Stabilization and passification of uncertain systems via static output feedback

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Part of a cooperation program funded by РФФИ CNRS

Passification and Simple Adaptive Control

- [S&CL 2008] Robust stabilization of LTV systems

  (and L_2 gain attenuation) via SAC

$$u(t) = w(t) - K(t)y(t) \quad , \quad \dot{K}(t) = Gy(t)y^T(t)\Gamma - \phi(K(t))$$

of uncertain system

$$\dot{x}(t) = A(\Delta(t))x(t) + B(\Delta(t))u(t) \quad , \quad y(t) = C(\Delta(t))x(t)$$

can be proved on the basis of the existence of

- a static output-feedback gain $u(t) = w(t) - F(\Delta(t))y(t)$
- a linear combination of inputs and outputs $z(t) = Gy(t) + D(\Delta(t))w(t)$

  that makes the closed-loop passive w.r.t. w/z.

- For given G the problem can be cast (at the expense of some conservatism)

  into a convex optimization problem (LMI).

How to design the G matrix ?

- Similar to a static output feedback design problem : BMI
- Suggested heuristic in 3 steps with $\mathbf{\Delta}_1 \subset \mathbf{\Delta}_2 \subset \mathbf{\Delta}_3$
  1. Design of a robustly stabilizing SOF F for $\Delta \in \mathbf{\Delta}_1$
  2. For that F, design of a robustly passified output
     $$z(t) = Gy(t) + D(\Delta)w(t) \text{ for } \Delta \in \mathbf{\Delta}_2$$
  3. For that G, find $F(\Delta)$ that proves robust stability of SAC for $\Delta \in \mathbf{\Delta}_3$

- Step 3 see [S&CL 2008]
- Step 1 LQG based heuristic with optimization [Ait Rami, El Ghaoui 1996]
- Step 2 is LMI and is done while minimizing $D(\Delta)$.

Considered type of uncertain systems

- State-space LTI systems

$$\dot{x}(t) = A(\delta)x(t) + B(\delta)u(t) \quad , \quad y(t) = Cx(t)$$

- with affine scalar uncertainties

$$\begin{bmatrix} A(\delta) & B(\delta) \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} + \sum_{i=1}^{N} \delta_i \begin{bmatrix} A_i & B_i \end{bmatrix}$$

- in intervals that include zero ($\underline{\delta}_i \leq 0, \overline{\delta}_i \geq 0$)

$$\boldsymbol{\Delta} = \left\{ \delta = \begin{pmatrix} \delta_1 & \dots & \delta_N \end{pmatrix} \ : \ \underline{\delta}_i \leq \delta_i \leq \overline{\delta}_i \right\}$$

- The finite set of vertices is denoted

$$\boldsymbol{\Delta}_v = \left\{ \delta = \begin{pmatrix} \delta_1 & \dots & \delta_N \end{pmatrix} \ : \ \delta_i \in \{\underline{\delta}_i, \overline{\delta}_i\} \right\}$$

❶ Design of a robustly stabilizing SOF F

■ Static output-feedback seen as structured state-feedback

- $\dot{x} = A(\delta)x + B(\delta)u, \;\; u = -\underbrace{FC}_{S}x.$
- [Gadewadikar et al. 2007] All such quadratically stabilizing S parameterized:

$$S = FC = R^{-1}(B^T(\delta)P + L(\delta)) \; ,$$

$$A^T(\delta)P + PA(\delta) - PB(\delta)R^{-1}B^T(\delta)P + Q + L(\delta)^T R^{-1} L(\delta) \leq 0 \; .$$

▲ Non convex, but...

- R and Q are LQR type matrices,

one can choose $R = \hat{R}$, $Q = \hat{Q}$ based on closed-loop requirement issues.

