Mixed LMI/Randomized Methods for Static Output Feedback Control Design

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Introduction

- Work in the framework of a joint project funded by CNRS and RFBR
- ”Robust and adaptive control of complex systems”

- Confrontation of LMI and Randomized techniques
- Get some insight on an LMI-based heuristic for SOF design
- Using hit-and-run methods to explore randomly the set of solutions.

- Useful algorithm
- Extensions to robust multi-objective problems
- Design sets of controllers

Outline

1. LMI-based heuristic for SOF design
2. Hit-and-run method
3. Proposed algorithms
4. Examples from the Compleib library
5. Conclusions & extensions

❶ LMI-based heuristic for SOF design

- Static state and output feedback

$$\dot{x} = Ax + Bu, \;\; y = Cx$$

$$u = K_s x \;\mid\; u = Ky$$

- Classical LMI results describing all SF gains

$$\mathcal{K}_{SF} = \left\{ K_s = SX^{-1} \;:\; AX + BS + XA^T + S^T B^T < \mathbf{0}, \;\; X > \mathbf{0} \right\}$$

- [ECC01] LMI result describing a subset of SOF gains

$$\mathcal{K}_{SOF}^{K_s} = \left\{ K = -F^{-1}Z \;:\; \mathcal{L}^{K_s}(P, Z, F) < \mathbf{0} \;,\; P > \mathbf{0} \right\}$$

where $\mathcal{L}^{K_s}(P, Z, F) =$

$$\begin{bmatrix} A^T P + PA & PB \\ B^T P & \mathbf{0} \end{bmatrix} + \begin{bmatrix} {K_s}^T \\ -\mathbf{1} \end{bmatrix} \begin{bmatrix} ZC & F \end{bmatrix} + \begin{bmatrix} C^T Z^T \\ F^T \end{bmatrix} \begin{bmatrix} K_s & -\mathbf{1} \end{bmatrix}$$

- [ECC01] LMI result describing a subset of SOF gains

$$\mathcal{K}_{SOF}^{K_s} = \left\{ K = -F^{-1}Z \ : \ \mathcal{L}^{K_s}(P,Z,F) < \mathbf{0} \ , \ P > \mathbf{0} \right\}$$

where $\mathcal{L}^{K_s}(P,Z,F) =$

$$\begin{bmatrix} A^TP + PA & PB \\ B^TP & \mathbf{0} \end{bmatrix} + \begin{bmatrix} {K_s}^T \\ -\mathbf{1} \end{bmatrix} \begin{bmatrix} ZC & F \end{bmatrix} + \begin{bmatrix} C^TZ^T \\ F^T \end{bmatrix} \begin{bmatrix} K_s & -\mathbf{1} \end{bmatrix}$$

- $K_s \notin \mathcal{K}_{SF} \ \Rightarrow \ \mathcal{K}_{SOF}^{K_s} = \emptyset$

$$\begin{bmatrix} \mathbf{1} & {K_s}^T \end{bmatrix} \mathcal{L}^{K_s}(P,Z,F) \begin{bmatrix} \mathbf{1} \\ K_s \end{bmatrix} = (A+BK_s)^TP + P(A+BK_s) < \mathbf{0}$$

- But $K_s \in \mathcal{K}_{SF} \ \not\Rightarrow \ \mathcal{K}_{SOF}^{K_s} \neq \emptyset$

  (P has to prove stability of $A + BK_s$ and $A + BKC$ simultaneously)

- [ECC01] LMI result describing a subset of SOF gains

$$\mathcal{K}_{SOF}^{K_s} = \left\{ K = -F^{-1}Z \ : \ \mathcal{L}^{K_s}(P,Z,F) < \mathbf{0} \ , \ P > \mathbf{0} \right\}$$

