Structured adaptive control, or how to solve LMIs with Simulink

Alexandru - Razvan LUZI
Dimitri PEAUCELLE

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Introduction

- Direct adaptive control:
Adaptation of control gains done directly based on measurements.

- $\neq$ Indirect adaptive control:
Estimator of model parameters + scheduled control gain

- Feedback-loop stabilizing gains, MRAC not considered

- Lyapunov based stability proofs, not gradient approximation 'MIT rule'

- Framework initiated by V.A. Yakubovich in the late 1960's

- Contributions: new adaptive control law with asymptotic structure
+ may solve LMIs

Plan

Passivity-based adaptive control of LTI systems

Theorem

The following two conditions are equivalent:

❶ *There exists a static control* $u(t) = Fy(t) + w(t)$ *for the system*

$$\dot{x}(t) = \mathbb{A}x(t) + \mathbb{B}u(t) \quad , \quad y(t) = \mathbb{C}x(t) \quad , \quad z(t) = y(t)$$

that makes the closed-loop strictly passive (with respect to w/z*).*

❷ *For all* $\Gamma \succ \mathbf{0}$ *the following adaptive control*

$$u(t) = K(t)y(t) + w(t) \quad , \quad \dot{K}(t) = -y(t)y^T(t)\Gamma$$

makes the closed-loop globally strictly-passive.

- Strict-passivity includes asymptotic stability of $x=0$
- Adaptive control converges to $K(\infty)$: strictly-passifying static gain

- Theorem for square systems - extensions exist for non-square systems
- Not all stabilizable systems are strictly-passifiable
  - modified adaptive laws exist for stabilizable systems

- Condition ❶ also reads in terms of matrix inequalities as

$$\exists Q \succ \mathbf{0} \quad : \quad (\mathbb{A}+\mathbb{B}F\mathbb{C})^T Q + Q(\mathbb{A}+\mathbb{B}F\mathbb{C}) \prec \mathbf{0} \ , \quad Q\mathbb{B} = \mathbb{C}^T$$

It happens to be an LMI constraint!

$$\exists Q \succ \mathbf{0} \quad : \quad \mathbb{A}^T Q + Q\mathbb{A} + \mathbb{C}^T(F^T+F)\mathbb{C} \prec \mathbf{0} \ , \quad Q\mathbb{B} = \mathbb{C}^T$$

Finding F solution to the LMI is equivalent to simulating the system with the adaptive control law and taking $F=K(\infty)$.

All LMIs define strict-passifiable systems

- Let us consider an example: LMIs for an upper bound on the H_∞ norm

$$\begin{bmatrix} A^T P + PA + C^T C & PB + C^T D \\ B^T P + D^T C & -\gamma^2 \mathbf{1} + D^T D \end{bmatrix} \prec \mathbf{0} \ , \quad P = P^T \succ \mathbf{0}.$$

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▲ Can be decomposed in a sum with elementary matrix variables

$$\mathbb{A} + \mathbb{B}_P^T \begin{bmatrix} \mathbf{0} & P & \mathbf{0} \\ P & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -P \end{bmatrix} \mathbb{B}_P + \mathbb{B}_\gamma^T (-\gamma^2 \mathbf{1}) \mathbb{B}_\gamma \prec \mathbf{0} \quad , \quad P = P^T$$

$$\mathbb{A} = \begin{bmatrix} C^T C & C^T D & \mathbf{0} \\ D^T C & D^T D & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} , \; \mathbb{B}_P = \begin{bmatrix} A & B & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} , \; \mathbb{B}_\gamma = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix}$$

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- Or equivalently when gathering all variables in a block-diagonal matrix

$$\mathbb{A} + \mathbb{B}^T F \mathbb{B} \prec \mathbf{0} \quad , \quad \mathbb{B} = \begin{bmatrix} \mathbb{B}_P & \mathbb{B}_\gamma \end{bmatrix}$$

with the structural equality constraints

$$F = \begin{bmatrix} F_P & \mathbf{0} \\ \mathbf{0} & F_\gamma \end{bmatrix} \quad , \quad F_P = \begin{bmatrix} \mathbf{0} & P & \mathbf{0} \\ P & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -P \end{bmatrix} \quad , \quad P = P^T \quad , \quad F_\gamma = -\gamma^2 \mathbf{1}$$

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● The constraint $\mathbb{A} + \mathbb{B}^T F \mathbb{B} \prec \mathbf{0}$ holds iff $(\mathbb{A}, \mathbb{B}, \mathbb{C} = \mathbb{B}^T)$ is strictly-passifiable by F.

