Some results related to modeling systems with uncertainty

Несколько результатов об моделирование систем с неопределённостями

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■ Outline

❶ Motivations for uncertain descriptor modeling

❷ Affine polytopic models

❸ LFT models - frequency dependent models

❹ Augmented descriptor models and conservatism reduction

➊ Motivations for uncertain descriptor modeling

■ Models issued from physics are naturally in descriptor form

$$E\dot{x} = Ax + Bu$$

● Example: mechanical system

$$M\ddot{\theta} + C\dot{\theta} + K\theta = u$$

M: inertia ; C: friction ; K: stiffness ; u external forces

$$\begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \begin{pmatrix} \ddot{\theta} \\ \dot{\theta} \end{pmatrix} = \begin{bmatrix} -C & -K \\ I & 0 \end{bmatrix} \begin{pmatrix} \dot{\theta} \\ \theta \end{pmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} u$$

● Example: robotic systems with Lagrange formulations [MG89]

E may not be invertible

● Example: networks of systems with algebraic constraints describing links

❶ Motivations for uncertain descriptor modeling

- Descriptor models can be converted to usual models
  - Example: mechanical system

$$\ddot{\theta} + M^{-1}C\dot{\theta} + M^{-1}K\theta = M^{-1}u$$

  - Assumes that M^{-1} is well-conditioned and known
  - If some parameters are unknown: $M(\Delta)$, $C(\Delta)$, $K(\Delta)$

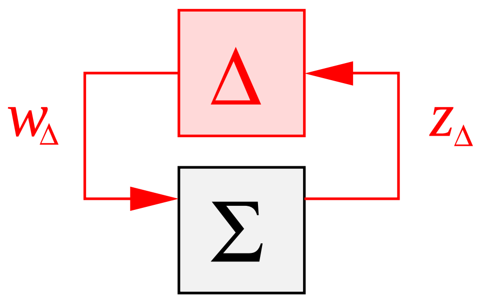
$$\ddot{\theta} + M^{-1}(\Delta)C(\Delta)\dot{\theta} + M^{-1}(\Delta)K(\Delta)\theta = M^{-1}(\Delta)u$$

  - Increased complexity of the model
- Descriptor models are preferable for describing systems with uncertainty

- Example: DC motor $I\dot{\omega} = bu$ regulated in speed $u = -\omega$
- Parameters are assumed uncertain $I = 1 + \delta_1$, $b = 1 + \delta_2$

$$(1+\delta_1)\dot{\omega} = -(1+\delta_2)\omega \quad \Rightarrow \quad \dot{\omega} = -\frac{1+\delta_2}{1+\delta_1}\omega$$

- Model is rational w.r.t. uncertainties: exists an LFT representation



$$-\frac{1+\delta_2}{1+\delta_1} = A + B_\Delta \Delta (\mathbf{I} - D_\Delta \Delta)^{-1} C_\Delta = \begin{bmatrix} A & B_\Delta \\ C_\Delta & D_\Delta \end{bmatrix} \star \Delta$$

- Can be build with Robust Control toolbox of Matlab or LFRT [Mag05]

- Building the LFT for $\dot{\omega} = -\frac{1+\delta_2}{1+\delta_1}\omega$
- 1st step: descriptor form with no denominators

$$\dot{\omega} + \delta_1\dot{\omega} = -\omega - \delta_2\omega$$

- 2nd step: all multiplications correspond to a feedback

$$\dot{\omega} + w_1 = -\omega - w_2 \quad : \quad \begin{array}{l} w_1 = \delta_1 z_1 \\ z_1 = \dot{\omega} \end{array} \quad , \quad \begin{array}{l} w_1 = \delta_2 z_2 \\ z_2 = \omega \end{array}$$

- 3rd step: descriptor LFT: $\Delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}$, $z_\Delta = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, $w_\Delta = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$,

$$\left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline -1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \begin{pmatrix} \dot{\omega} \\ z_\Delta \end{pmatrix} = \left[\begin{array}{c|cc} -1 & -1 & -1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right] \begin{pmatrix} \omega \\ w_\Delta \end{pmatrix} \quad , \quad w_\Delta = \Delta z_\Delta$$