where $\mathcal{L}^{K_s}(P,Z,F) =$

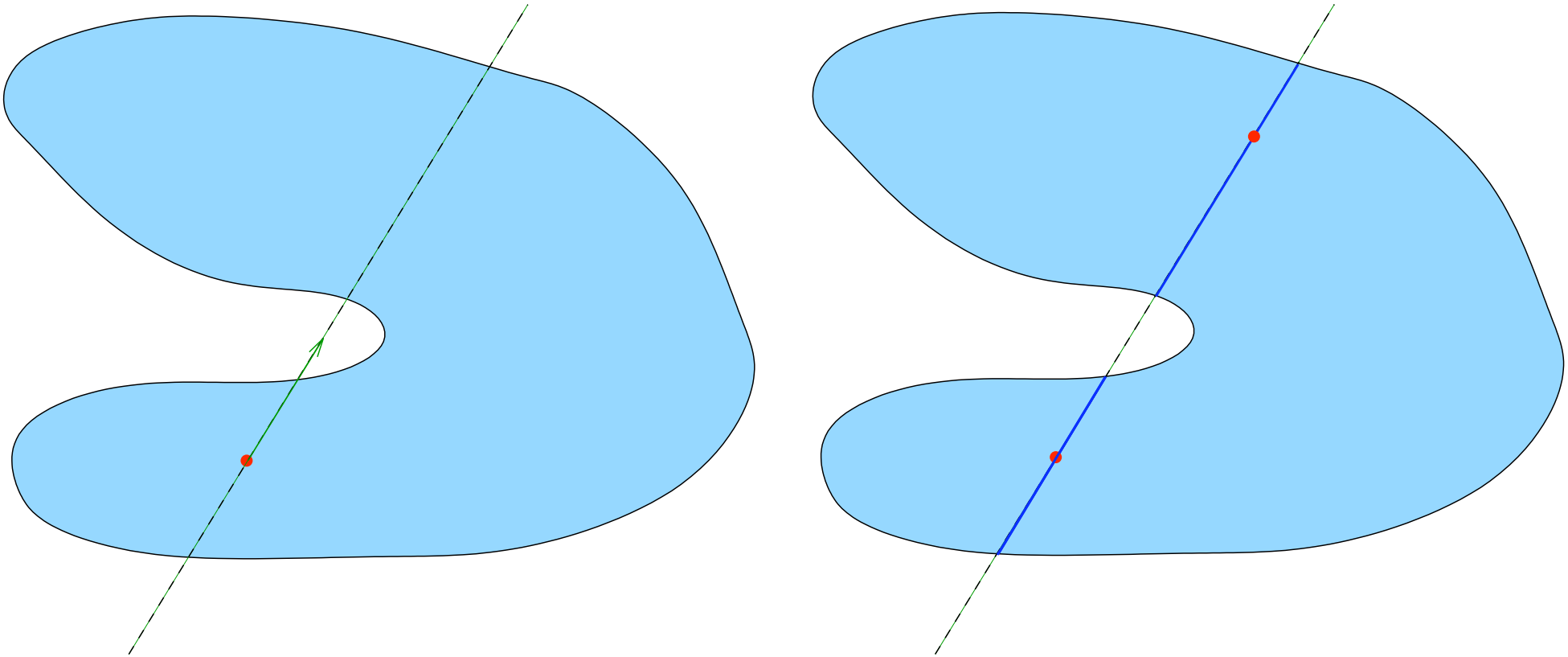
$$\begin{bmatrix} A^TP + PA & PB \\ B^TP & \mathbf{0} \end{bmatrix} + \begin{bmatrix} {K_s}^T \\ -\mathbf{1} \end{bmatrix} \begin{bmatrix} ZC & F \end{bmatrix} + \begin{bmatrix} C^TZ^T \\ F^T \end{bmatrix} \begin{bmatrix} K_s & -\mathbf{1} \end{bmatrix}$$

- $K_s \notin \mathcal{K}_{SF} \ \Rightarrow \ \mathcal{K}_{SOF}^{K_s} = \emptyset$ but $K_s \in \mathcal{K}_{SF} \not\Rightarrow \mathcal{K}_{SOF}^{K_s} \neq \emptyset$
- All SOF gains are represented: $\bigcup_{K_s \in \mathcal{K}_{SF}} \mathcal{K}_{SOF}^{K_s} = \mathcal{K}_{SOF}$

