# Robust $H_2$ perfomance of discrete-time periodic systems: LMIs with reduced dimensions

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## Introduction

### Linear discrete-time periodic systems

- $N$-periodic state-space systems $x_{k+1} = A_k x_k$ with $A_{k+N} = A_k$
- Used to model
  - linear systems with periodic parametric changes
  - sampled linearized dynamics of NL system along periodic trajectory
  - multi-rate sampled-data systems
  - ...
- Periodic models also used for control of LTI systems
  - Periodic control may have better performances than static control
- As for all linear models, uncertainties should be included:

$$x_{k+1} = A_k(\Delta_k) x_k$$

## LMI-based results for uncertain periodic systems?

- Extensions of existing results for LTI systems

  [Bittanti, Colaneri, De Souza, Trofino, Farges...]
- High numerical burden - can it be reduced ?
- Conservatism - how can it be reduced ?

## Outline

1. About modeling of $N$-periodic systems: lifted, descriptor, and other models
2. Existing $H_2$-performance LMI analysis results
3. Contribution: less conservative with limited number of decision variables
4. A numerical example



# ❶ About modeling of $N$-periodic systems

## Various models related to various state "definition"

- $N$-periodic system with performance inputs/outputs

$$x_{k+1} = A_k x_k + B_k w_k \ , \ z_k = C_k x_k + D_k w_k$$

- Instantaneous state $\{ \ x_k \in \mathrm{R}^{n_k} \ \}_{k=0,1\dots}$
- The instantaneous state may be of varying dimensions
- Any $x_{i_0 \in \{1 \dots N\}}$ can define initial condition
- The trajectory over period $i$ is defined by

$$\hat{x}_{i_0,i} = \text{vec}\begin{pmatrix} x_{i_0+Ni} & \cdots & x_{i_0+N(i+1)-1} \end{pmatrix}$$

- Overall dimensions of system

$$x_k \in \mathrm{R}^{n_k} \ , \ n = \textstyle\sum_{k=1}^N n_k$$

$$w_k \in \mathrm{R}^{m_k} \ , \ m = \textstyle\sum_{k=1}^N m_k \ , \ z_k \in \mathrm{R}^{p_k} \ , \ p = \textstyle\sum_{k=1}^N p_k$$

## ❶ About modeling of $N$-periodic systems

**Various models related to various state "definition"**

- $N$-periodic : $x_{k+1} = A_k x_k + B_k w_k \ , \ z_k = C_k x_k + D_k w_k$
- "Lifted" LTI system with vectors of all instantaneous input/outputs over a period

$$x_{i_0+N(i+1)} = \Phi_{i_0,N} x_{i_0+Ni} + \Psi_{i_0} \hat{w}_{i_0,i} \ , \ \hat{z}_{i_0,i} = \Upsilon_{i_0} x_{i_0+Ni} + \Xi_{i_0} \hat{w}_{i_0,i}$$

$$\hat{w}_{i_0,i} = \text{vec} \begin{pmatrix} w_{i_0+Ni} & \cdots & w_{i_0+N(i+1)-1} \end{pmatrix}$$
$$\hat{z}_{i_0,i} = \text{vec} \begin{pmatrix} z_{i_0+Ni} & \cdots & z_{i_0+N(i+1)-1} \end{pmatrix}$$

$$\Phi_{i_0,j} = A_{i_0+N-1} \cdots A_{i_0+N-j} \ , \ \Psi_{i_0} = \begin{bmatrix} \Phi_{i_0,N-1} B_{i_0} & \cdots & B_{i_0+N-1} \end{bmatrix}$$

▲ One representative state for each period $\{ \ x_{i_0+iN} \in \mathrm{R}^{n_{i_0}} \ \}_{i=0,1\ldots}$

▲ Smaller order model - with products between data matrices

▲ Model is dependent of the choice of $i_0 \in \{1 \ldots N\}$



## ❶ About modeling of $N$-periodic systems

**Various models related to various state "definition"**

- $N$-periodic : $x_{k+1} = A_k x_k + B_k w_k \ , \ z_k = C_k x_k + D_k w_k$
- "Cyclic LTI" model $\tilde{x}_{i_0,k+1} = \tilde{A}_{i_0} \tilde{x}_{i_0,k} + \tilde{B}_{i_0} \tilde{w}_{i_0,k}$

$$\tilde{A}_{i_0} = \begin{bmatrix} 0 & \cdots & 0 & A_{i_0+N-1} \\ A_{i_0} & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & A_{i_0+N-2} & 0 \end{bmatrix} \quad \tilde{x}_{i_0,k} = \begin{pmatrix} \tilde{x}_{i_0,k,1} \\ \vdots \\ \tilde{x}_{i_0,k,N-1} \end{pmatrix}$$

where $\tilde{x}_{i_0,k,j} = x_{i_0+k}$ if $j \equiv k[N]$ and otherwise $\tilde{x}_{i_0,k,j} = 0$

▲ Switching state-vector containing all states over one period

$$\left\{ \ \sum_{k=iN}^{i(N+1)-1} \tilde{x}_{i_0,k} = \hat{x}_i \in \mathrm{R}^n \ \right\}_{i=0,1\ldots}$$

