LMI-based design of robust adaptive control for linear systems with time-varying uncertanties

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Introduction

Simple adaptive control

$$u(t) = K(t)y(t) + w(t) \ , \ \dot{K}(t) = -Gy(t)y^T(t)\Gamma - \phi(K(t))\Gamma$$

- Simple or direct adaptive control [Fradkov, Kaufman et al, Ioannou, Barkana]
- Adaptation does not need parameter measurement or estimation.
- Properties achieved thanks to closed-loop passification
  (almost passive systems [Barkana])

$$\exists F, G \ : \ (A + BFC)^T P + P(A + BFC) < 0 \ , \ PB = C^T G^T$$

- [Yaesh'06], [ROCOND'06] First LMI-based results for proving robustness
- [Ben Yamin'07], [ALCOSP'07] LMI-based results for L_2-gain attenuation

Simple adaptive control

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- SAC is claimed to be robust and to adapt to parametric variations.
- This work is devoted to:
  - For a class of uncertain systems, provide robust stability conditions.
  - Conditions are inspired of Robust Control results for uncertain linear systems.
  - They are in LMI form, therefore numerically testable.
- Outline:
  1. Problem statement : Properties of SAC and definition of uncertain systems
  2. Main result : LMI formulas for proving existence of an attraction domain
  3. Additional result : LMI formulas for estimating the attraction domain
  4. Some clues about the proofs

❶ Problem statement

Simple adaptive control

$$u(t) = K(t)y(t) + w(t) \ , \ \dot{K}(t) = -Gy(t)y^T(t)\Gamma - \phi(K(t))\Gamma$$

- If $\Gamma = 0$: SAC reduces to SOF ($K(t) = K(0) = F$).
Stability depends of the choice of F.
- For $\Gamma > 0$ and appropriate choices of G and ϕ:
SAC may be stabilizing whatever initial conditions $K(0)$.
If $K(t)$ converges to a fixed point, then $K(\infty) = F$ is stablizing SOF.
- For this presentation: G is assumed given
and ϕ is dead-zone type, defined by $\phi(K) = \psi(\text{Tr}(K^T K))K$ where

$$\begin{cases} \psi(k) = 0 & \forall\, 0 \leq k < \alpha \\ \psi(k) = \frac{k-\alpha}{\beta-k} & \forall\, \alpha \leq k < \beta \end{cases}$$

Simple adaptive control

$$u(t) = K(t)y(t) + w(t) \ , \quad \dot{K}(t) = -Gy(t)y^T(t)\Gamma - \phi(K(t))\Gamma$$

- ϕ is dead-zone type, defined by $\phi(K) = \psi(\text{Tr}(K^TK))K$ where

$$\left\{ \begin{array}{ll} \psi(k) = 0 & \forall\, 0 \leq k < \alpha \\ \psi(k) = \frac{k-\alpha}{\beta-k} & \forall\, \alpha \leq k < \beta \end{array} \right.$$

- The term $-Gyy^T\Gamma$ has properties for "driving" K to stabilizing values.
- The term $-\phi(K)\Gamma$ prevents K to grow too large ($\text{Tr}(K^TK) < \beta$).
- α should be large to keep the adaptation free.
- α and β are assumed given accordingly to implementation constraints.

Linear system with affine time-varying uncertainties

$$\dot{x} = \overbrace{\left[A_0 + \sum_{p=1}^{\bar{p}} \delta_p A_p \right]}^{A(\delta)} x + \overbrace{\left[B_0 + \sum_{p=1}^{\bar{p}} \delta_p B_p \right]}^{B(\delta)} u \ , \ y = \overbrace{\left[C_0 + \sum_{p=1}^{\bar{p}} \delta_p C_p \right]}^{C(\delta)} x \ .$$

- Bounded uncertainties $\underline{\delta}_p \leq \delta_p(t) \leq \overline{\delta}_p$,
- with bounded time derivatives $\underline{\vartheta}_p \leq \dot{\delta}_p(t) \leq \overline{\vartheta}_p$.
- Δ is the set of all uncertainties,
- $\bar{\Delta}$ the set of $2^{2\bar{p}}$ vertices $\delta_p \in \{\underline{\delta}_p, \overline{\delta}_p\}$, $\vartheta_p \in \{\underline{\vartheta}_p, \overline{\vartheta}_p\}$.