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  $(\mathbb{A}, \mathbb{B}, \mathbb{C} = \mathbb{B}^T)$ is strictly-passifiable by F.
- LMI converted to strict-passification problem, with equality constraints.

- Procedure applies to any LMI: concludes with search of passifying

$$F = \begin{bmatrix} F_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & F_N \end{bmatrix}$$

for a (symmetric) system $(\mathbb{A}, \mathbb{B}, \mathbb{C} = \mathbb{B}^T)$
with additional structural equality constraints that can be compacted in

$$\mathrm{vec}(F_i) = S_i x_i \quad \Leftrightarrow \quad U_i \mathrm{vec}(F_i) = 0$$

where $\mathrm{vec}(F_i)$ is the vector composed of stacked columns of F_i,
x_i are vectors of independent scalar decision variables and $U_i = S_i^{\perp}$.

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- When starting from the canonical representation $L_0 + \sum_j \hat{x}_j L_j \prec \mathbf{0}$,
  then the structural constraints are all of the type

$$F_j = \begin{bmatrix} \hat{x}_j \mathbf{1}_{r_{j1}} & \mathbf{0} \\ \mathbf{0} & -\hat{x}_j \mathbf{1}_{r_{j2}} \end{bmatrix}$$

  and $r_{j1} + r_{j2}$ can be very large and U_j is huge.
- F and U_is expected to be smaller when matrix representation.

Block-diagonal adaptive control with asymptotic structure

Theorem

The following two conditions are equivalent:

❶ *There exists a decentralized static control* $u_i(t) = F_i y_i(t) + w_i(t)$ *satisfying the structural constraints* $U_i \mathrm{vec}(F_i) = 0$ *for the system*

$$\dot{x}(t) = \mathbb{A}x(t) + \sum \mathbb{B}_i u_i(t) \quad , \quad y_i(t) = \mathbb{C}_i x(t) \quad , \quad z(t) = y(t)$$

that makes the closed-loop strictly passive (with respect to w/z*).*

❷ *For all* $\Gamma_i \succ \mathbf{0}$, $\alpha_i > 0$ *the following adaptive control*

$$\begin{aligned} u_i(t) &= K_i(t) y_i(t) + w_i(t) \quad , \\ \dot{K}_i(t) &= -y_i(t) y_i^T(t) \Gamma_i - \alpha_i \cdot \mathrm{mat}\left(U_i^T U_i \cdot \mathrm{vec}(K_i(t))\right) \Gamma_i \end{aligned}$$

makes the closed-loop globally strictly-passive.

'mat' is the function such that $\mathrm{mat}(\mathrm{vec}(F)) = F$.

Proof of ❶ ⇒ ❷

- ❶ reads as

$$\exists F, \exists Q \succ \mathbf{0} \quad : \quad \begin{array}{c} (\mathbb{A}+\mathbb{B}F\mathbb{C})^T Q + Q(\mathbb{A}+\mathbb{B}F\mathbb{C}) < \mathbf{0}, \quad Q\mathbb{B} = \mathbb{C}^T, \\ F = \mathrm{diag}\left[\begin{array}{ccc} \cdots & F_i & \cdots \end{array}\right], \quad U_i \cdot \mathrm{vec}(F_i) = 0 \end{array} \tag{1}$$

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- Let the Lyapunov function for the non-linear system (with adaptive law)

$$V(x,K) = \frac{1}{2}\left(x^T Q x + \sum_i \mathrm{Tr}\left((K_i - F_i)\Gamma^{-1}(K_i - F_i)^T \right) \right)$$

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- After manipulations and using $Q\mathbb{B} = \mathbb{C}^T$, $U_i \cdot \mathrm{vec}(F_i) = 0$, we get:

$$\dot{V}(x,K) = x^T(\mathbb{A}+\mathbb{B}F\mathbb{C})^T Q x + w^T z - \sum_i \alpha_i (U_i \cdot \mathrm{vec}(K_i))^T (U_i \cdot \mathrm{vec}(K_i)).$$