▲ Yet, it remains non convex (Riccati type equation)

$$S = FC = \hat{R}^{-1}(B^T(\delta)P + L(\delta)) \; ,$$

$$A^T(\delta)P + PA(\delta) - PB(\delta)\hat{R}^{-1}B^T(\delta)P + \hat{Q} + L(\delta)^T \hat{R}^{-1} L(\delta) \leq 0 \; .$$

● [Ait Rami, El Ghaoui 1996] If the pairs $(A(\delta), B(\delta))$ are stabilizable for all values in the finite set $\Delta \in \mathbf{\Delta}_v$ and if $(\tilde{Q}(\delta), \tilde{R}(\delta))$ are given positive definite matrices, then the solution to the following LMI maximization problem

$$\{P^*(\delta)\}_{\delta\in\mathbf{\Delta}_v} = \arg\max \textstyle\sum_{\delta\in\mathbf{\Delta}_v} \mathrm{Trace}(P(\delta))$$

$$\forall\delta \in \mathbf{\Delta}_v \quad \begin{bmatrix} A^T(\delta)P(\delta) + P(\delta)A(\delta) + \tilde{Q}(\delta) & P(\delta)B(\delta) \\ B^T(\delta)P(\delta) & \tilde{R}(\delta) \end{bmatrix} \geq 0$$

are positive semi-definite $P^*(\delta) \geq 0$ and solution to the Riccati equations

$$A^T(\delta)P^*(\delta) + P^*(\delta)A(\delta) - P^*(\delta)B(\delta)\tilde{R}^{-1}(\delta)B^T(\delta)P^*(\delta) + \tilde{Q}(\delta) = 0.$$

▲ In that special case: Riccati problem $\equiv$ convex optimization.

❶ Design of a robustly stabilizing SOF F

▲ Equivalence no more holds in considered case, but strategy may succeed:

$$\max \text{Trace}(P)$$

$$FC = \hat{R}^{-1}(B^T(\delta)P + L(\delta))$$

$$\tilde{Q}(\delta) = \hat{Q} + L(\delta)^T\hat{R}^{-1}L(\delta)$$

$$\begin{bmatrix} A^T(\delta)P + PA(\delta) + \tilde{Q}(\delta) & PB(\delta) \\ B^T(\delta)P & \hat{R} \end{bmatrix} \geq 0$$

▲ Convexity achieved by choosing a priori $\mu > 0$ such that

$$(1+\mu)\hat{Q} \geq \tilde{Q} \;, \quad \begin{bmatrix} \mu\hat{Q} & L(\delta) \\ L^T(\delta) & \hat{R} \end{bmatrix} \geq 0$$

● Affine w.r.t. δ : test on vertices $\delta \in \mathbf{\Delta}_v$ equivalent to robust test $\delta \in \mathbf{\Delta}$

❶ Design of a robustly stabilizing SOF F

Proposed static output-feedback design heuristic

1-a- Choose desired closed-loop poles $\lambda = \begin{pmatrix} \lambda_1 & \dots & \lambda_n \end{pmatrix}$

1-b- Apply [Johnson 1988] to get $\hat{Q}$ and $\hat{R}$ such that the nominal system ($\delta = 0$) has optimal LQR state-feedback with poles in λ.

2- Choose some $\mu > 0$ and solve the LMI maximization problem.

3- Check (LMI) if the closed-loop system is quadratically stable for that F.

- ▲ No guarantee of success
- ● Based on practical design requirements
- ● The tests are all convex optimizations (LMIs)

❷ Design of a robustly passified output for $F = \hat{F}$ given

Passified output with minimal feedthrough gain

$$\dot{x} = \underbrace{(A(\delta) - B(\delta)\hat{F}C)}_{A(\hat{F},\delta)\ \text{stable}} x + B(\delta)w \ , \quad z = GCx + D(\delta)w$$

- Passified output exists: take $G \sim 0$ and $D > 0$

(such that $\dot{V}(x(t)) < w^T(t)z(t)$ where $V(t)$ is Lyapunov function)

