Integral Quadratic Separation Framework

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Outline

❶ Topological separation & related theory

- Well-posedness definition and main result
- Relations with Lyapunov theory
- The case of linear uncertain systems : quadratic separation
- The Lur'e problem
- Relations with IQC framework

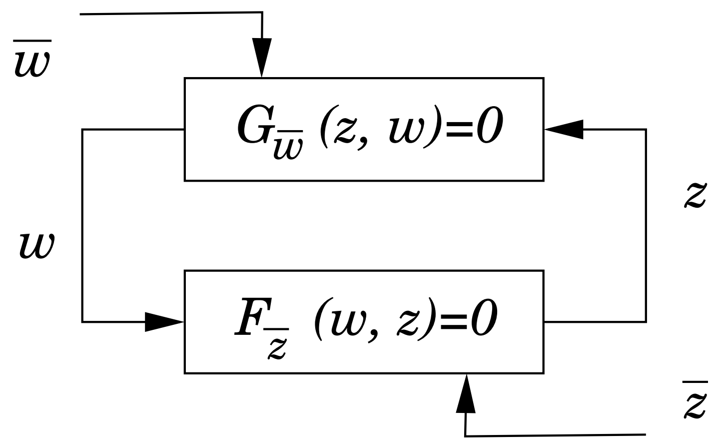
❷ Integral Quadratic Separation (IQS) for the descriptor case

❸ Performances in the IQS framework

❹ System augmentation : a way towards conservatism reduction

❺ The Romuald toolbox

Well-posedness



Well-Posedness:

Bounded $(\bar{w}, \bar{z}) \Rightarrow$ unique bounded (w, z)

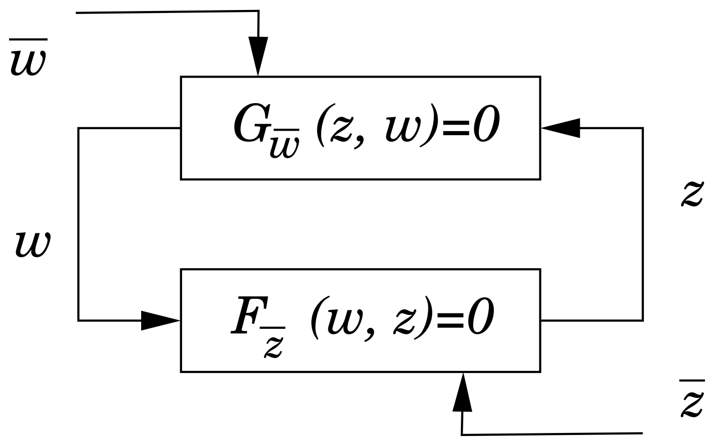
- In case $\overbrace{z = Aw + \bar{z}}^{F}$ and $\overbrace{w = \Delta z + \bar{w}}^{G}$ are linear applications

Well-posedness : $(1 - A\Delta)$ non-singular

- What if $\Delta = \Delta \in \mathbf{\Delta}$ is uncertain ?
- If $A = T(j\omega)$ is an LTI system ?
- If G is non-linear ?

...

Well-posedness & topological separation



Well-Posedness:

Bounded $(\bar{w}, \bar{z})$

$$\Rightarrow \exists!(w,z) \;,\; \exists \gamma \;:\; \left\| \begin{array}{c} w \\ z \end{array} \right\| \le \gamma \left\| \begin{array}{c} \bar{w} \\ \bar{z} \end{array} \right\|$$

- [Safonov 80] $\exists \theta$ topological separator:

$$\mathcal{G}^I(\bar{w}) = \{(w,z) \;:\; G_{\bar{w}}(z,w)=0\} \subset \{(w,z) \;:\; \theta(w,z) \le \phi_2(||\bar{w}||)\}$$
$$\mathcal{F}(\bar{z}) = \{(w,z) \;:\; F_{\bar{z}}(w,z)=0\} \subset \{(w,z) \;:\; \theta(w,z) > -\phi_1(||\bar{z}||)\}$$

- ϕ_1 and ϕ_2 are positive functions. When $\bar{w}=0$, $\bar{z}=0$ separation reads as

$$\mathcal{G}^I(0) = \{(w,z) \;:\; G_0(z,w)=0\} \subset \{(w,z) \;:\; \theta(w,z) \le 0\}$$
$$\mathcal{F}(0) = \{(w,z) \;:\; F_0(w,z)=0\} \subset \{(w,z) \;:\; \theta(w,z) > 0\}$$

■ For dynamic systems $\dot{x} = f(x)$, topological separation $\equiv$ Lyapunov theory

$$\overbrace{z(t) = f(w(t)) + \bar{z}(t)}^{F} \;,\; \overbrace{\underbrace{w(t)}_{x(t)} = \int_0^t \underbrace{z(\tau)}_{\dot{x}(t)}\,d\tau + \bar{w}(t)}^{G}$$

▲ $\bar{w}$: contains information on initial conditions ($x(0) = 0$ by convention)

● Well-posedness $\Rightarrow$ for zero initial conditions and zero perturbations :

$w = z = 0$ (equilibrium point).

● Well-posedness (global stability)

$\Rightarrow$ whatever bounded perturbations the state remains close to equilibrium

■ For dynamic systems $\dot{x} = f(x)$, topological separation $\equiv$ Lyapunov theory

$$\overbrace{z(t) = f(w(t)) + \bar{z}(t)}^{F} \;,\; \overbrace{\underbrace{w(t)}_{x(t)} = \int_0^t \underbrace{z(\tau)}_{\dot{x}(t)} d\tau + \bar{w}(t)}^{G}$$

● Assume a Lyapunov function $V(0) = 0 \,,\, V(x) > 0 \,,\, \dot{V}(x) < 0$

▲ Topological separation w.r.t. $\mathcal{G}^I(0)$ is obtained with

$$\theta(w = x, z = \dot{x}) = \int_0^\infty -\frac{\partial V}{\partial x}(x(\tau))\dot{x}(\tau)d\tau = \lim_{t\to\infty} -V(x(t)) < 0$$

