedness is satisfied for all ξ in the simplex Ξ.

- Large LMI conditions and large H matrix.
- H can be artificially increased by adding artificial rows/columns in $\mathcal{E}_1$ and $\mathcal{A}$.
- Unnecessary degrees of freedom ?
- On examples such tests encounter numerical problems.

① Interpretation of slack variable via Finsler Lemma

- H is such that, for some τ, the following quantity is negligible

$$\begin{bmatrix} \mathcal{E}_1^*(\xi) \\ -\mathcal{A}^*(\xi) \end{bmatrix} \left(H^* - \tau \begin{bmatrix} \mathcal{E}_1(\xi) & -\mathcal{A}(\xi) \end{bmatrix} \right) \quad , \quad \forall \xi \in \Xi$$

- If $\begin{bmatrix} \mathcal{E}_1(\xi) & -\mathcal{A}(\xi) \end{bmatrix}$ have zero columns, one can choose the same for H.
- Reduces number of decision variables.

② Factorization of uncertain rows

- If $\mathcal{E}_1$ is square without uncertainties.

- Algorithm for factorization as

$$\begin{bmatrix} \mathcal{E}_1^{-1}\mathcal{A}^{[i]} \\ 1_b \end{bmatrix} = \underbrace{\begin{bmatrix} \mathcal{B}_1 & \mathcal{B}_2 \\ 0_{b,\underline{a}} & 1_b \end{bmatrix}}_{\mathcal{D}} \begin{bmatrix} \mathcal{C}^{[i]} \\ 1_b \end{bmatrix}$$

where $\mathcal{B}_2$ gathers all the rows without uncertainties,
$\mathcal{C}^{[i]}$ gathers the uncertain rows (nb rows$(\mathcal{C}^{[i]}) = \underline{a} \leq a =$ nb rows$(\mathcal{A}^{[i]})$),
and $\mathcal{B}_1^*\mathcal{B}_1 = 1_{\underline{a}}$, $\mathcal{B}_1^*\mathcal{B}_2 = 0$.

- Computation of the factorization: less than 1% of LMI test.

- If $\Theta^{[i]}$ satisfy the IQC conditions w.r.t. $\mathbb{W}$ and $\exists H$ s.t. for all vertices

$$\mathcal{D}^*\Theta^{[i]}\mathcal{D} > \hat{H}\begin{bmatrix} 1_{\underline{a}} & -\mathcal{C}^{[i]} \end{bmatrix} + \begin{bmatrix} 1_{\underline{a}} \\ -\mathcal{C}^{[i]\prime} \end{bmatrix}\hat{H}^*$$

well-posedness is satisfied for all ξ in the simplex Ξ.

- No conservatism compared to general slack variable result.
- Size of LMIs reduced from $(a+b)\times(a+b)$ to $(\underline{a}+b)\times(\underline{a}+b)$
- Size of variable $\hat{H}$ also reduced by factor $(a-\underline{a})$ compared to H
- Suppressed unnecessary degrees of freedom
- One can expect improved numerical properties

③ Case of unique separator

- If Θ satisfy the IQC conditions w.r.t. $\mathbb{W}$ and $\exists H$ s.t. for all vertices

$$\Theta > H \left[\begin{array}{cc} \mathcal{E}_1^{[i]} & -\mathcal{A}^{[i]} \end{array} \right] + \left[\begin{array}{cc} \mathcal{E}_1^{[i]} & -\mathcal{A}^{[i]} \end{array} \right]^* H^*$$

well-posedness is satisfied for all ξ in the simplex Ξ.

▲ Is the slack variable H usefull in that case ?

▲ If $\mathcal{E}_1$ is square and not affected by uncertainties: it is not.

- If Θ satisfy the IQC conditions w.r.t. $\mathbb{W}$
  and if $\left[\begin{array}{cc} 1_a & 0 \end{array} \right] \Theta \left[\begin{array}{cc} 1_a & 0 \end{array} \right]^* \leq 0$ (which is the case for known sets $\mathbb{W}$)
  s.t. for all vertices

$$\left[\begin{array}{c} \mathcal{E}_1^{-1}\mathcal{A}^{[i]} \\ 1 \end{array} \right]^* \Theta \left[\begin{array}{c} \mathcal{E}_1^{-1}\mathcal{A}^{[i]} \\ 1 \end{array} \right] > 0$$

well-posedness is satisfied for all ξ in the simplex Ξ.