- For SAC, one rather needs $D \sim 0$

- Solve LMI optimization problem with $D(\delta) = \sum_{i=1}^{N} \delta_i D_i$

$$\min \sum_{\delta \in \mathbf{\Delta}_v} \text{Trace}(D(\delta))$$

$$1 \leq H \ , \quad \forall \delta \in \mathbf{\Delta}_v$$

$$\begin{bmatrix} A^T(\hat{F},\delta)H + HA(\hat{F},\delta) & HB(\delta) \\ B^T(\delta)H & 0 \end{bmatrix} \leq \begin{bmatrix} -\epsilon 1 & C^T G^T \\ GC & D(\delta) + D^T(\delta) \end{bmatrix}$$

- Longitudinal angle of aircraft

$$\begin{array}{rcl} \dot{\vartheta} & = & \omega_z \\ \dot{\omega}_z & = & -a^{\alpha}_{mz}\vartheta - a^{\omega z}_{mz}\omega_z + a^{\alpha}_{mz}\Theta + a^{\delta}_{mz}u \\ \dot{\Theta} & = & -a^{\alpha}_{y}\vartheta + a^{\alpha}_{y}\Theta \end{array} \quad , \quad y = \begin{pmatrix} \vartheta \\ \omega_z \end{pmatrix}$$

30% variation on state matrix coefficients and 50% variation on input gain

- Small size example to test properties of the heuristic

- Test of the influence of tuning parameter μ

$$\lambda = \begin{pmatrix} -5 & -5 & -5 \end{pmatrix}$$

μ	F	$\|\mathcal{R}\|$	G	$\max(D(\delta))$
0.01	$\begin{bmatrix} -0.95 & -0.03 \end{bmatrix}$	351	$\begin{bmatrix} 3.88 & -7.38 \end{bmatrix}$	12.68
0.1	$\begin{bmatrix} -2.43 & -0.31 \end{bmatrix}$	348	$\begin{bmatrix} 0.83 & -6.46 \end{bmatrix}$	1.93
1	$\begin{bmatrix} -4.94 & -0.97 \end{bmatrix}$	337	$\begin{bmatrix} -0.50 & -5.94 \end{bmatrix}$	0.67
10	$\begin{bmatrix} -4.95 & -0.97 \end{bmatrix}$	336	$\begin{bmatrix} -0.50 & -5.94 \end{bmatrix}$	0.67
100	$\begin{bmatrix} -4.93 & -0.97 \end{bmatrix}$	336	$\begin{bmatrix} -0.50 & -5.94 \end{bmatrix}$	0.67

- $\|\mathcal{R}\| \neq 0$: LMI maximization does not converge to Riccati equality
- Found robustly stabilizing F and passified outputs with limited feedthrough.

Numerical example

■ Test of the influence of tuning parameter λ

$$\mu = 1 \ , \ \ \lambda = \hat{\lambda} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