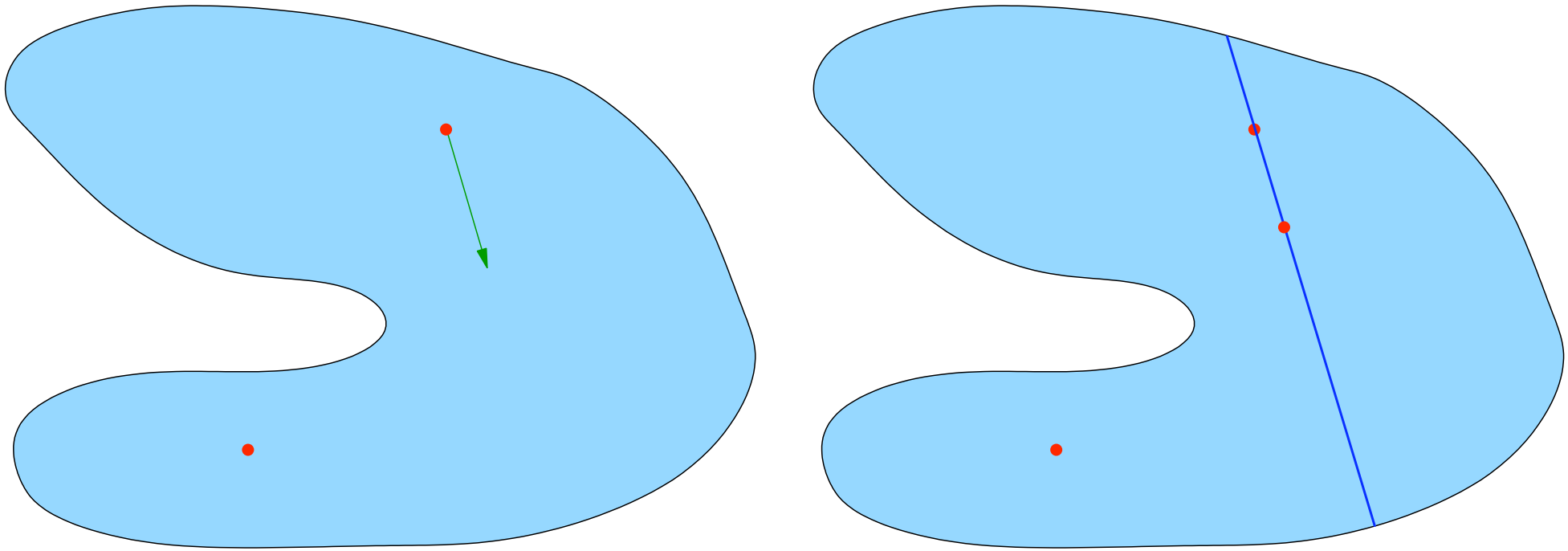
$$(K \in \mathcal{K}_{SOF} \ \Rightarrow \ KC \in \mathcal{K}_{SF} \text{ and } K \in \mathcal{K}_{SOF}^{KC})$$

- Aim: generate finite set $\{K_{s,i=1\ldots N}\} \subset \mathcal{K}_{SF}$
  and get approximation $\bigcup_{i=1\ldots N} \mathcal{K}_{SOF}^{K_{si}} \subset \mathcal{K}_{SOF}$.

- Starting from a feasible point, explore sets in random directions



- Iterate to get the needed number of points



- Intervals in the (random) direction obtained using a boundary oracle.
- New point is taken random in the interval.

- Considered sets

$$\mathcal{K}_{SF} = \{K_s : \ A + BK_s \ \text{Hurwitz}\}$$

$$\mathcal{K}_{SOF} = \{K : \ A + BKC \ \text{Hurwitz}\}$$

- From a known gain K_i,
- search in random direction $D = Y/||Y||$ (`Y=randn(m,p)`),
- find the set (for example using `fsolve`):

$$\Theta_{K_i} = \{\theta : \ f(\theta) = \max \mathrm{R}\lambda(A + B(K_i + \theta D)C) < 0\}$$

- Take a random value $\theta_i \in \Theta_{K_i}$ (with `rand`) and get a new point

$$K_{i+1} = K_i + \theta_i D$$

❸ Proposed algorithms

▲ How efficient is the following procedure for SOF design ?

"Take $K_s \in \mathcal{K}_{SF}$ *and solve* $\mathcal{L}^{K_s} < \mathbf{0}$"

■ Algorithm 1:

- 1- Find one value $K_s \in \mathcal{K}_{SF}$ by solving the corresponding LMI problem.
- 2- Using K_s as a starting point, generate $\{K_{s,i=1\ldots N}\} \subset \mathcal{K}_{SF}$ using H&R
- 3- Check if the LMIs $\mathcal{L}^{K_{s,i}} < \mathbf{0}$ is feasible.