▲ Topological separation w.r.t. $\mathcal{F}(0)$ does hold as well

$$\theta(w, z = f(w)) = \int_0^\infty -\dot{V}(w(\tau))d\tau > 0$$

■ For linear systems : quadratic Lyapunov function, *i.e. quadratic separator*

$$\overbrace{z(t) = Aw(t) + \bar{z}(t)}^{F} \;,\; \overbrace{\underbrace{w(t)}_{x(t)} = \int_0^t \underbrace{z(\tau)}_{\dot{x}(t)} d\tau + \bar{w}(t)}^{G}$$

● A possible separator based on quadratic Lyapunov function $V(x) = x^T P x$

$$\theta(w,z) = \int_0^\infty \begin{pmatrix} z^T(\tau) & w^T(\tau) \end{pmatrix} \begin{bmatrix} 0 & -P \\ -P & 0 \end{bmatrix} \begin{pmatrix} z(\tau) \\ w(\tau) \end{pmatrix} d\tau$$

▲ Quadratic separation w.r.t. $\mathcal{G}^I(0)$:

$$\lim_{t\to\infty} -x^T(t)Px(t) \leq 0 \;,\; \textit{i.e. } P > 0$$

▲ Quadratic separation w.r.t. $\mathcal{F}(0)$ guaranteed if

$$\forall t > 0 \;,\; -2w^T(t)PAw(t) > 0 \;,\; \textit{i.e. } A^T P + PA < 0$$

Topological separation : alternative to Lyapunov theory

- Needs to manipulate systems in a new form
- Suited for systems described as feedback connected blocs

Any linear system with rational dependence w.r.t. parameters writes as such

$$\dot{x} = (A + B_\Delta \Delta (1 - D_\Delta \Delta)^{-1} C_\Delta) x \quad \overset{\text{LFT}}{\longleftrightarrow} \quad \begin{cases} \dot{x} = Ax + B_\Delta w_\Delta \\ z_\Delta = C_\Delta x + D_\Delta w_\Delta \\ w_\Delta = \Delta z_\Delta \end{cases}$$

- Finding a topological separator is *a priori*

  as complicated as finding a Lyapunov function
- Allows to deal with several features simultaneously in a unified way

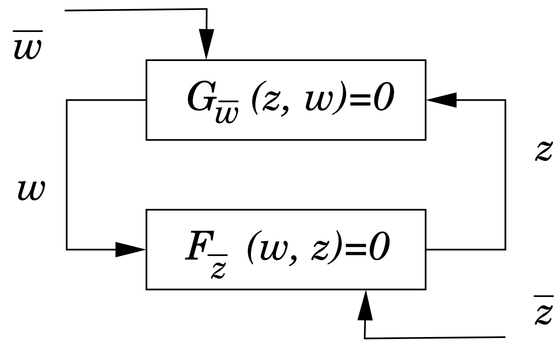
Quadratic separation [Iwasaki & Hara 1998]

- If $F(w) = Aw$ is a linear transformation
  and $G = \Delta$ is an uncertain operator defined as $\Delta \in \mathbf{\Delta}$ convex set
  it is necessary and sufficient to look for a quadratic separator

$$\theta(z, w) = \int_0^{\infty} \begin{pmatrix} z^T & w^T \end{pmatrix} \Theta \begin{pmatrix} z \\ w \end{pmatrix} d\tau$$

- If $F(w) = A(\omega)w$ is a linear parameter dependent transformation
  and $G = \Delta$ is an uncertain operator defined as $\Delta \in \mathbf{\Delta}$ convex set
  necessary and sufficient to look for a parameter-dependent quadratic separator

$$\theta(z, w) = \int_0^{\infty} \begin{pmatrix} z^T & w^T \end{pmatrix} \Theta(\omega) \begin{pmatrix} z \\ w \end{pmatrix} d\tau$$

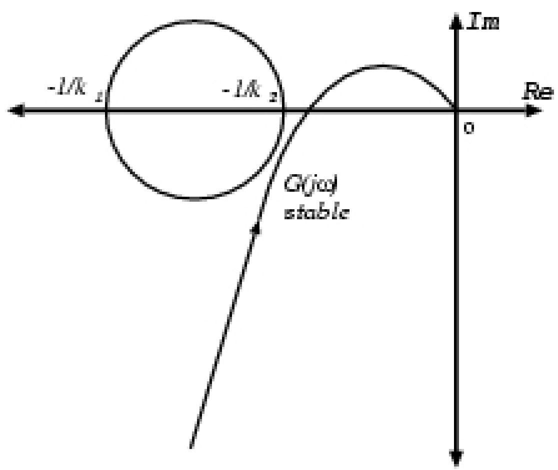


A well-known example : the Lur'e problem

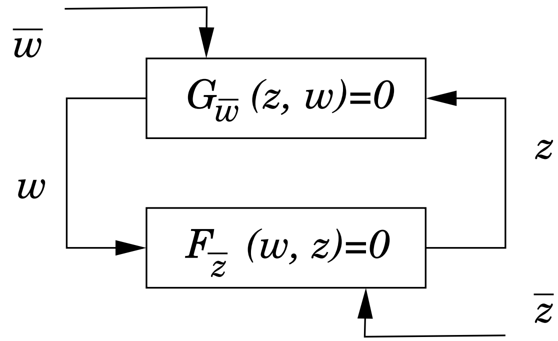
- $F = T(j\omega)$ is a transfer function
- $G(z)/z \in [\ -k_1,\ \ -k_2\ ]$ is a sector-bounded gain

  (*i.e.* the inverse graph of G is in $[\ -1/k_1\ ,\ \ -1/k_2\ ]$)

- Circle criterion : exists a quadratic separator (circle) for all ω



Another example : parameter-dependent Lyapunov function



- $F = A(\delta)$ parameter-dependent LTI state-space model
- $G = \mathcal{I}$ is an integrator

- Necessary and sufficient to have

$$\Theta(\delta) = \begin{bmatrix} 0 & -P(\delta) \\ -P(\delta) & 0 \end{bmatrix}$$

Direct relation with the IQC framework

- $F = T(j\omega)$ is a transfer matrix
- $G = \Delta$ is an operator known to satisfy an Integral Quadratic Constraint (IQC)

$$\int_{-\infty}^{+\infty} \begin{bmatrix} 1 & \Delta^*(j\omega) \end{bmatrix} \Pi(\omega) \begin{bmatrix} 1 \\ \Delta(j\omega) \end{bmatrix} d\omega \leq 0$$