▲ Model is dependent of the choice of $i_0 \in \{1 \ldots N\}$

## ❶ About modeling of $N$-periodic systems

**Various models related to various state "definition"**

- Proposed descriptor model: $\begin{bmatrix} M_{i_0} \\ N_{i_0} \end{bmatrix} q_{i_0,i} = \begin{pmatrix} 0 \\ \hat{z}_{i_0,i} \end{pmatrix}$

$$\begin{bmatrix} M_{i_0} \\ N_{i_0} \end{bmatrix} = \left[\begin{array}{cccc|ccc} A_{i_0} & -1_{n_{i_0+1}} & & 0 & B_{i_0} & & 0 \\ & \ddots & \ddots & & & \ddots & \\ 0 & & A_{i_0+N-1} & -1_{n_{i_0}} & 0 & & B_{i_0+N-1} \\ \hline C_{i_0} & 0 & & 0 & D_{i_0} & & 0 \\ & \ddots & \ddots & & & \ddots & \\ 0 & & C_{i_0+N-1} & 0 & 0 & & D_{i_0+N-1} \end{array}\right]$$

- ▲ $q_{i_0,i} = \text{vec}\left( x_{i_0+iN} \quad \cdots \quad x_{i_0+(i+1)N} \;\middle|\; \hat{w}_{i_0,i} \right)$
- ▲ Model is dependent of the choice of $i_0 \in \{1 \dots N\}$



## ❷ Existing $H_2$-performance analysis results

**Stability analysis results**

- Exist for all types of models
- Stability tests independent of choice of $i_0$
- Most results with periodic Lyapunov function $V_k = x_k^T P_k x_k$ , $P_{k+N} = P_k$.
  - ▲ nb decisions variables proportional to $n^2$
- For "lifted" models one Lyapunov function $V = x_{i_0+Ni}^T P x_{i_0+Ni}$
  - ▲ nb decisions variables proportional to $n_{i_0}^2$
  - ▲ "Lifted" models not suitable for robustness

- Proposed descriptor model makes robustness results possible

  with reduced decision variables of Lyapunov function $V = x_{i_0+Ni}^T P x_{i_0+Ni}$

## ❷ Existing $H_2$-performance analysis results

**Robust $H_2$-performance results**

- For polytopic uncertain periodic system $\sum_{v=1}^{\bar{v}} \zeta_v = 1 \;,\; \zeta_v \geq 0$

$$A_k(\Delta) = \sum_{v=1}^{\bar{v}} \zeta_v A_k^{[v]} \;,\; B_k(\Delta) = \sum_{v=1}^{\bar{v}} \zeta_v B_k^{[v]} \;\; \ldots$$

- Upper bound on robust $H_2$ $\quad \gamma_{qs} = \min \sum_{k=0}^{N-1} \text{Trace}(T_k^{[v]})$

$$\begin{array}{c} {A_k^{[v]}}^T P_{k+1} A_k^{[v]} - P_k + {C_k^{[v]}}^T C_k^{[v]} < 0 \\ {B_k^{[v]}}^T P_{k+1} B_k^{[v]} - T_k^{[v]} + {D_k^{[v]}}^T D_k^{[v]} < 0 \end{array}$$

- ▲ Based on "quadratic stability" framework
- ▲ Maybe very conservative because parameter-independent Lyapunov fct



## ❷ Existing $H_2$-performance analysis results

**Robust $H_2$-performance results**

- Upper bound on robust $H_2$ $\quad \gamma_{f} = \min \sum_{k=0}^{N-1} \text{Trace}(T_k^{[v]})$

$$\begin{bmatrix} -P_k^{[v]} + {C_k^{[v]}}^T C_k^{[v]} & 0 \\ 0 & P_{k+1}^{[v]} \end{bmatrix} + \left\langle \begin{bmatrix} \tilde{F}_{1k} \\ \tilde{F}_{2k} \end{bmatrix} \begin{bmatrix} A_k^{[v]} & -1 \end{bmatrix} \right\rangle < 0$$

$$\begin{bmatrix} -T_k^{[v]} + {D_k^{[v]}}^T D_k^{[v]} & 0 \\ 0 & P_{k+1}^{[v]} \end{bmatrix} + \left\langle \begin{bmatrix} \tilde{F}_{3k} \\ \tilde{F}_{4k} \end{bmatrix} \begin{bmatrix} B_k^{[v]} & -1 \end{bmatrix} \right\rangle < 0$$

- ▲ Based on "slack variables" framework [Geromel 98, SCL 00], [Farges 05]
- ▲ Much less conservative
- ▲ Many more decisions variables - proportional to $n^2\bar{v}$
- ▲ Independent of an $i_0$