● N_{sof}: number of cases when the LMIs are feasible at step • 3

● N_{sof}/N: efficiency of the procedure to find and SOF gain

❸ Proposed algorithms

■ Algorithm 1:

- 1- Find one value $K_s \in \mathcal{K}_{SF}$ by solving the corresponding LMI problem.
- 2- Using K_s as a starting point, generate $\{K_{s,i=1\ldots N}\} \subset \mathcal{K}_{SF}$ using H&R
- 3- Check if the LMI $\mathcal{L}^{K_{s,i}} < \mathbf{0}$ is feasible.

● N_{sof}: number of cases when the LMIs are feasible at step • 3

● Each solution of the LMIs at step • 3 gives an SOF K_j.

▲ Is the set $\{K_{j=1\ldots N_{sof}}\}$ a "good" subset of $\mathcal{K}_{SF}$?

■ Algorithm 2:

- 1- Find one value $K_s \in \mathcal{K}_{SF}$ by solving the corresponding LMI problem.
- 2- Using K_s as a starting point, generate $\{K_{s,i}\} \subset \mathcal{K}_{SF}$ using H&R
  and stop as soon as one value is such that $\mathcal{L}^{K_{s,i}} < \mathbf{0}$ is feasible.
- 3- Using K as a stating point to generate $\{K_{i=1\ldots N}\} \subset \mathcal{K}_{SOF}$ using H&R

● $\{K_{j=1\ldots N_{sof}}\}$ and $\{K_{i=1\ldots N}\}$ are two subsets of $\mathcal{K}_{SOF}$ for comparison.

❹ Examples from the Compleib library

- Compleib `www.complib.de` examples of SOF design problems
- All 53 non open-loop stable problems were tested
- $N_{sof} = 0$ (algorithm 1 fails) only for one example (`AC10`, known to be hard)
- $N_{sof}/N < 5\%$ for 11 problems (bad conditioning = num pb in LMIs)