- Stability of the closed-loop is guaranteed if for all ω

$$\begin{bmatrix} T^*(j\omega) & 1 \end{bmatrix} \Pi(\omega) \begin{bmatrix} T(j\omega) \\ 1 \end{bmatrix} > 0$$

- Knowing Δ how to choose $\Pi = \Theta$? (*i.e.* the quadratic separator)

Plenty of results in μ-analysis and IQC theory

D-scalings, DG-scalings etc. but still, conservative

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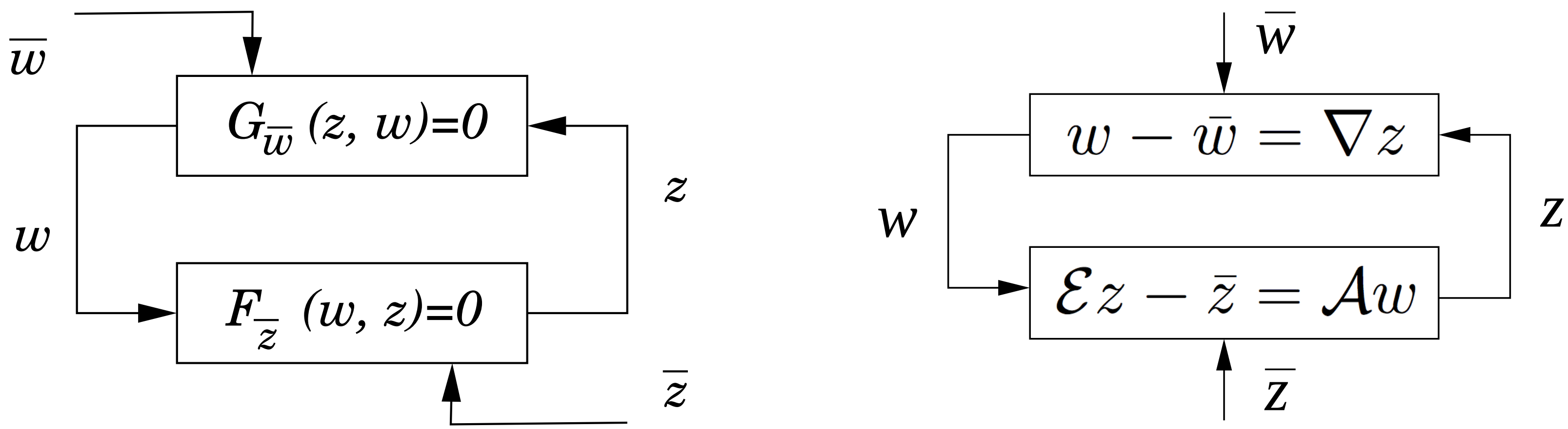

- Linear implicit application in feedback loop with an uncertain operator

$$\underbrace{\mathcal{E}z(t) = \mathcal{A}w(t)}_{F} \;,\; \underbrace{w(t) = [\nabla z](t)}_{G} \quad \nabla \in \mathbb{W}$$

- ∇ is bloc-diagonal contains scalar, full-bloc, LTI and LTV uncertainties and other operators such as integrators, delays...

- Result also extend to time-varying linear applications $\mathcal{E}(t)$, $\mathcal{A}(t)$ and to polytopic linear applications $\begin{bmatrix} \mathcal{E}(\xi) & \mathcal{A}(\xi) \end{bmatrix} \in \mathcal{CO}\left\{\begin{bmatrix} \mathcal{E}^{[i]} & \mathcal{A}^{[i]} \end{bmatrix}\right\}$.

■ Integral Quadratic Separation [Automatica'08, CDC'08]

- For the case of linear application with uncertain operator

$$\mathcal{E} z(t) = \mathcal{A} w(t) \ , \ w(t) = [\nabla z](t) \quad \nabla \in \mathbb{W}$$

where $\mathcal{E} = \mathcal{E}_1 \mathcal{E}_2$ with $\mathcal{E}_1$ full column rank,

- Integral Quadratic Separator (IQS) : $\exists \Theta$, matrix, solution of LMI

$$\begin{bmatrix} \mathcal{E}_1 & -\mathcal{A} \end{bmatrix}^{\perp *} \Theta \begin{bmatrix} \mathcal{E}_1 & -\mathcal{A} \end{bmatrix}^{\perp} > 0$$

and Integral Quadratic Constraint (IQC) $\forall \nabla \in \mathbb{W}$

$$\int_0^\infty \begin{pmatrix} \mathcal{E}_2 z(t) \\ [\nabla z](t) \end{pmatrix}^* \Theta \begin{pmatrix} \mathcal{E}_2 z(t) \\ [\nabla z](t) \end{pmatrix} dt \leq 0$$

- For given $\mathbb{W}$, there exist (conservative) LMI conditions for Θ solution to IQC

$$\int_0^\infty \begin{pmatrix} \mathcal{E}_2 z(t) \\ [\nabla z](t) \end{pmatrix}^* \Theta \begin{pmatrix} \mathcal{E}_2 z(t) \\ [\nabla z](t) \end{pmatrix} dt \leq 0$$

- Θ is build out of IQS for elementary blocs of ∇
- Improved DG-scalings, full-bloc S-procedure, vertex separators...
- Building Θ and related LMIs is tedious but can be automatized

  `www.laas.fr/OLOCEP/romuloc/`

- It is conservative except in few special cases [Meinsma et al., 1997].