Proof of ❶ ⇒ ❷ (continued)

$$\dot{V}(x, K) = x^T(\mathbb{A} + \mathbb{B}F\mathbb{C})^T Q x + w^T z - \sum_i \alpha_i (U_i \cdot \text{vec}(K_i))^T (U_i \cdot \text{vec}(K_i)).$$

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- First term is strictly negative due to (1), until $x = 0$,
- Last term is strictly negative, until $U_i \cdot \mathrm{vec}(K_i) = 0$

- If no perturbations ($w = 0$) the system converges to the attractor

$$\mathcal{A} = \{(x,K) \ : \ x = 0 \ , \ U_i \cdot \mathrm{vec}(K_i) = 0\}$$

- On the attractor $\dot{K}_i = 0$: the gains $K_i(\infty)$ are constant.

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- For initial conditions at equilibrium and nonzero perturbations

$$0 \leq V(x(t),K(t)) = \int_0^t \dot{V}(x,K)d\tau < \int_0^t w^T z \; d\tau$$

$\Rightarrow$ the system is strictly passive.

Proof of ❷ ⇒ ❶

- The system with adaptive control is globally asymptotically stable, it converges to a asymptotically stable equilibrium: $F_i = K_i(\infty)$ are stabilizing gains.

- Same reasoning holds for passivity.

Summary

■ All LMI problems are equivalent to static output-feedback strict-passification problems with structure constraints:

- gain F is block-diagonal
- sub-blocks should satisfy $U_i \text{vec}(F_i) = 0$.

■ If a structured strict-passification problem admits solutions, the block-diagonal adaptive law with asymptotic structure will converge to one of these.

● The LMIs can be solved by simulating the adaptive controlled systems.

▲ If the system converges $K_i(\infty)$ contain solutions of the LMIs.

▲ If does not converges the LMIs are infeasible.

Numerical example

- Consider the transfer function:

$$G(s) = \frac{s^2 + s + 1}{s^2 + s + 2}$$

- Problem: compute the H_∞ norm (or at least an upper bound).

▲ In Matlab: `norm(G, Inf, 1e-4) = 1.3251`

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▲ LMI problem converted to adaptive passification

$$\dot{K}_i = -y_i y_i^T \Gamma_i - \alpha_i \cdot \mathrm{mat}\left(U_i^T U_i \cdot \mathrm{vec}(K_i)\right) \Gamma_i \quad , \quad y_1 \in \mathrm{R}^6 \quad , \quad y_2 \in \mathrm{R}$$

with structural asymptotic constraints :

$$F_1 = \begin{bmatrix} \mathbf{0} & P & \mathbf{0} \\ P^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -P \end{bmatrix} \quad , \quad P = P^T \in \mathbb{R}^{2\times 2} \quad , \quad F_2 = -\gamma^2 \mathbf{1} = -\gamma^2 .$$

- Parameters for simulating the adaptive law (simulation in Simulink)
  - Initial conditions $x = (1 \ldots 1)^T$ and $K_i = \mathbf{0}$
  - $\Gamma_1 = 1000 \cdot \mathbf{1}$, $\Gamma_2 = 10$, $\alpha_1 = \alpha_2 = 1$

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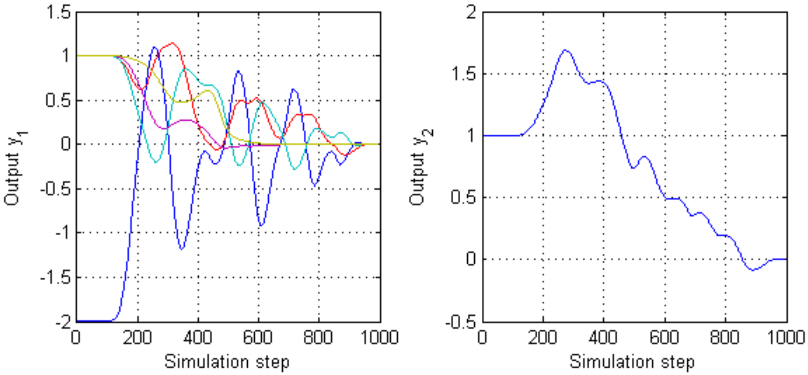
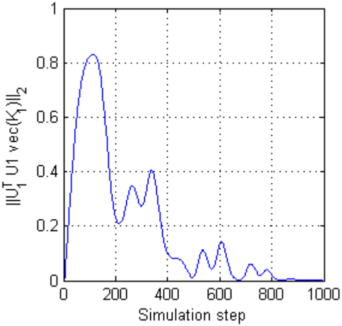
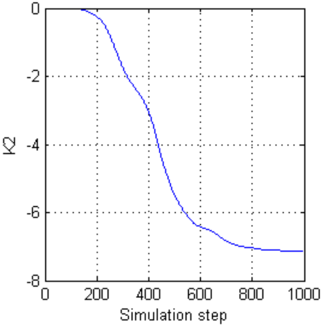
  - Initial conditions $x = (1 \dots 1)^T$ and $K_i = \mathbf{0}$
  - $\Gamma_1 = 1000 \cdot \mathbf{1}$, $\Gamma_2 = 10$, $\alpha_1 = \alpha_2 = 1$
- Convergence to zero of the 'outputs' y_i



