# Evaluating regions of attraction of LTI systems with saturation in IQS framework

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- Saturated control of a linear system

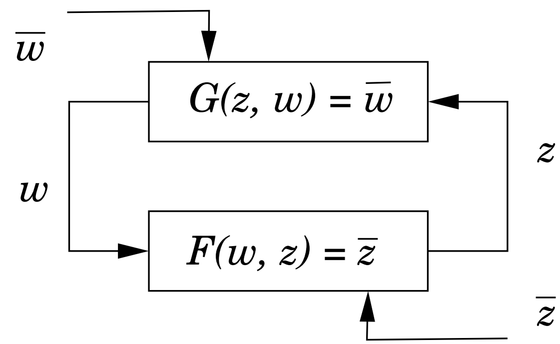
$$\dot{x} = Ax + Bu \ , \ \ u = \text{sat}(Ky) \ , \ \ y = Cx$$

- Assume $K$ designed for the linear system (no saturation)
- System with saturation: Stability is (in general) only local
- Problem: find (largest possible) set of $x(0)$ such that $x(\infty) = 0$

- Goal of this presentation : formalize the problem in the IQS framework

- Can ”system augmentation” relaxations provide less conservative results ?

- Well-posedness of a feedback loop




- Uniqueness and boundedness of internal signals for all bounded disturbances

$$\exists \gamma : \forall (\bar{w}, \bar{z}) \in L_2 \times L_2 \;, \quad \left\| \begin{array}{c} w - w_0 \\ z - z_0 \end{array} \right\| \leq \gamma \left\| \begin{array}{c} \bar{w} \\ \bar{z} \end{array} \right\| , \quad \begin{array}{c} G(z_0, w_0) = 0 \\ F(w_0, z_0) = 0 \end{array}$$

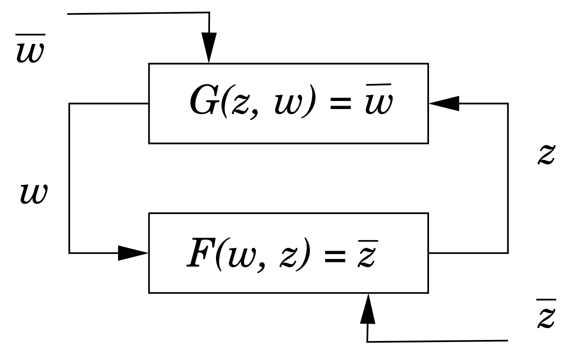
- **iff** exists a topological separator $\theta$
- Negative on the inverse graph of one component
- Positive definite on the graph of the other component of the loop

$$\mathcal{G}^I(\bar{w}) = \{(w,z) \; : \; G(z,w) = \bar{w}\} \subset \{(w,z) \; : \; \theta(w,z) \leq \phi_2(||\bar{w}||)\}$$

$$\mathcal{F}(\bar{z}) = \{(w,z) \; : \; F(w,z) = \bar{z}\} \subset \{(w,z) \; : \; \theta(w,z) > -\phi_1(||\bar{z}||)\}$$

- Issues: How to choose $\theta$ ? How to test the separation inequalities ?

- Well-posedness of a feedback loop




- In case of causal $G(z,w)$ : $w = \Delta z$ , $\Delta \in \mathcal{RH}_\infty^{m \times l}$

and stable proper LTI $F(w,z)$ : $z = H(s)w$

- Necessary and sufficient (lossless) choice of separator

$$\theta(w,z) = \|w\|^2 - \gamma^2 \|z\|^2$$

- Separation inequalities:

$$\theta(w,z) = \|w\|^2 - \gamma^2 \|z\|^2 \leq 0 \, , \, \forall w = \Delta z \;\; \Leftrightarrow \;\; \|\Delta\|_\infty^2 \leq \gamma^2$$

$$\theta(w,z) = \|w\|^2 - \gamma^2 \|z\|^2 > 0 \, , \, \forall z = H(s)w \;\; \Leftrightarrow \;\; \|H\|_\infty^2 < \frac{1}{\gamma^2}$$