Robust analysis in IQS framework:

- 1- Write the robust analysis problem as a well-posedness problem

$$\mathcal{E}z = \mathcal{A}w \;\; , \;\; w = \nabla z = \begin{bmatrix} \nabla_1 & & 0 \\ & \ddots & \\ 0 & & \nabla_{\bar{\text{J}}} \end{bmatrix} z$$

- 2- Build Integral Quadratic Separators for each elementary bloc ∇_j
- 3- Apply the IQS results to get (conservative) LMIs

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■ Induced L_2 norm (H_∞ in the LTI case)

$$E\dot{x} = Ax + Bv \ , \ \ g = Cx + Dv$$

▲ Prove that system is asymptotically stable

▲ and $\|g\| < \gamma\|v\|$ for zero initial conditions $x(0) = 0$

(strict upper bound on the L_2 gain attenuation)

● Equivalent to well-posedness with respect to

Integrator with zero initial conditions $x(t) = [\mathcal{I}_1 \dot{x}](t) = \int_0^t \dot{x}(\tau)d\tau$

and signals such that $\|v\| \leq \frac{1}{\gamma}\|g\|$

■ Induced L_2 norm

$$E\dot{x} = Ax + Bv \ , \ \ g = Cx + Dv$$

▲ Define ∇_{n2n} the fictitious non-causal uncertain operator such that

$$v = \nabla_{n2n} g \quad \text{iff} \quad \|v\| \leq \frac{1}{\gamma}\|g\|$$

● Induced L_2 norm problem is equivalent to well-posedness of

$$\underbrace{\begin{bmatrix} E & 0 \\ 0 & 1 \end{bmatrix}}_{\mathcal{E}} \underbrace{\begin{pmatrix} \dot{x} \\ g \end{pmatrix}}_{z} = \underbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}_{\mathcal{A}} \underbrace{\begin{pmatrix} x \\ v \end{pmatrix}}_{w} \ , \ \ \nabla = \begin{bmatrix} \mathcal{I}_1 & 0 \\ 0 & \nabla_{n2n} \end{bmatrix}$$

Induced L_2 norm

$$\underbrace{\begin{bmatrix} E & 0 \\ 0 & 1 \end{bmatrix}}_{\mathcal{E}} \underbrace{\begin{pmatrix} \dot{x} \\ g \end{pmatrix}}_{z} = \underbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}_{\mathcal{A}} \underbrace{\begin{pmatrix} x \\ v \end{pmatrix}}_{w} \quad , \quad \nabla = \begin{bmatrix} \mathcal{I}_1 & 0 \\ 0 & \nabla_{n2n} \end{bmatrix}$$

- Elementary IQS for bloc $\mathcal{I}_1$ is

$$\Theta_{\mathcal{I}_1} = \begin{bmatrix} 0 & -P \\ -P & 0 \end{bmatrix} \quad : \quad P > 0$$

Indeed (recall $x(t) = [\mathcal{I}_1 \dot{x}](t) = \int_0^t \dot{x}(\tau) d\tau$ and $x(0) = 0$)

$$\left\langle \begin{pmatrix} \dot{x} \\ \mathcal{I}_1 \dot{x} \end{pmatrix} \middle| \Theta_{\mathcal{I}_1} \begin{pmatrix} \dot{x} \\ \mathcal{I}_1 \dot{x} \end{pmatrix} \right\rangle_T = -x^*(T) P x(T) \leq 0$$

Induced L_2 norm

$$\underbrace{\begin{bmatrix} E & 0 \\ 0 & 1 \end{bmatrix}}_{\mathcal{E}} \underbrace{\begin{pmatrix} \dot{x} \\ g \end{pmatrix}}_{z} = \underbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}_{\mathcal{A}} \underbrace{\begin{pmatrix} x \\ v \end{pmatrix}}_{w} \quad , \quad \nabla = \begin{bmatrix} \mathcal{I}_1 & 0 \\ 0 & \nabla_{n2n} \end{bmatrix}$$

- Elementary IQS for bloc ∇_{n2n} is (small gain theorem)

$$\Theta_{\nabla_{n2n}} = \begin{bmatrix} -\tau 1 & 0 \\ 0 & \tau\gamma^2 1 \end{bmatrix} \quad : \quad \tau > 0$$

Indeed (recall $v = \nabla_{n2n} g$ iff $\|v\| \leq \frac{1}{\gamma}\|g\|$)

$$\left\langle \begin{pmatrix} g \\ \nabla_{n2n} g \end{pmatrix} \middle| \Theta_{\nabla_{n2n}} \begin{pmatrix} g \\ \nabla_{n2n} g \end{pmatrix} \right\rangle = \tau(-\|g\|^2 + \gamma^2\|v\|^2) \leq 0$$

- Apply IQS and get (for non-descriptor case $E = 1$)

$$P > 0 \quad , \quad \tau > 0$$

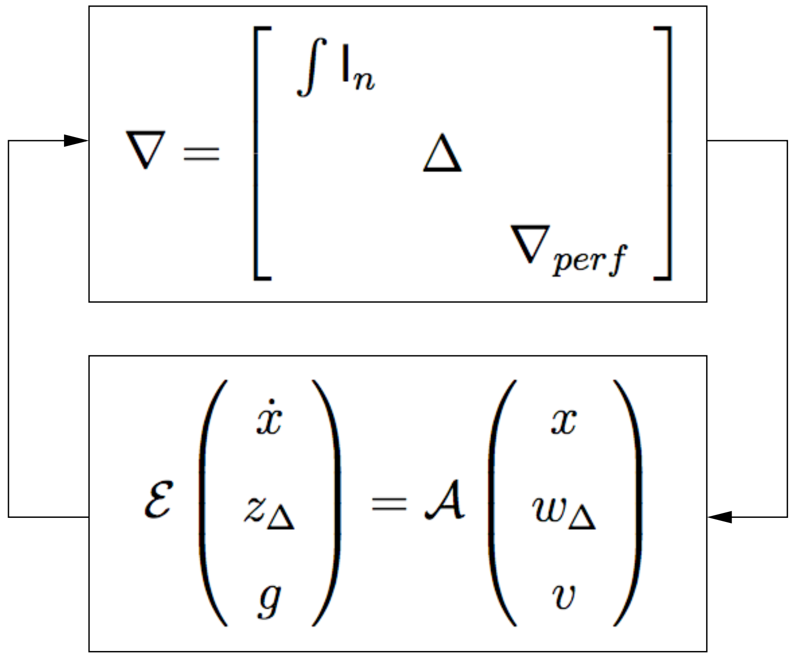
$$\begin{bmatrix} A^*P + PA + \tau C^*C & PB + \tau C^*D \\ B^*P + \tau D^*C & -\tau\gamma^2 1 + \tau D^*D \end{bmatrix} < 0$$

which is the classical H_∞ result.