## ❷ Existing $H_2$-performance analysis results

**Robust $H_2$-performance results**

- Upper bound on robust $H_2$ $\quad \gamma_f = \min \sum_{k=0}^{N-1} \text{Trace}(T_k^{[v]})$

$$\begin{bmatrix} -P_k^{[v]} + C_k^{[v]T} C_k^{[v]} & 0 \\ 0 & P_{k+1}^{[v]} \end{bmatrix} + \left\langle \begin{bmatrix} \tilde{F}_{1k} \\ \tilde{F}_{2k} \end{bmatrix} \begin{bmatrix} A_k^{[v]} & -1 \end{bmatrix} \right\rangle < 0$$

$$\begin{bmatrix} -T_k^{[v]} + D_k^{[v]T} D_k^{[v]} & 0 \\ 0 & P_{k+1}^{[v]} \end{bmatrix} + \left\langle \begin{bmatrix} \tilde{F}_{3k} \\ \tilde{F}_{4k} \end{bmatrix} \begin{bmatrix} B_k^{[v]} & -1 \end{bmatrix} \right\rangle < 0$$

- Slack variables define a virtual periodic system with same $H_2$ performance

$$\eta_{k+1} = \tilde{G}_k \eta_k + \tilde{H}_k w_k \;,\; z_k = C_k(\Delta)\eta_k + D_k(\Delta) w_k$$

$$\begin{bmatrix} \tilde{F}_{1k} \\ \tilde{F}_{2k} \end{bmatrix} = \begin{bmatrix} \tilde{G}_k{}^T \\ -1 \end{bmatrix} (-\tilde{F}_{2k}) \;,\; \begin{bmatrix} \tilde{F}_{3k} \\ \tilde{F}_{4k} \end{bmatrix} = \begin{bmatrix} \tilde{H}_k{}^T \\ -1 \end{bmatrix} (-\tilde{F}_{4k})$$



## ❸ Main result

**Robust $H_2$-performance results**

- Upper bound on robust $H_2$ $\quad \gamma_{i_0} = \min \sum_{k=0}^{N-1} \text{Trace}(T_k^{[v]})$

$$\left[\begin{array}{ccc|c} -P^{[v]} & 0 & 0 & W^{[v]} \\ 0 & 0_{n-n_{i_0}} & 0 & 0 \\ 0 & 0 & P^{[v]} & 0 \\ \hline W^{[v]T} & 0 & 0 & -T^{[v]} \end{array}\right] + N_{i_0}^{[v]T} N_{i_0}^{[v]} + \left\langle F M_{i_0}^{[v]} \right\rangle < 0$$

- Proved to be less conservative
- Less decisions variables - proportional to $(n_{i_0} + m)^2 \bar{v}$
- Dependent of $i_0$, best upper-bound $\quad \min_{i_0=\{1\dots N\}} \gamma_{i_0}$

## Robust $H_2$-performance results

- Upper bound on robust $H_2$ $\quad \gamma_{i_0} = \min \sum_{k=0}^{N-1} \text{Trace}(T_k^{[v]})$

$$\mathcal{L}(P^{[v]}, W^{[v]}, T^{[v]}) + N_{i_0}^{[v]^T} N_{i_0}^{[v]} + \left\langle F M_{i_0}^{[v]} \right\rangle < 0$$

▲ $F$ defines a virtual <u>non-causal</u> periodic system with same $H_2$ performance

$$\left[\begin{array}{cccc|ccc} \multicolumn{7}{c}{F} \\ \hline C_{i_0} & 0 & & 0 & D_{i_0} & & 0 \\ & \ddots & \ddots & & & \ddots & \\ 0 & & C_{i_0+N-1} & 0 & 0 & & D_{i_0+N-1} \end{array}\right] \left(\begin{array}{c} x_{i_0+iN} \\ \vdots \\ x_{i_0+(i+1)N} \\ \hline \hat{w}_{i_0,i} \end{array}\right) = \left(\begin{array}{c} 0 \\ \hat{z}_{i_0,i} \end{array}\right)$$

- Other upper-bounds that can be defined

▲ $\gamma_{i_0}^d$ : with structure on $F$ to make the virtual system causal, yet dynamic

▲ $\gamma_{i_0}^s$ : with structure on $F$ to make the virtual system static





## Dimensions of the example

$n_k = 2,\ p_k = 2,\ m_k = 1,\ N = 3,\ \bar{v} = 4$

- Lower bound on worst case $H_2$ performance evaluated on a grid

| | $\gamma_{qs}$ | $\gamma_f$ | $\gamma_{wc}$ |
|---|---|---|---|
| | 19.8482 | 7.7257 | 6.8430 |
| nb vars/rows | 25/45 | 103/99 | $\infty$ |

| $i_0$ | $\gamma_{nc}$ | $\gamma_d$ | $\gamma_s$ |
|---|---|---|---|
| 1 | 7.4100 | 8.1173 | 8.2347 |
| 2 | 7.4003 | 8.1236 | 8.2607 |
| 3 | 7.1730 | 7.4646 | 8.3482 |
| nb vars/rows | 127/57 | 109/57 | 91/57 |

## Key results

- New descriptor type modeling
- Less conservative LMI results
- Size of the LMI problem maintained

## Future work

- Extensions for other performance criteria
- Usage of virtual non-causal system
- Feedback control design