- Choice of an Integral Quadratic Separator

$$\theta(w,z) = \left\langle \begin{pmatrix} z \\ w \end{pmatrix} \middle| \Theta \begin{pmatrix} z \\ w \end{pmatrix} \right\rangle = \int_0^{\infty} \begin{pmatrix} z^T(t) & w^T(t) \end{pmatrix} \Theta(t) \begin{pmatrix} z(t) \\ w(t) \end{pmatrix} dt$$

- Identical choice to IQC framework [Megretski, Rantzer, Jönsson]

$$\theta(w,z) = \int_{-\infty}^{+\infty} \begin{pmatrix} z^T(j\omega) & w^T(j\omega) \end{pmatrix} \Pi(j\omega) \begin{pmatrix} z(j\omega) \\ w(j\omega) \end{pmatrix} d\omega$$

- $\Pi$ is called a multiplier. $\theta(w,z) \leq 0$ is called an IQC.
- Conservatism reduction in IQC framework : $\omega$-dependent multipliers:

$$\Pi(j\omega) = \begin{bmatrix} \mathbf{1} & \Psi_1(j\omega)^* & \cdots & \Psi_r(j\omega)^* \end{bmatrix} \hat{\Pi} \begin{bmatrix} \mathbf{1} \\ \Psi_1(j\omega) \\ \vdots \\ \Psi_r(j\omega) \end{bmatrix}$$

- Main IQS result (both for $\omega$ or $t$ or $k$ dependent signals)

- IQS is **necessary and sufficient** under assumptions (proof based on [Iwasaki 2001])
  - One component is a linear application, can be descriptor form $F(w,z) = \mathcal{A}w - \mathcal{E}z$
    - can be time-varying $\mathcal{A}(t)w(t) - \mathcal{E}(t)z(t)$ or frequency dep. $\hat{\mathcal{A}}(\omega)\hat{w}(\omega) - \hat{\mathcal{E}}(\omega)\hat{z}(\omega)$
    - $\mathcal{A}(t)$, $\mathcal{E}(t)$ are bounded and $\mathcal{E}(t) = \mathcal{E}_1(t)\mathcal{E}_2$ where $\mathcal{E}_1(t)$ is full column rank
  - The other component can be defined in a set

$$G(z,w) = \nabla(z) - w \ , \ \nabla \in \mathbb{W}$$

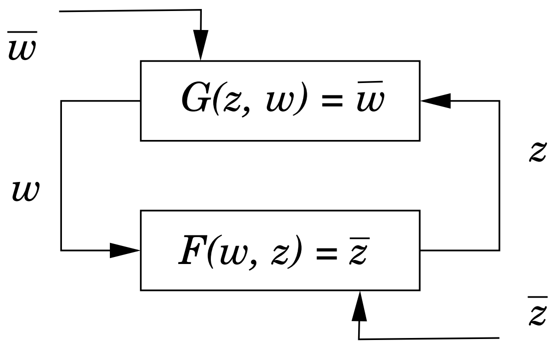
    - $\mathbb{W}$ must have a linear-like property

$$\forall (z_1, z_2) \ , \ \forall \nabla \in \mathbb{W} \ , \ \exists \tilde{\nabla} \in \mathbb{W} \ : \ \nabla(z_1) - \nabla(z_2) = \tilde{\nabla}(z_1 - z_2)$$

    - $\mathbb{W}$ need not to be causal

- The matrix $\Theta$ must satisfy an IQC over $\mathbb{W}$ + an LMI involving ($\mathcal{E}$, $\mathcal{A}$)

- Global stability of a non-linear system $\dot{x} = f(x,t)$



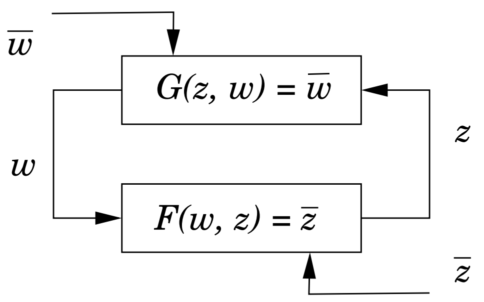

$$G(z = \dot{x}, w = x) = \int_0^t z(\tau) d\tau - w(t),$$
$$F(w,z,t) = f(w,t) - z(t)$$