- No difficulty to generate LMIs for descriptor case
- No difficulty to handle systems with uncertainties, time-delays...

Generic robust performance analysis problem:

- Well-posedness of



$$\nabla = \begin{bmatrix} \int \mathbf{I}_n & & \\ & \Delta & \\ & & \nabla_{perf} \end{bmatrix}$$

$$\mathcal{E}\begin{pmatrix} \dot{x} \\ z_\Delta \\ g \end{pmatrix} = \mathcal{A}\begin{pmatrix} x \\ w_\Delta \\ v \end{pmatrix}$$

- $\int 1_n$ integrator
- Δ matrix of uncertainties
- ∇_{perf} operator related to performances
  (induced L_2, H_∞, H_2, impulse-to-norm, norm-to-peak, impulse-to-peak)

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■ Towards less-conservative conditions: System augmentation

▲ Example of stability of uncertain system with parametric uncertainty ($\dot{\delta} = 0$)

$$\dot{x} = (A + \delta B_\Delta (1 - \delta D_\Delta)^{-1} C_\Delta) x$$

▲ Corresponds to well-posedness of

$$\begin{pmatrix} \dot{x} \\ z_\Delta \end{pmatrix} = \begin{bmatrix} A & B_\Delta \\ C_\Delta & D_\Delta \end{bmatrix} \begin{pmatrix} x \\ w_\Delta \end{pmatrix} \quad , \quad \nabla = \begin{bmatrix} \mathcal{I}_1 1_n & 0 \\ 0 & \delta 1_m \end{bmatrix}$$

▲ [Meinsma] rule indicates results may be conservative

- Well-posedness of

$$\begin{pmatrix} \dot{x} \\ z_\Delta \end{pmatrix} = \begin{bmatrix} A & B_\Delta \\ C_\Delta & D_\Delta \end{bmatrix} \begin{pmatrix} x \\ w_\Delta \end{pmatrix} \quad , \quad \nabla = \begin{bmatrix} \mathcal{I}1_n & 0 \\ 0 & \delta 1_m \end{bmatrix}$$

- adding the fact that $\dot{w}_\Delta = \delta \dot{z}_\Delta$, is also equivalent to well-posedness of

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -C_\Delta & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{pmatrix} \dot{w}_\Delta \\ \dot{x} \\ \hline z_\Delta \\ \dot{z}_\Delta \end{pmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & A & B_\Delta & 0 \\ 0 & C_\Delta & D_\Delta & 0 \\ 0 & 0 & 0 & D_\Delta \\ 1 & 0 & -1 & 0 \end{array}\right] \begin{pmatrix} w_\Delta \\ x \\ \hline w_\Delta \\ \dot{w}_\Delta \end{pmatrix}$$

$$\nabla = \begin{bmatrix} \mathcal{I}1_{m+n} & 0 \\ 0 & \delta 1_{2m} \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -C_\Delta & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{pmatrix} \dot{w}_\Delta \\ \dot{x} \\ \hline z_\Delta \\ \dot{z}_\Delta \end{pmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & A & B_\Delta & 0 \\ 0 & C_\Delta & D_\Delta & 0 \\ 0 & 0 & 0 & D_\Delta \\ 1 & 0 & -1 & 0 \end{array}\right] \begin{pmatrix} w_\Delta \\ x \\ \hline w_\Delta \\ \dot{w}_\Delta \end{pmatrix}$$

$$\nabla = \begin{bmatrix} \mathcal{I}1_{m+n} & 0 \\ 0 & \delta 1_{2m} \end{bmatrix}$$

- It is descriptor model.
- More decisions variables in the separator (increased dimensions of ∇)
- Bigger LMI conditions ($m+n$ rows)

- Lyapunov function is with respect to the augmented state

(vector involved in the integrator operator)

$$\begin{pmatrix} w_\Delta^* & x^* \end{pmatrix} P \begin{pmatrix} w_\Delta \\ x \end{pmatrix}$$

- Recalling that

$$w_\Delta = \delta(1 - \delta D_\Delta)^{-1} C_\Delta x$$

the result corresponds to looking for a parameter dependent Lyapunov function

$$x^* \begin{bmatrix} \delta(1 - \delta D_\Delta)^{-1} C_\Delta \\ 1 \end{bmatrix}^* P \begin{bmatrix} \delta(1 - \delta D_\Delta)^{-1} C_\Delta \\ 1 \end{bmatrix} x$$

- Proves to be less conservative than for LMIs obtained on original system.

❹ System augmentation and conservatism reduction

■ Towards less-conservative conditions: System augmentation

- Adding more equations for higher derivatives of the state:

  less conservative LMI conditions

- Same technique works for time varying uncertainties

  (if known bounds on derivatives)

- Has been applied successfully to time-delay systems [Gouaisbaut]:

  gives sequences of LMI conditions with decreasing conservatism

▲ Related to SOS representations of positive polynomials [Sato 2009]:

  conservatism decreases as the order of the representation is augmented

- No need to manipulate by hand LMIs (Schur complements etc.), polynomials...

▲ Does conservatism vanishes? Exactly? Asymptotically?

▲ Is it possible to cope with non-linearities in the same way?