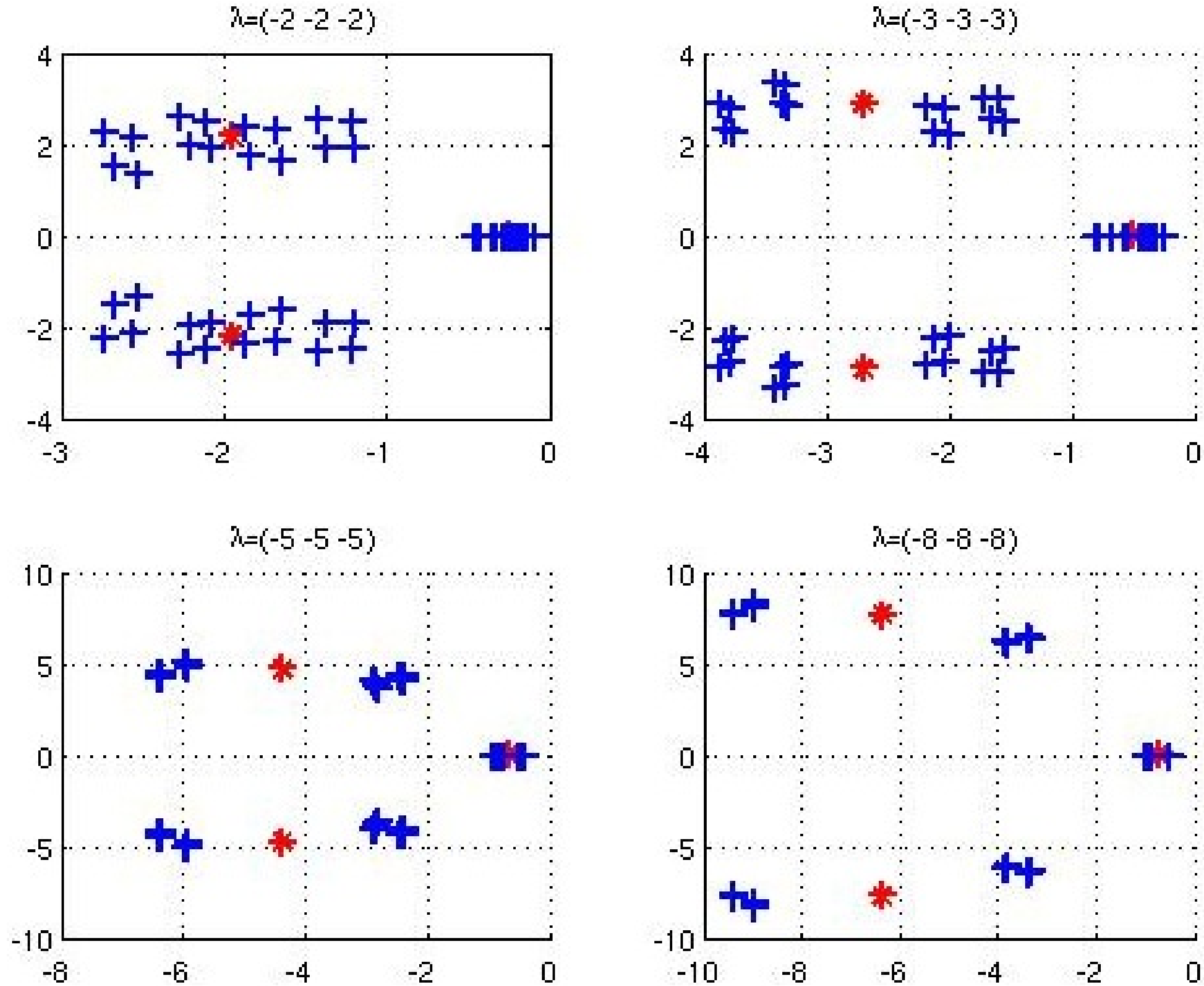
$\hat{\lambda}$	F	$\|\mathcal{R}\|$	G	$\max(D(\delta))$
-1	infeasible			
-2	$\begin{bmatrix} -0.36 & -0.25 \end{bmatrix}$	1.95031	$\begin{bmatrix} -3.68 & -7.38 \end{bmatrix}$	2.66
-3	$\begin{bmatrix} -1.39 & -0.49 \end{bmatrix}$	16.2933	$\begin{bmatrix} -2.02 & -6.29 \end{bmatrix}$	1.30
-5	$\begin{bmatrix} -4.94 & -0.97 \end{bmatrix}$	337.377	$\begin{bmatrix} -0.50 & -5.94 \end{bmatrix}$	0.67
-8	$\begin{bmatrix} -12.89 & -1.51 \end{bmatrix}$	6072.42	$\begin{bmatrix} 0.93 & -5.84 \end{bmatrix}$	0.43

▲ Failure for $\hat{\lambda} = -1$: sensitivity of heuristic to choice of LQR matrices $\hat{Q}$, $\hat{R}$