- $\bar{w}$ plays the role of the initial conditions, $\bar{z}$ are external disturbances
- Well-posedness: for all bounded initial conditions and all bounded disturbances,
  the state remains bounded around the equilibrium $\equiv$ global stability

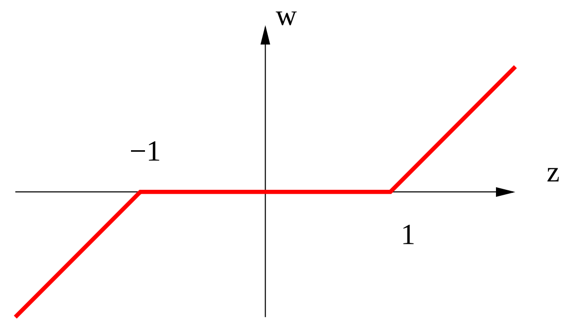
- For linear systems $\dot{x}(t) = A(t)x(t)$, $\nabla = s^{-1}\mathbf{1}$
- IQS: $\theta(w,z) = \int_0^\infty \begin{pmatrix} z^T(t) & w^T(t) \end{pmatrix} \begin{bmatrix} \mathbf{0} & -P(t) \\ -P(t) & -\dot{P}(t) \end{bmatrix} \begin{pmatrix} z(t) \\ w(t) \end{pmatrix} dt$
  - $\theta(w,z) \leq 0$ for all $G(z,w) = 0$ **iff** $P(t) \geq \mathbf{0}$
  - $\theta(w,z) > 0$ for all $F(w,z) = 0$ **iff** $A^T(t)P(t) + P(t)A(t) + \dot{P}(t) < \mathbf{0}$

# Examples - Topological Separation and Lyapunov

- Global stability of a system with a dead-zone




$$G_1(\dot{x},x) = \int_0^t \dot{x}(\tau)d\tau - x(t),$$
$$G_2(g,v) = \mathrm{dz}(g(t)) - v(t),$$
$$F_1(x,v,\dot{x},t) = f_1(x,v,t) - \dot{x}(t),$$
$$F_2(x,v,g,t) = f_2(x,v,t) - g(t)$$




- IQS applies for linear $f_1$, $f_2$

- Dead-zone embedded in a sector uncertainty $\mathbb{W}_\infty = \{\nabla_\infty \ : \ 0 \leq \nabla_\infty(g) \leq g\}$

$$\mathcal{G}_2^I = \{(v,g) \ : \ G_2(g,v) = 0\} \subset \{(v,g) \ : \ v = \nabla_\infty(g) \ , \ \nabla_\infty \in \mathbb{W}_\infty\}$$

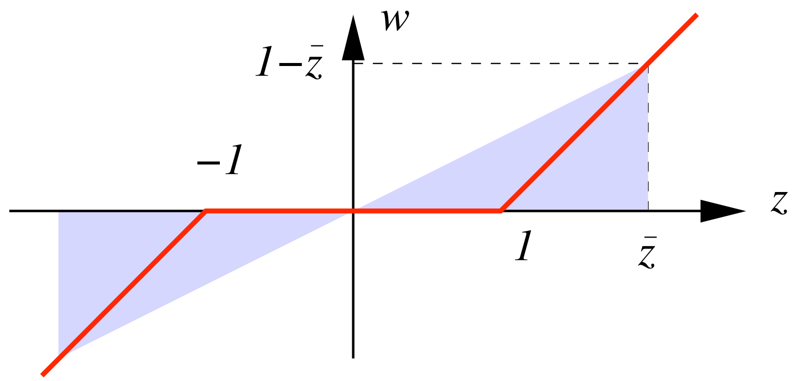
- This is the only source of conservatism

- LMI conditions obtained for the IQS defined by

$$\Theta = \left[\begin{array}{cc|cc} \mathbf{0} & \mathbf{0} & -P & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -p_1 \\ \hline -P & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -p_1 & \mathbf{0} & 2p_1 \end{array}\right], \quad \begin{array}{l} P > \mathbf{0}, \\ p_1 > 0. \end{array}$$