Outline

■ Freely distributed software to test the theoretical results

● Existing software : RoMulOC

`www.laas.fr/OLOCEP/romuloc`

▲ Contains some of the analysis results plus some state-feedback features

● Currently developed software : Romuald

▲ Dedicated to analysis of descriptor systems

▲ Fully coded using the quadratic separation theory

▲ Allows systematic system augmentation

▲ First preliminary tests currently done for satellite and plane applications

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>> quiz = ctrpb( OrderOfAugmentation ) + h2 (usys);
>> result = solvesdp( quiz )
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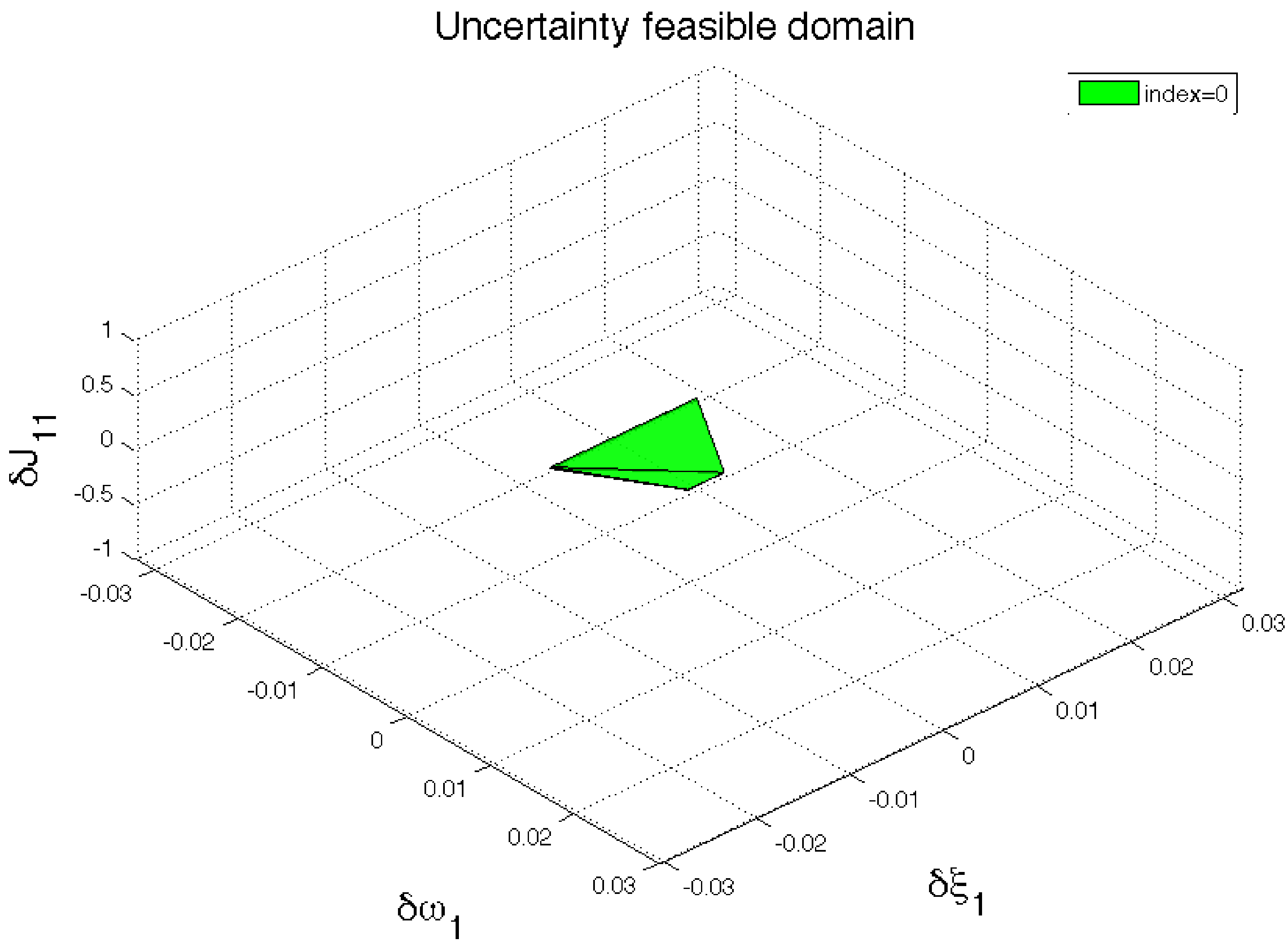
Uncertainty feasible domain
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δJ11
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0.5
0
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-1
δω1
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-0.02
-0.01
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0.03
δξ1
-0.03
-0.02
-0.01
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0.01
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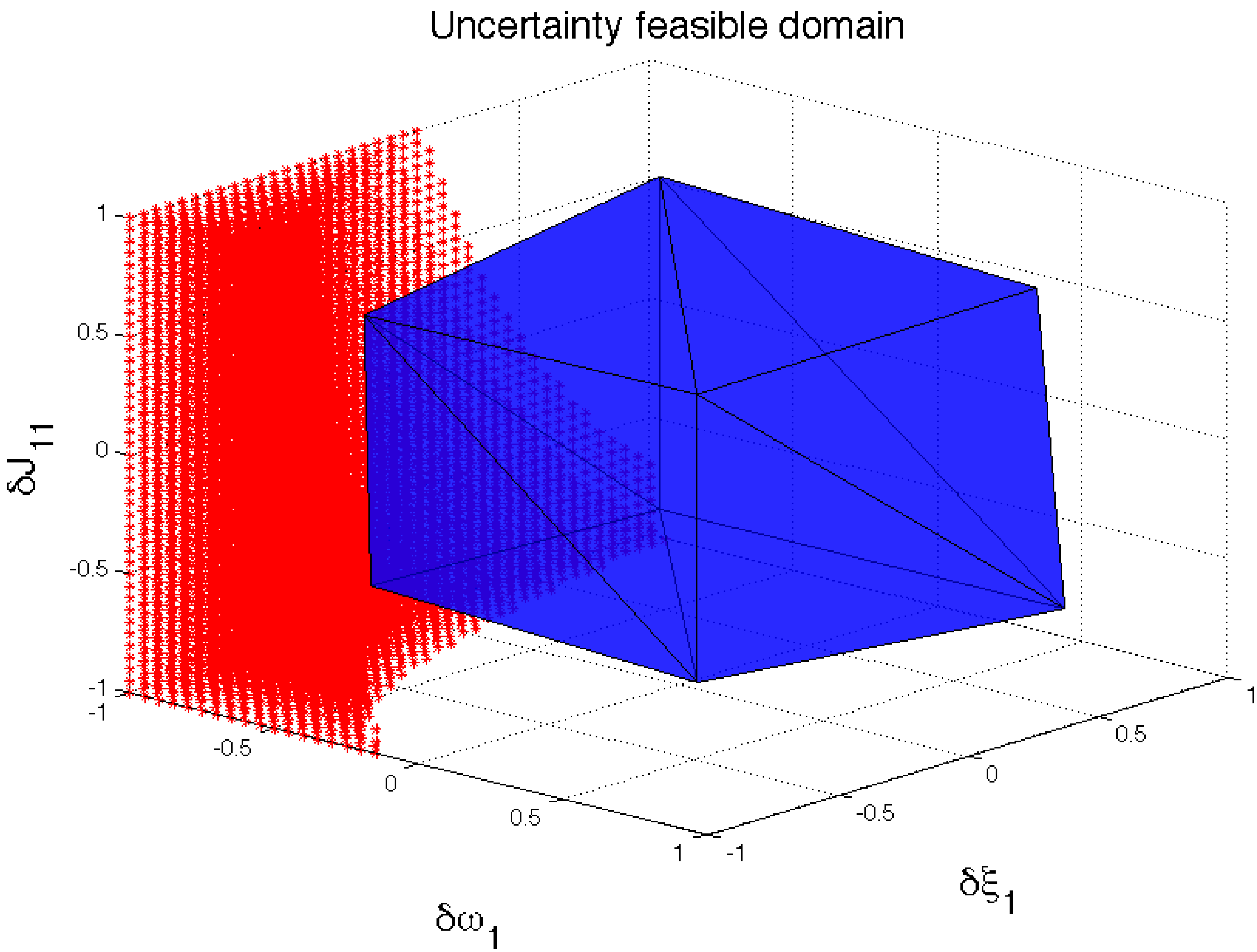
Uncertainty feasible domain
δJ_{11}
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0
-0.5
-1
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$\delta\omega_1$
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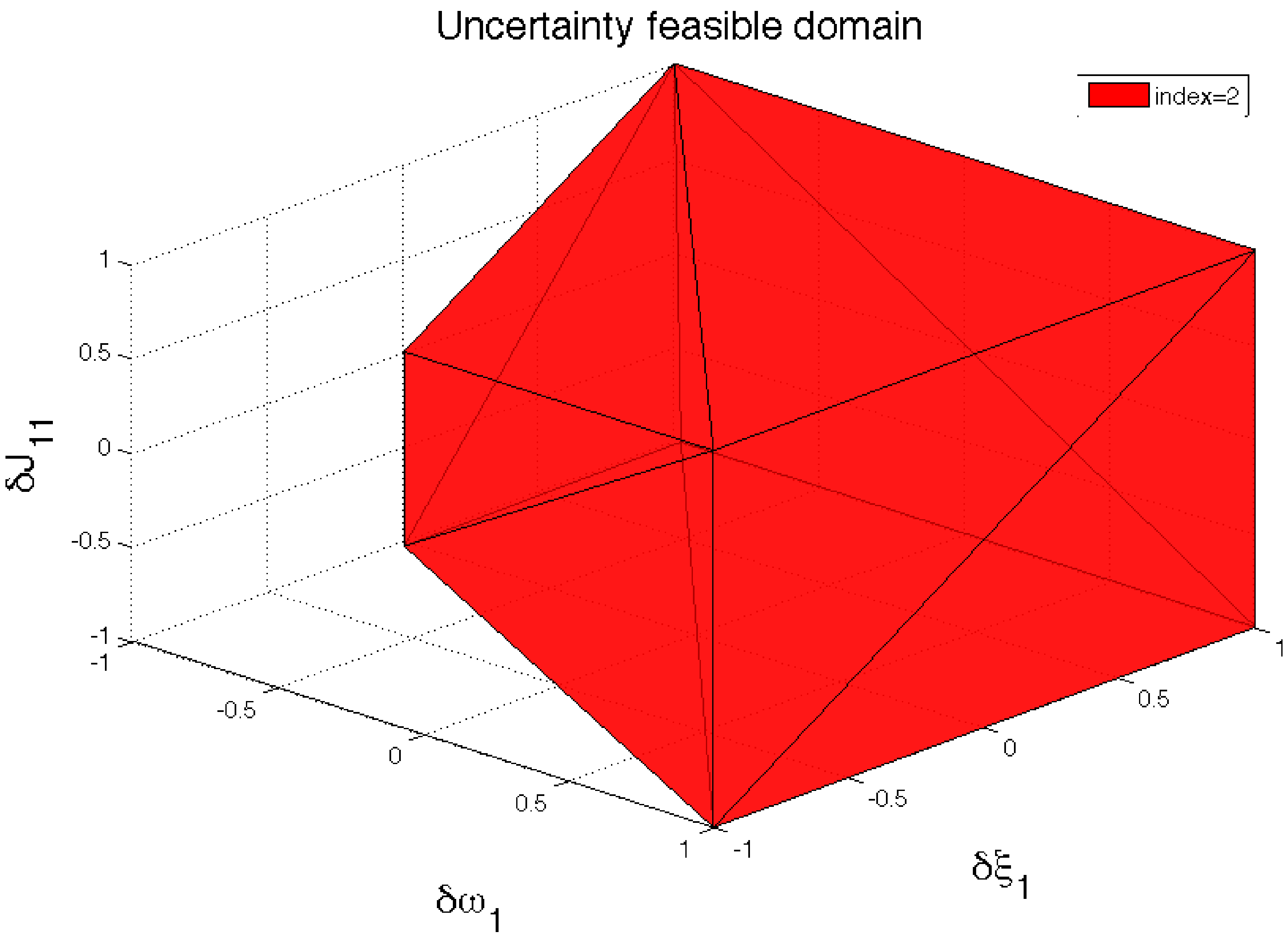
-0.5
0
0.5
1
$\delta\xi_1$