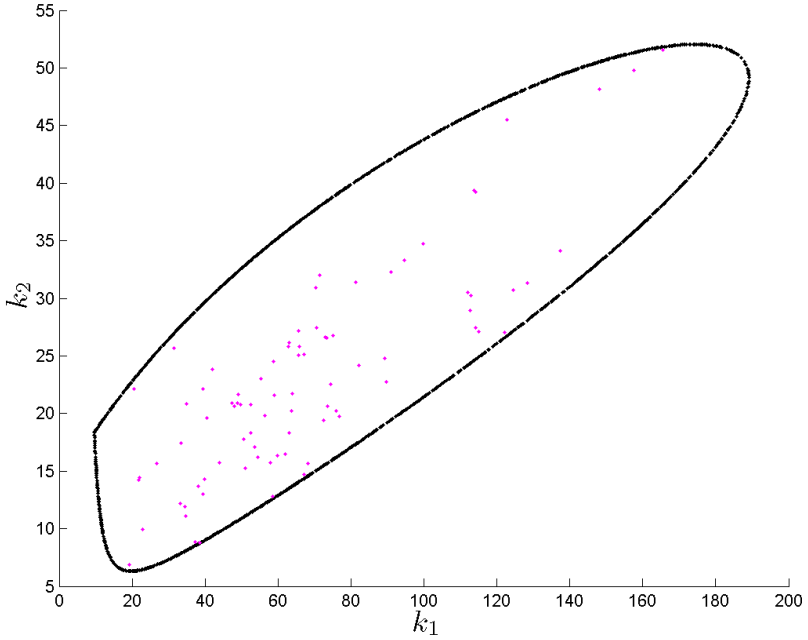
❹ Examples from the Compleib library

Ex.	n_x	n_u	n_y	OLS	N_{sof}/N
AC1	5	3	3	OLMS	240/1000
AC2	5	3	3	OLMS	334/1000
AC5	4	2	2	OLNS	283/1000
AC9	10	4	5	OLNS	34/1000
AC10	55	2	2	OLNS	*
AC11	5	2	4	OLNS	996/1000
AC12	4	3	4	OLNS	997/1000
AC13	28	3	4	OLNS	39/1000
AC14	40	3	4	OLNS	13/1000
AC18	10	2	2	OLNS	17/1000
HE1	4	2	1	OLNS	93/1000
HE3	8	4	6	OLNS	12/1000
HE4	8	4	6	OLNS	1000/1000
HE5	4	2	2	OLNS	13/1000
HE6	20	4	6	OLNS	321/1000
HE7	20	4	6	OLNS	327/1000
DIS2	3	2	2	OLNS	842/1000
DIS4	6	4	6	OLNS	1000/1000
DIS5	4	2	2	OLNS	725/1000
JE2	21	3	3	OLMS	94/1000
JE3	24	3	6	OLMS	44/1000
REA1	4	2	3	OLNS	999/1000
REA2	4	2	2	OLNS	554/1000
REA3	12	1	3	OLNS	965/1000

❹ Examples from the Compleib library

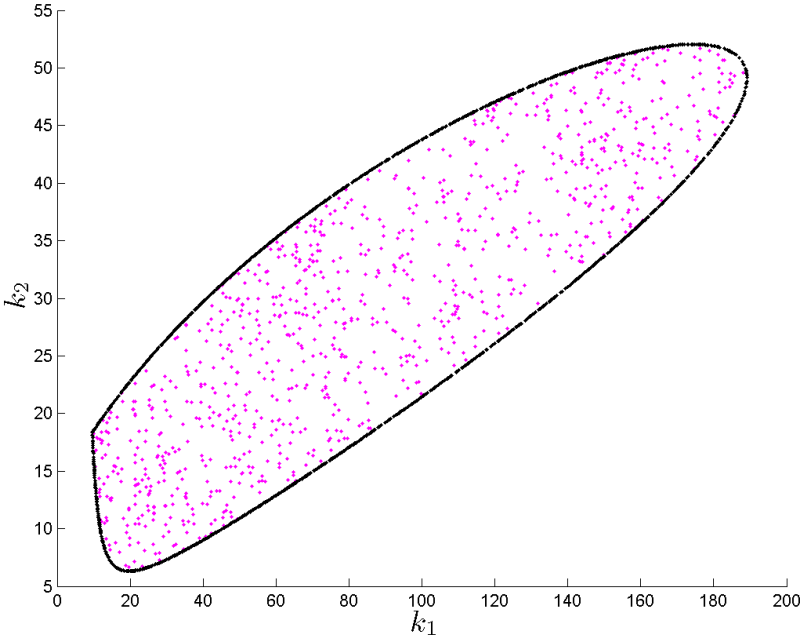
Ex.	n_x	n_u	n_y	OLS	N_{sof}/N
WEC1	10	3	4	OLNS	775/1000
BDT2	82	4	4	OLMS	43/1000
IH	21	11	10	OLMS	63/1000
CSE2	60	2	30	OLNS	10/10
PAS	5	1	3	OLMS	236/1000
TF1	7	2	4	OLMS	81/1000
TF2	7	2	3	OLMS	189/1000
TF3	7	2	3	OLMS	6/1000
NN1	3	1	2	OLNS	629/1000
NN2	2	1	1	OLMS	1000/1000
NN5	7	1	2	OLNS	84/1000
NN6	9	1	4	OLNS	983/1000
NN7	9	1	4	OLNS	620/1000
NN9	5	3	2	OLNS	7/1000
NN12	6	2	2	OLNS	28/1000
NN13	6	2	2	OLNS	77/1000
NN14	6	2	2	OLNS	44/1000
NN15	3	2	2	OLMS	821/1000
NN16	8	4	4	OLMS	61/1000
NN17	3	2	1	OLNS	125/1000
HF2D10	5	2	3	OLNS	991/1000
HF2D11	5	2	3	OLNS	993/1000
HF2D14	5	2	4	OLNS	1000/1000
HF2D15	5	2	4	OLNS	1000/1000
HF2D16	5	2	4	OLNS	998/1000
HF2D17	5	2	4	OLNS	1000/1000
HF2D18	5	2	2	OLNS	755/1000
TMD	6	2	4	OLNS	654/1000
FS	5	1	3	OLNS	977/1000