- ▲ Can model some non-linear systems with parametric uncertainties.
- ▲ Nominal system ($\delta = 0$) assumed to have SOF passifiability property:

$$\exists F_0, \ \exists D \ : \ \dot{x} = (A_0 + B_0F_0C_0)x + B_0w \ , \ z = GC_0x + Dw \ \text{ is passive}$$

THM 1 Existence of an attraction domain

- For a chosen scalar $\epsilon > 0$ solve the LMI problem

$$\mathcal{L}_1(H_1, H_2, P_{p=0\dots\bar{p}}, F_{p=0\dots\bar{p}}, D_{p=0\dots\bar{p}}, R_{p=0\dots\bar{p}}, T_{p=0\dots\bar{p}}) < \mathrm{O}$$

- If feasible, the solution is such that

▲ $F(\delta) = F_0 + \sum_{p=1}^{\bar{p}} \delta_p F_p$ is a robustly stabilizing parameter-dependent SOF and global asymptotic stability of the origin is proved with Lyapunov function $V(x,\delta) = x^T P(\delta) x$ where $P(\delta) = P_0 + \sum_{p=1}^{\bar{p}} \delta_p P_p$.

▲ $\exists Q$ such that $x^T Q x \leq 1$ is a robust attraction domain for SAC and global asymptotic convergence to that domain is proved with Lyapunov function $W(x,K,\delta) = V(x,\delta) + \mathrm{Tr}\left((K - F(\delta))\Gamma^{-1}(K - F(\delta))^T\right)$.

- In both cases the closed-loop systems have passivity properties with respect to output $z = GC(\delta)x + D(\delta)w$ where $D(\delta) = D_0 + \sum_{p=1}^{\bar{p}} \delta_p D_p$.

THM 1 Existence of an attraction domain

- For a chosen scalar $\epsilon > 0$ solve the LMI problem

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- If LMIs are feasible for some ϵ then they also hold for any $\hat{\epsilon} \in]0\ \epsilon]$.

▲ Needed to test the conditions for a small enough ϵ.

▲ ϵ is related to exponential stability.

- Numerical complexity

▲ Nb of variables of the LMIs is proportional to $n^2\bar{p}$

▲ Size of LMI constraints is proportional to $n2^{2\bar{p}}$

▲ Using YALMIP/SeDuMi computation time is < 1min for $n < 10$, $\bar{p} < 5$.

THM 2 Estimating the attraction domain

- Given a solution to LMIs of THM 1, solve the LMI optimization problem

$$\tau^* = \min \tau \ : \ \mathcal{L}_2(\tau, S_{p=1\ldots\bar{p}}, \epsilon_{p=1\ldots\bar{p}}, \Gamma^{-1}) < \mathbf{O}$$

Attraction domains are $x^T Q x \leq 1$ with Q such that $\tau^* Q \leq P(\delta)$ for all $\delta \in \bar{\Delta}$.

- Asymptotic stability of the origin ($\tau^* = 0$) either if
  - $\dot{\delta} = 0$, constant parametric uncertainties [IJACSP'08]
  - $F(\delta) = F_0$, exists a unique SOF for all uncertainties [Kaufman et al. 1994]
- Arbitrarily small attraction domain: $\tau^* \longrightarrow 0$ if $\Gamma^{-1} \longrightarrow \mathbf{O}$
  - But Γ should not be too large for implementation purpose

$$u(t) = K(t)y(t) + w(t) \ , \ \dot{K}(t) = -Gy(t)y^T(t)\Gamma - \phi(K(t))\Gamma$$

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Attraction domains are $x^T Q x \leq 1$ with Q such that $\tau^* Q \leq P(\delta)$ for all $\delta \in \bar{\Delta}$.