Uncertainty feasible domain
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-1
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0
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0.5
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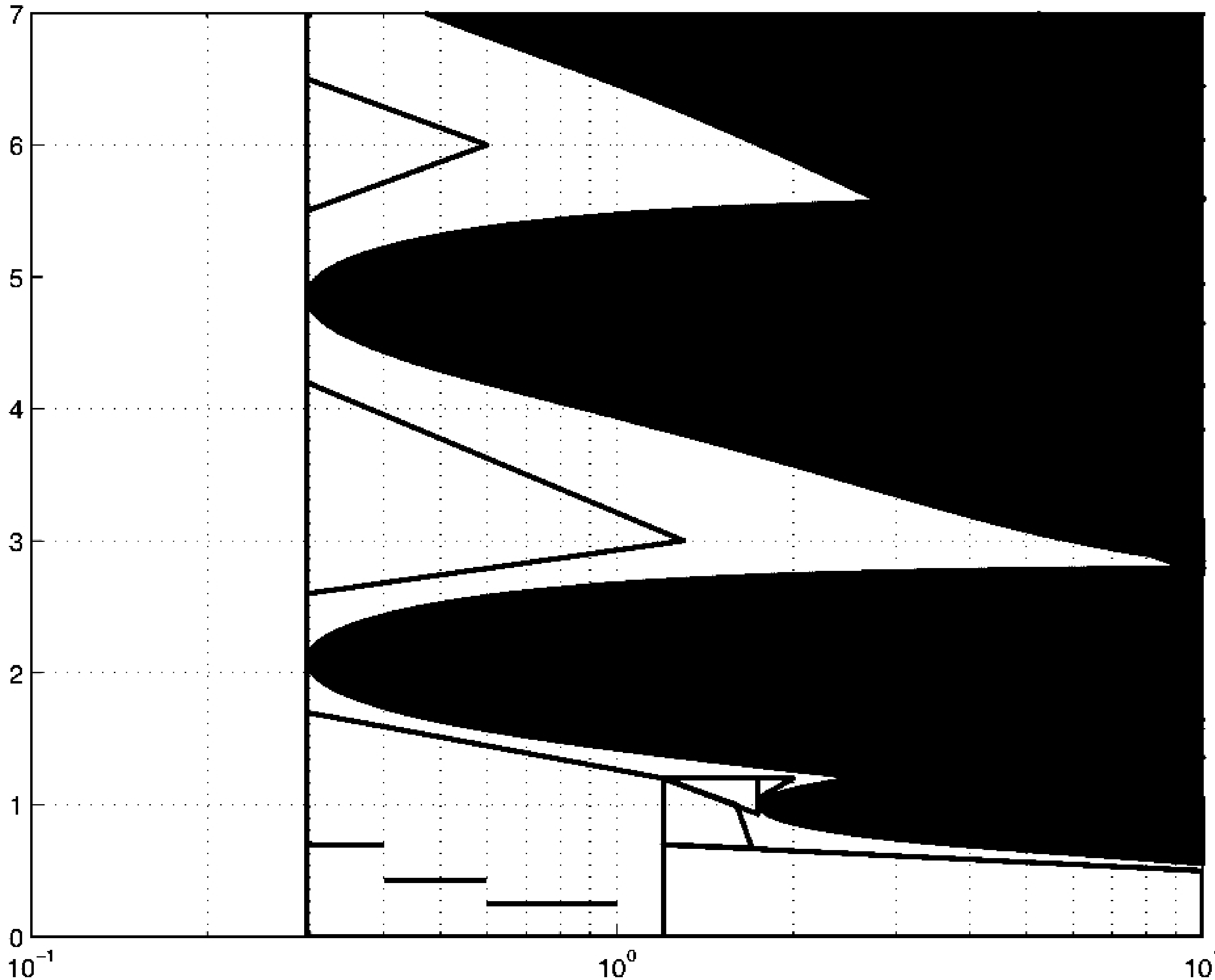
δJ11
δω1
δξ1