- last step: invert the left-hand side matrix

- LFT for $\dot{\omega} = -\frac{1+\delta_2}{1+\delta_1}\omega$

$$-\frac{1+\delta_2}{1+\delta_1} = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline -1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]^{-1} \left[\begin{array}{c|cc} -1 & -1 & -1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right] \star \left[\begin{array}{cc} \delta_1 & 0 \\ 0 & \delta_2 \end{array}\right]$$

$$= \left[\begin{array}{c|cc} -1 & -1 & -1 \\ \hline -1 & -1 & -1 \\ 1 & 0 & 0 \end{array}\right] \star \left[\begin{array}{cc} \delta_1 & 0 \\ 0 & \delta_2 \end{array}\right]$$

$$= -1 + \left[\begin{array}{cc} -1 & -1 \end{array}\right] \left[\begin{array}{cc} \delta_1 & 0 \\ 0 & \delta_2 \end{array}\right] \left(\mathbf{I} - \left[\begin{array}{cc} -1 & -1 \\ 0 & 0 \end{array}\right] \left[\begin{array}{cc} \delta_1 & 0 \\ 0 & \delta_2 \end{array}\right] \right)^{-1} \left[\begin{array}{c} -1 \\ 1 \end{array}\right]$$

$$= -1 - \left[\begin{array}{cc} \delta_1 & \delta_2 \end{array}\right] \left[\begin{array}{cc} 1+\delta_1 & \delta_2 \\ 0 & 1 \end{array}\right]^{-1} \left[\begin{array}{c} -1 \\ 1 \end{array}\right]$$

$$= -1 - \left[\begin{array}{cc} \delta_1 & \delta_2 \end{array}\right] \left[\begin{array}{c} -\frac{1+\delta_2}{1+\delta_1} \\ 1 \end{array}\right] = -1 + \frac{\delta_1 - \delta_2}{1+\delta_1} = -\frac{1+\delta_2}{1+\delta_1}$$

- The example shows the interest of descriptor models,

  even if only for technical manipulations

■ All fractional models have affine descriptor representations [MAS03]

▲ Proof: All fractional models can be converted to an LFT

$$\begin{pmatrix} \dot{x} \\ z_\Delta \end{pmatrix} = \begin{bmatrix} A & B_\Delta \\ C_\Delta & D_\Delta \end{bmatrix} \begin{pmatrix} x \\ w_\Delta \end{pmatrix} \quad , \quad w_\Delta = \Delta z_\Delta$$

▲ the LFT gives the affine descriptor form:

$$\begin{bmatrix} \mathsf{I} & -B_\Delta \Delta \\ \mathsf{O} & \mathsf{I} - D_\Delta \Delta \end{bmatrix} \begin{pmatrix} \dot{x} \\ z_\Delta \end{pmatrix} = \begin{bmatrix} A \\ C_\Delta \end{bmatrix} x$$

● Can give representations of smaller dimensions

▲ Example

$$\begin{bmatrix} 1 & \delta_1 & \delta_2 \\ 0 & 1+\delta_1 & \delta_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \dot{\omega} \\ z_1 \\ z_2 \end{pmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \omega \quad \Leftrightarrow \quad (1+\delta_1)\dot{\omega} = -(1+\delta_2)\omega$$

■ General descriptor models

$$E_{xx}\dot{x} + E_{x\pi}\pi = Ax + Bu$$
$$y + E_{xy}\dot{x} + E_{y\pi}\pi = Cx + Du$$

- x: state ; u: inputs
- π: linearly constrained signals
- E_{xx} and A may not be square

▲ Can be converted to $\hat{E}\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u$ with $\hat{E}$ and $\hat{A}$ square and $\dot{\hat{x}} = \begin{pmatrix} \dot{x} \\ \pi \\ \lambda \end{pmatrix}$

▲ Not recommend: increased size of the model

Outline

1. Motivations for uncertain descriptor modeling
2. Affine polytopic models
3. LFT models - frequency dependent models
4. Augmented descriptor models and conservatism reduction