- Minimizing the attraction domain: solve $\tau^* Q \leq P(\delta)$ for all $\delta \in \bar{\Delta}$
  - with $\max \lambda \ : \ \lambda \mathbf{I} \leq Q$ - then minimizes largest semi-axis
  - with $\max \mathrm{Tr}(Q)$ - then minimizes mean value of all semi-axes

❹ Some clues about the proofs

THM 1 Robust passivity properties by relaxation of equality constraints

$$\mathcal{L}_1(H_1, H_2, P_{p=0\ldots\bar{p}}, F_{p=0\ldots\bar{p}}, D_{p=0\ldots\bar{p}}, R_{p=0\ldots\bar{p}}, T_{p=0\ldots\bar{p}}) < \mathrm{O}$$
$$\Rightarrow \quad \left(P(\delta)B(\delta) - C^T(\delta)G^T\right)\left(P(\delta)B(\delta) - C^T(\delta)G^T\right)^T \leq R(\delta)$$

made possible thanks to feed-through gain (shunt) $z = Gy + D(\delta)w$.

THM 1 Robust passivity properties thanks to properties of corrective term ϕ

$$\mathcal{L}_1(H_1, H_2, P_{p=0\ldots\bar{p}}, F_{p=0\ldots\bar{p}}, D_{p=0\ldots\bar{p}}, R_{p=0\ldots\bar{p}}, T_{p=0\ldots\bar{p}}) < \mathrm{O}$$
$$\Rightarrow \quad \dot{W}(x,K,\delta) \leq z^T w - \tfrac{\epsilon}{2} x^T P(\delta) x + U(K,\delta) - \mathrm{Tr}\left(\dot{F}(\delta)\Gamma^{-1}(K - F(\delta))^T\right)$$

where $U(K,\delta) = \frac{1}{2} y^T(K^T K - \beta \mathbf{I})y - \mathrm{Tr}\left(\phi(K)(K - F(\delta))^T\right) \leq 0$.

❹ Some clues about the proofs

THM 1 Negative derivative of Lyapunov function for large enough x:

$$\mathcal{L}_1(H_1, H_2, P_{p=0\ldots\bar{p}}, F_{p=0\ldots\bar{p}}, D_{p=0\ldots\bar{p}}, R_{p=0\ldots\bar{p}}, T_{p=0\ldots\bar{p}}) < \mathrm{O}$$
$$\Rightarrow \quad \dot{W}(x,K,\delta) \leq z^T w - \tfrac{\epsilon}{2} x^T P(\delta) x - \mathrm{Tr}\left(\dot{F}(\delta)\Gamma^{-1}(K - F(\delta))^T\right)$$

where δ, $\dot{\delta}$ and K are bounded ($\mathrm{Tr}(KK^T) \leq \beta$).

THM 2 Estimating the attraction domain

$$\mathcal{L}_2(\tau, S_{p=1\ldots\bar{p}}, \epsilon_{p=1\ldots\bar{p}}, \Gamma^{-1}) < \mathrm{O}$$
$$\Rightarrow \quad \tau \geq \tfrac{2}{\epsilon}\mathrm{Tr}\left(\dot{F}(\delta)\Gamma^{-1}(K - F(\delta))^T\right) \quad , \quad \forall \delta \in \Delta, \forall K \text{bounded}$$

Conclusions

Novel robustness results

- LMI-based: use of efficient numerical tools [YALMIP, SeDuMi...]
- Guaranteed robustness to uncertainties on all data $(A(\delta), B(\delta), C(\delta))$
- Estimated attraction domain in case of time-vaying uncertainties

Future work

- Validations of the theoretical results on examples
- Heuristics for the design of G matrix
- SAC applied to dynamic output-feedback
- ...