▲ $\|\mathcal{R}\|$ increases as robust pole location is made harder

Numerical example

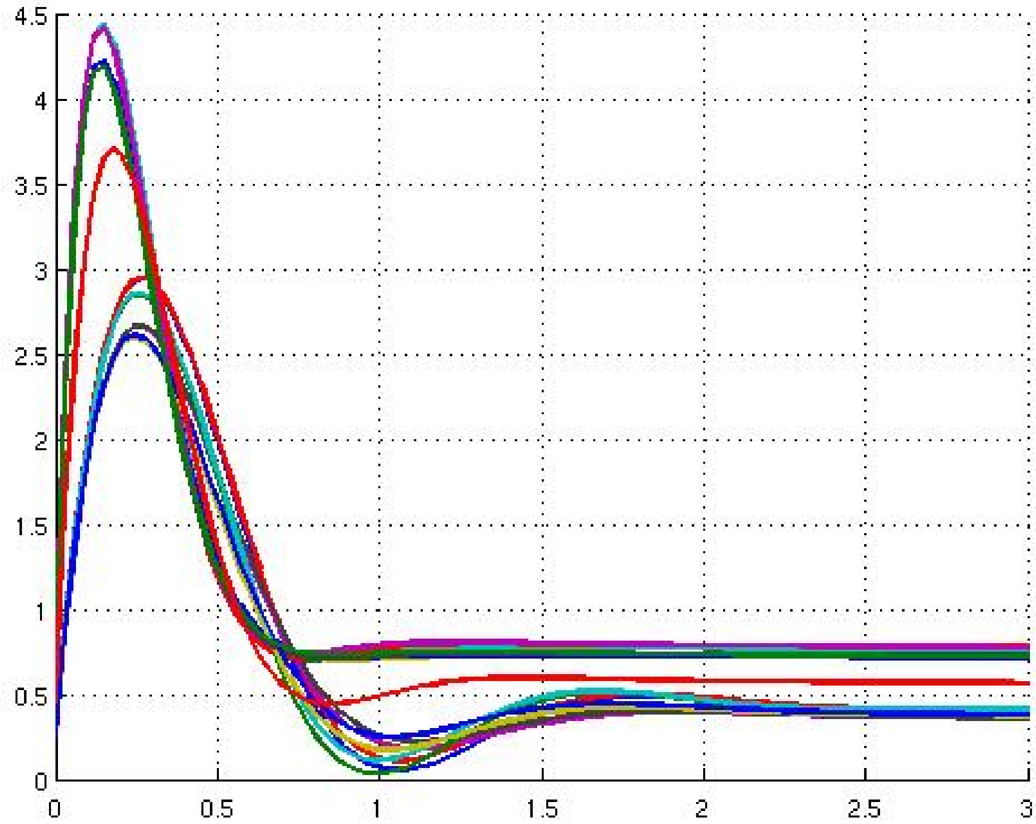
- Poles for closed-loop nominal system (✳) and at vertices (+)



- Not at prescribed location (impossible) but follow requirements.

Numerical example

- Step responses of the passive output $z = Gy + D(\delta)w$ for all vertices (results obtained with $\mu = 1$ and $\lambda = (-5\ -5\ -5)$)



Conclusions

- **Heuristic algorithm for static output feedback stabilization**
  - Successful on examples from COMPLib [Leibfritz 2004] (nominal case)
  - No guarantee and dependent of μ, $\hat{Q}$, $\hat{R}$
  - Need for more testing and extensions for robust stabilization case

- **Heuristic for the design of passifiable output**
  - Successful first tests
  - Needed of validation for robust analysis of obtained SAC
  - Need for a methodology to choose the sets $\mathbf{\Delta}_1 \subset \mathbf{\Delta}_2 \subset \mathbf{\Delta}_3$

- **Other heuristics and comparisons: yet to come**

Thank you