▲ This non conservative case $\equiv$ usual “quadratic stability” framework.

Example of impulse-to-norm performance

- LMIs for the general slack variable result ($E = 1$ for simplicity)

$$\left[\begin{array}{cccc|cc} -P^{[i]} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -P^{[i]} & 0 \\ 0 & 0 & -\tau^{[i]}1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\tau^{[i]}1 & 0 & 0 \\ \hline 0 & -P^{[i]} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & Q^{[i]} \end{array}\right]$$

$$> H \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 0 & -B^{[i]} \\ 0 & 1 & 0 & 0 & -A^{[i]} & 0 \\ 0 & 0 & 1 & 0 & 0 & -D^{[i]} \\ 0 & 0 & 0 & 1 & -C^{[i]} & 0 \end{array}\right] + \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 0 & -B^{[i]} \\ 0 & 1 & 0 & 0 & -A^{[i]} & 0 \\ 0 & 0 & 1 & 0 & 0 & -D^{[i]} \\ 0 & 0 & 0 & 1 & -C^{[i]} & 0 \end{array}\right]^* H^*$$

$$P^{[i]} > 0 \ , \ \text{trace}(Q^{[i]}) \leq \tau^{[i]}\gamma^2$$

Example of impulse-to-norm performance

- If B and D are not affected by uncertainty
- Factorization of rows gives following non conservative LMIs

$$\left[\begin{array}{cc|cc} 0 & 0 & -P^{[i]} & 0 \\ 0 & -\tau^{[i]}1 & 0 & 0 \\ \hline -P^{[i]} & 0 & 0 & 0 \\ 0 & 0 & 0 & -B^*P^{[i]}B-\tau^{[i]}D^*D+Q^{[i]} \end{array}\right]$$

$$> \hat{H}\left[\begin{array}{cc|cc} 1 & 0 & -A^{[i]} & 0 \\ 0 & 1 & -C^{[i]} & 0 \end{array}\right] + \left[\begin{array}{cc|cc} 1 & 0 & -A^{[i]} & 0 \\ 0 & 1 & -C^{[i]} & 0 \end{array}\right]^* \hat{H}^*$$

$$P^{[i]} > 0 \;,\; \text{trace}(Q^{[i]}) \leq \tau^{[i]}\gamma^2$$

- Noticing the zero columns, gives following non conservative LMIs

$$\left[\begin{array}{cc|c} 0 & 0 & -P^{[i]} \\ 0 & -\tau^{[i]}1 & 0 \\ \hline -P^{[i]} & 0 & 0 \end{array}\right] > \tilde{H} \left[\begin{array}{cc|c} 1 & 0 & -A^{[i]} \\ 0 & 1 & -C^{[i]} \end{array}\right] + \left[\begin{array}{cc|c} 1 & 0 & -A^{[i]} \\ 0 & 1 & -C^{[i]} \end{array}\right]^* \tilde{H}^*$$

$$Q^{[i]} > B^* P^{[i]} B + \tau^{[i]} D^* D$$

$$P^{[i]} > 0 \ , \ \text{trace}(Q^{[i]}) \leq \tau^{[i]} \gamma^2$$

Example of impulse-to-norm performance

- Removed slack variables when solving for a unique separator ("quadratic stability")

$$A^{[i]*}P + PA^{[i]} + \tau C^{[i]*}C^{[i]} < 0\ , \qquad P > 0\ ,$$
$$Q < B^{[i]*}PB^{[i]} + \tau D^{[i]*}D^{[i]}\ , \qquad \text{trace}(Q) < \tau\gamma^2\ .$$

▲ Note that when $D = 0$,
impulse-to-norm is equivalent to H_2 norm performance.

Conclusions

- Integral Quadratic Separation and Slack Variables
  - SV's can be used in the general IQS framework

    (includes issues such as performances, robustness, time-delay...

    in descriptor form)
  - Proposed methodology for coding efficiently the SV results

    (removed unnecessary degrees of freedom and reduced size of LMIs)

- Future work
  - Results currently coded in a toolbox for robust analysis

    Romuald (Matlab/YALMIP based tool - will be freely distributed)

    (extension of RoMulOC www.laas.fr/OLOCEP/romuloc)
  - Preliminary tests done on a satellite attitude control example
  - More testing on real medium size applications to come

Thank you