- Launcher in ballistic phase : attitude control
  - neglected atmospheric friction, sloshing modes, ext. perturbation, axes coupling: $I\ddot{\theta} = T$
  - Saturated actuator: $T = \text{sat}_{\bar{T}}(u) = u - \bar{T}\text{dz}(\frac{1}{\bar{T}}u)$
  - PD control $u = -K_P\theta - K_D\dot{\theta}$
- Global stability LMI test fails
  - Sector uncertainty includes $\nabla_\infty = 1$ for which the system is $I\ddot{\theta} = 0$ (unstable)
  - LMI test succeeds (whatever $\bar{g} < \infty$) if dead-zone is restricted to belong to

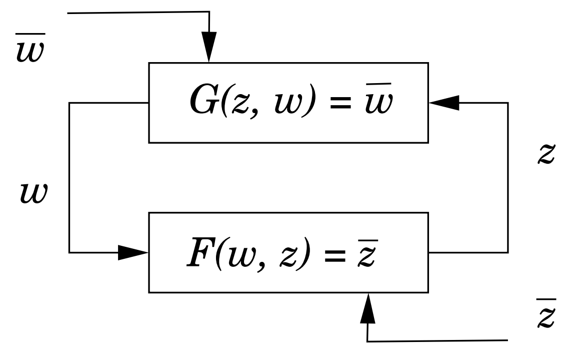
$$\mathbb{W}_{\bar{g}} = \{\nabla_{\bar{g}} \;:\; 0 \leq \nabla_{\bar{g}}(g) \leq \frac{1-\bar{g}}{\bar{g}} g\}$$




  - Useful if one can prove for constrained $x(0)$ that $|g(\theta)| \leq \bar{g}$ holds $\forall\theta \geq 0$.
- How can one prove local properties in IQS framework ?

# Initial conditions dependent IQS

- Well-posedness of a feedback loop




- Uniqueness and boundedness of internal signals for all bounded disturbances

$$\exists \gamma : \forall (\bar{w}, \bar{z}) \in L_2 \times L_2 \;, \quad \left\| \begin{array}{c} w - w_0 \\ z - z_0 \end{array} \right\| \leq \gamma \left\| \begin{array}{c} \bar{w} \\ \bar{z} \end{array} \right\| , \quad \begin{array}{c} G(z_0, w_0) = 0 \\ F(w_0, z_0) = 0 \end{array}$$

- How to introduce initial conditions $x(0)$ and “final” conditions $g(\theta)$ in IQS framework?

- Square-root of the Dirac operator: linear operator such that

$$x \mapsto \varphi_\theta x \quad : \quad \begin{array}{l} < \varphi_\theta x | M \varphi_\theta x >= \int_0^\infty \varphi_\theta x^T(t) M \varphi_\theta x(t) dt = x^T(\theta) M x(\theta) \\ < \varphi_{\theta_1} x | M \varphi_{\theta_2} x >= 0 \text{ if } \theta_1 \neq \theta_2 \end{array}$$

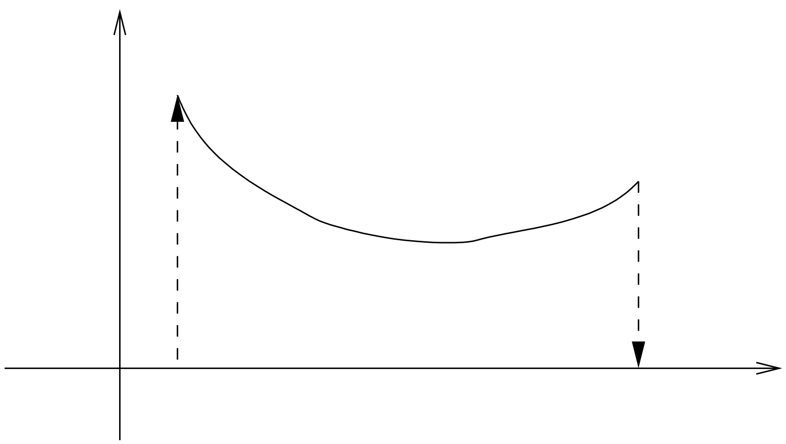
- Such operator is also used for PDE to describe states on the boundary