❹ Examples from the Compleib library



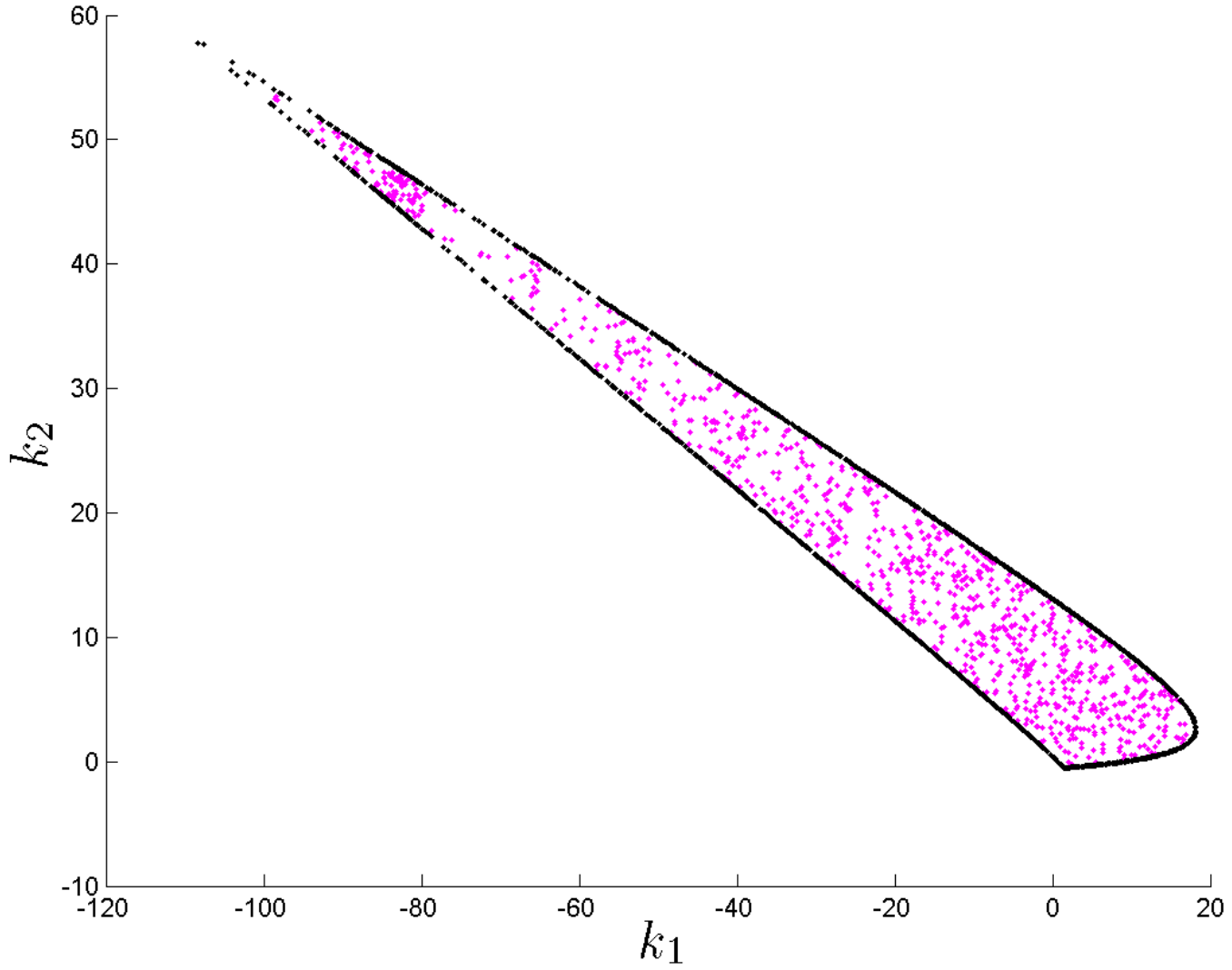
`NN5` SOF gains with Algo 1.

rather good distribution



`NN5` SOF gains with Algo 2.

very good distribution



AC7 SOF gains with Algo 2.

Conclusions

- Rather good properties of the studied LMI condition
- Combining LMI with H&R gives efficient algorithms for SOF design

  (attested on Compleib examples)
- Algorithms allow to describe as a finite set the interior of $\mathcal{K}_{SOF}$

  (one may choose the “best” one by inspection)
- LMI based technique: extends to robust/multi-objective problems easily

  (results to be submitted soon)