7
6
5
4
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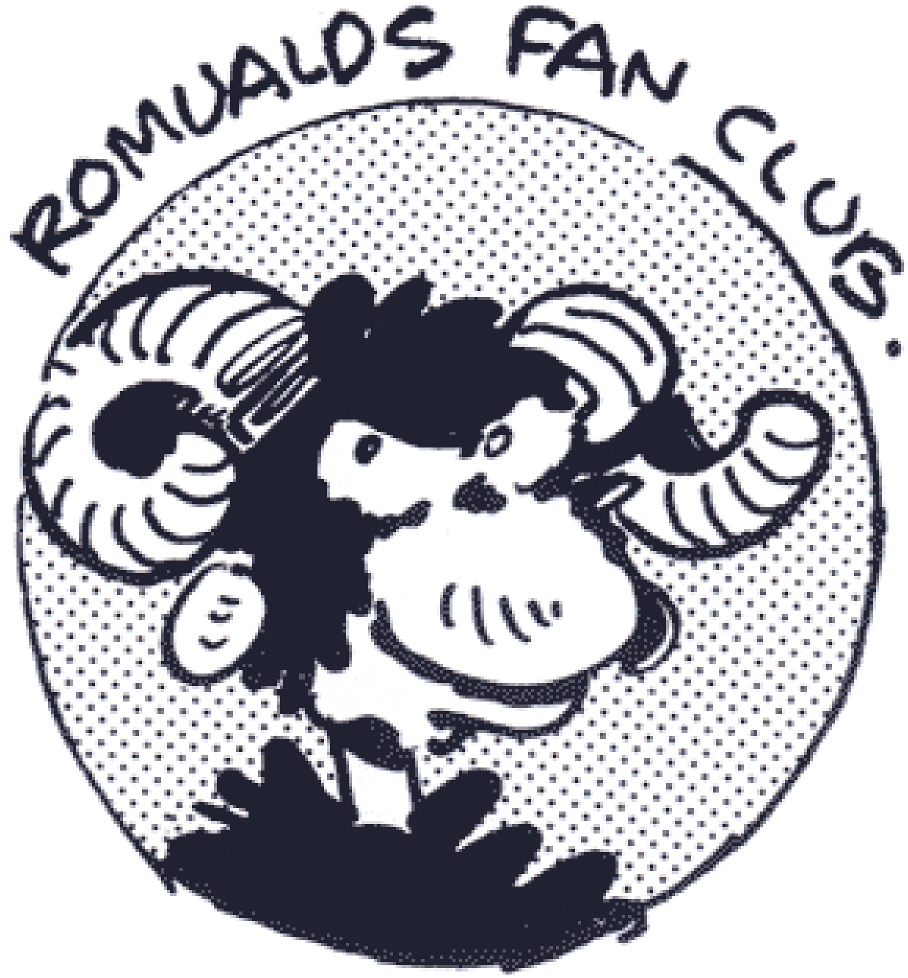
10^{1}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