Convergence to structured values of the adapted gains K_i

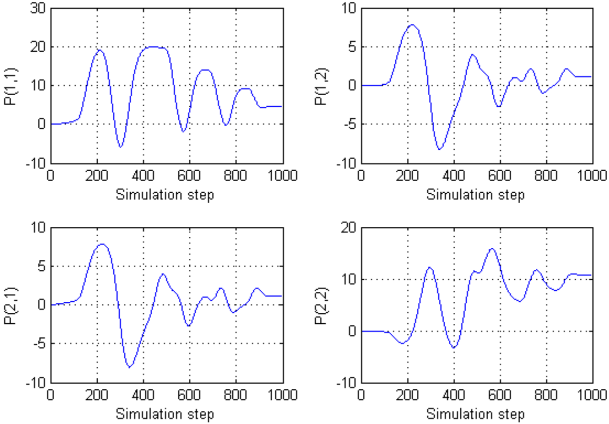




$$\mathbf{K}_1(\infty) = \begin{bmatrix} 0 & 0 & 4.6330 & 1.0671 & 0 & 0 \\ 0 & 0 & 1.0671 & 10.7960 & 0 & 0 \\ 4.6330 & 1.0671 & 0 & 0 & 0 & 0 \\ 1.0671 & 10.7960 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4.6330 & -1.0671 \\ 0 & 0 & 0 & 0 & -1.0671 & -10.7960 \end{bmatrix}$$

$$\mathbf{K}_2(\infty) = -7.1307$$

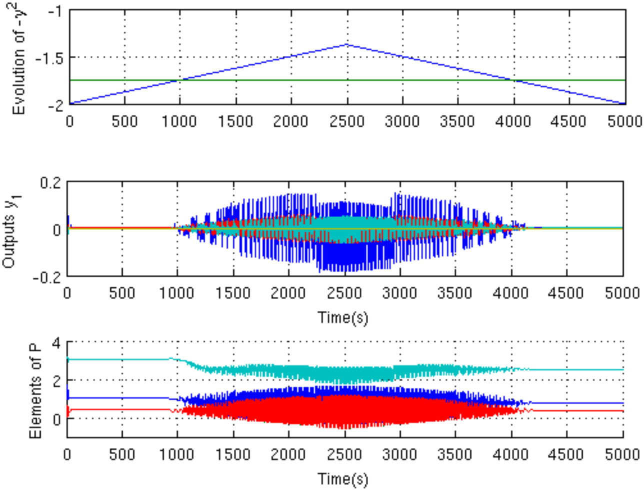
- Evolution of the $(1:2, 3:4)$ elements of K_1 that converge to P



- Solution of the LMIs

$$P = \begin{bmatrix} 4.6330 & 1.0671 \\ 1.0671 & 10.7960 \end{bmatrix}, \quad \gamma = 2.6703 \geq 1.3251 = \gamma_{opt}$$

- Test for feasible / unfeasible cases
- Only K_1 is adapated, γ is slowly linearly modified



- Unstable behavior when $\gamma < 1.3251 = \gamma_{opt}$.

Conclusions et perspectives

- LMI feasibility problems can be solved by simulating systems

- Need for a parser to convert LMIs to adaptive control problem
- Simulation time is large - what is the best implementation ?
- Is simulation time polynomial w.r.t. size of problem ?

- What about LMI optimization problems ?

- Decreasing parameters until system becomes unstable ?
- Minimizing gap with dual LMI problem (it works).
- Other ?

- Solving time-varying LMI problems ?