- System with initial and final conditions writes as

$$\left(\begin{array}{c} \varphi_0 x \\ \mathcal{T}_\theta \dot{x} \\ \hline \mathcal{T}_\theta g \\ \hline \varphi_\theta g \end{array}\right) = \left[\begin{array}{cc|c|c} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ A & \mathbf{0} & B & \mathbf{0} \\ \hline C & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & C & \mathbf{0} & \mathbf{0} \end{array}\right] \left(\begin{array}{c} \mathcal{T}_\theta x \\ \varphi_\theta x \\ \hline \mathcal{T}_\theta v \\ \hline \varphi_0 x \end{array}\right)$$

- $\mathcal{T}_\theta x$ is the truncated signal such that $\mathcal{T}_\theta x(t) = x(t)$ for $t \leq \theta$ and $= 0$ for $t > \theta$.
- The integration operator is redefined as a mapping

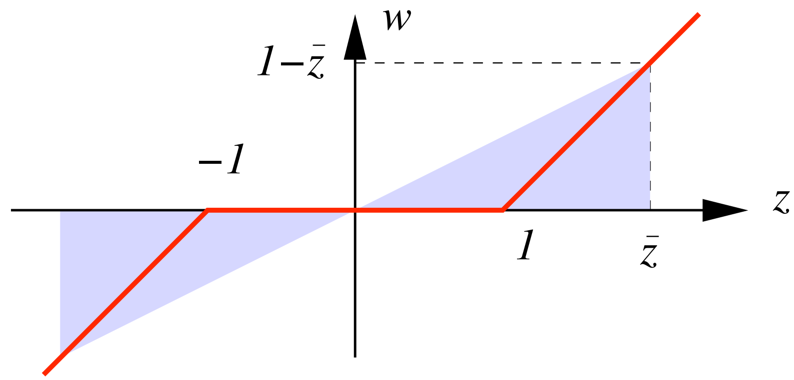
$$\left(\begin{array}{c} \mathcal{T}_\theta x \\ \varphi_\theta x \end{array}\right) = \mathcal{I} \left(\begin{array}{c} \varphi_0 x \\ \mathcal{T}_\theta \dot{x} \end{array}\right)$$



$$\begin{pmatrix} \varphi_0 x \\ \mathcal{T}_\theta \dot{x} \\ \hline \mathcal{T}_\theta g \\ \hline \varphi_\theta g \end{pmatrix} = \left[\begin{array}{cc|c|c} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ A & \mathbf{0} & B & \mathbf{0} \\ \hline C & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & C & \mathbf{0} & \mathbf{0} \end{array}\right] \begin{pmatrix} \mathcal{T}_\theta x \\ \varphi_\theta x \\ \hline \mathcal{T}_\theta v \\ \hline \varphi_0 x \end{pmatrix}$$

- Restricted sector constraint assumed to hold up to $t = \theta$:

$$\mathcal{T}_\theta v = \nabla_{\bar{g}} \mathcal{T}_\theta g$$



$$\begin{pmatrix} \varphi_0 x \\ \mathcal{T}_\theta \dot{x} \\ \hline \mathcal{T}_\theta g \\ \hline \varphi_\theta g \end{pmatrix} = \left[\begin{array}{cc|c|c} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ A & \mathbf{0} & B & \mathbf{0} \\ \hline C & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & C & \mathbf{0} & \mathbf{0} \end{array}\right] \begin{pmatrix} \mathcal{T}_\theta x \\ \varphi_\theta x \\ \hline \mathcal{T}_\theta v \\ \hline \varphi_0 x \end{pmatrix}$$

- Goal is to prove the restricted sector condition holds strictly at time $\theta$ (whatever $\theta$)
- i.e. find sets $1 \geq x^T(0) Q x(0) = < \varphi_0 x | Q \varphi_0 x >$ s.t. $|g(\theta)| = \|\varphi_\theta g\| < \bar{g}$
- reformulated as well posedness problem where $\varphi_0 x = \nabla_{ci} \varphi_\theta g$ defined by

$$w_{ci} = \nabla_{ci} z_{zi} \quad : \quad \bar{g}^2 < w_{ci} | Q w_{ci} > \leq \|z_{ci}\|^2$$

$$\begin{pmatrix} \varphi_0 x \\ \mathcal{T}_\theta \dot{x} \\ \hline \mathcal{T}_\theta g \\ \hline \varphi_\theta g \end{pmatrix} = \left[\begin{array}{cc|c|c} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ A & \mathbf{0} & B & \mathbf{0} \\ \hline C & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & C & \mathbf{0} & \mathbf{0} \end{array}\right] \begin{pmatrix} \mathcal{T}_\theta x \\ \varphi_\theta x \\ \hline \mathcal{T}_\theta v \\ \hline \varphi_0 x \end{pmatrix}$$

- Problem defined in this way is a well-posedness problem with $\nabla$ composed of 3 blocs

$$\nabla = \begin{bmatrix} \mathcal{I} & & \\ & \nabla_{\bar{g}} & \\ & & \nabla_{ci} \end{bmatrix}$$

- IQS framework applies and gives conservative LMI conditions
- Equivalent to LaSalle invariance principle with $V(x) = x^T Q x$ (ellipsoidal sets of IC)

- How to reduce conservatism ?
  - Needed a description of the dead-zone better than sector uncertainty
  - Needed to have dead-zone dependent sets of initial conditions
- Both features derived via descriptor modeling of system augmented with $\dot{v}$ and $\dot{g}$

$$v = \mathrm{dz}(g) \ : \ \begin{cases} \text{if } g > 1 & v = g - 1 & \dot{v} = \dot{g} \\ \text{if } |g| \geq 1 & v = 0 & \dot{v} = 0 \\ \text{if } g < -1 & v = g + 1 & \dot{v} = \dot{g} \end{cases}$$

  - For IQS, link between $\dot{v}$ and $\dot{g}$ is embedded in $\dot{v} = \nabla_{\{0,1\}}\dot{g}$, with $\nabla_{\{0,1\}} \in \{0, 1\}$.
  - Also needed to specify that $v$ is the integral of $\dot{v}$ (thus descriptor form)

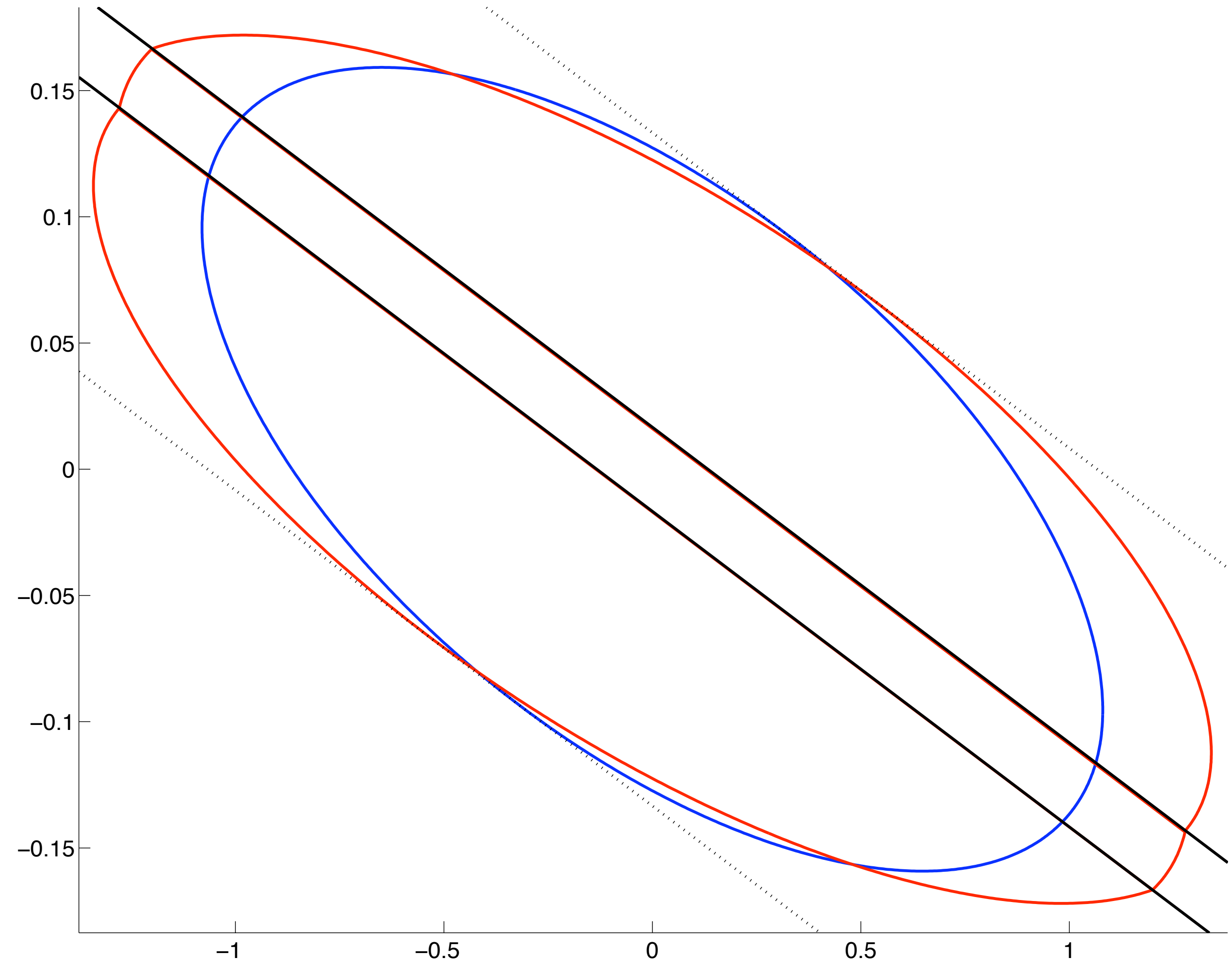
# System augmentation with derivatives

$$
\left[\begin{array}{cccc|c|c|c|c}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -C & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\left(\begin{array}{c}
\varphi_0 x \\ \varphi_0 v \\ \mathcal{T}_\theta \dot{x} \\ \mathcal{T}_\theta \dot{v} \\ \hline \mathcal{T}_\theta g \\ \hline \varphi_\theta g \\ \hline \mathcal{T}_\theta \dot{g} \\ \hline \varphi_\theta g
\end{array}\right)
=
\left[\begin{array}{cccc|c|c|c|cc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
A & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & C & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & C & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0
\end{array}\right]
\left(\begin{array}{c}
\mathcal{T}_\theta x \\ \mathcal{T}_\theta v \\ \varphi_\theta x \\ \varphi_\theta v \\ \hline \mathcal{T}_\theta v \\ \hline \varphi_\theta v \\ \hline \mathcal{T}_\theta \dot{v} \\ \hline \varphi_0 x \\ \varphi_0 v
\end{array}\right)
$$

- Problem defined in this way is a well-posedness problem with $\nabla$ composed of 5 blocs
- IQS framework applies and gives less conservative LMI conditions
- Equivalent to LaSalle invariance principle with

$$
V(x) = \begin{pmatrix} x \\ v \end{pmatrix}^T Q_a \begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} x \\ \mathrm{dz}(Cx) \end{pmatrix}^T Q_a \begin{pmatrix} x \\ \mathrm{dz}(Cx) \end{pmatrix}
$$

- LMIs tested on the launcher example




- Sets of initial conditions for which $|g(\theta)| \leq 8$ is guaranteed
- Improvement thanks to piecewise quadratic sets of initial conditions

- IQS framework can handle local stability issues
  - Provides LMI tests - conservative
  - System augmentation + descriptor modeling = reduction of conservatism

- Prospectives
  - Improved construction of the IQS $\equiv$ "generalized sector conditions"
  - Further system augmentation with higher derivatives (?)
  - Simultaneous handling of saturation, uncertainties, delays...
  - Hybrid systems ?