- All fractional models have affine descriptor representations

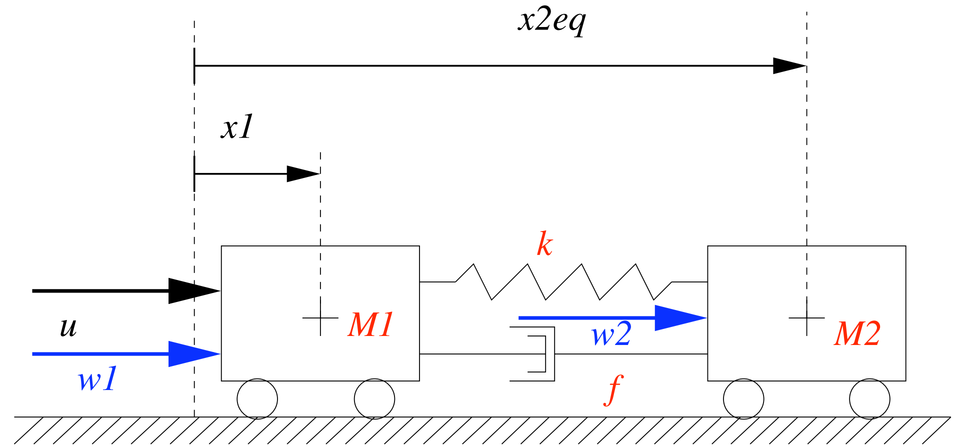
$$E_{xx}(\Delta)\dot{x} + E_{x\pi}(\Delta)\pi = A(\Delta)x + B(\Delta)u$$
$$y + E_{xy}(\Delta)\dot{x} + E_{y\pi}(\Delta)\pi = C(\Delta)x + D(\Delta)u$$

- Models also used for polynomial non-linear models [CTF02]
- General affine descriptor data

$$\begin{bmatrix} -E_{xx}(\Delta) & -E_{x\pi}(\Delta) & A(\Delta) & B(\Delta) \\ -E_{yx}(\Delta) & -E_{y\pi}(\Delta) & C(\Delta) & D(\Delta) \end{bmatrix} = M(\Delta)$$

- Different classes of affine models
- intervals: $\underline{M} \preceq M(\Delta) \preceq \overline{M}$ (element-wise $\underline{m}_{ij} \leq m_{ij}(\Delta) \leq \overline{m}_{ij}$)
- parallelotopes: $M(\Delta) = M^{|0|} + \sum_{p=1}^{\bar{p}} \delta_p M^{|p|} \quad : \quad |\delta_p| \leq 1$
- polytopes: $M(\Delta) \in co\left\{ M^{[v=1\dots\bar{v}]} \right\}$
- intervals $\subset$ parallelotopes $\subset$ polytopes

■ Example: two mass spring system



$$M_1\ddot{x}_1 + f(\dot{x}_1 - \dot{x}_2) + k(x_1 - x_2) = u + w_1$$

$$M_2\ddot{x}_2 + f(\dot{x}_2 - \dot{x}_1) + k(x_2 - x_1) = w_2$$

● General affine data model

$$\underbrace{\begin{bmatrix} M_1 & 0 & f & -f & k & -k \\ 0 & M_2 & -f & f & -k & k \end{bmatrix}}_{=M(\Delta)} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \dot{x}_1 \\ \dot{x}_2 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} u + w_1 \\ w_2 \end{pmatrix}$$

❷ Affine polytopic models

Example: two mass spring system - continued

- 4 uncertain parameters $\underline{M}_i \leq M_i \leq \overline{M}_i, \quad \underline{f} \leq f \leq \overline{f}, \quad \underline{k} \leq k \leq \overline{k}$

$$M(\Delta) = \begin{bmatrix} M_1 & 0 & f & -f & k & -k \\ 0 & M_2 & -f & f & -k & k \end{bmatrix}$$

- Interval model generates conservatism (elements assumed independent)

$$\begin{bmatrix} \underline{M_1} & 0 & \underline{f} & -\underline{f} & \underline{k} & -\underline{k} \\ 0 & \underline{M_2} & -\underline{f} & \underline{f} & -\underline{k} & \underline{k} \end{bmatrix}$$
$$\preceq \begin{bmatrix} \underline{M_1} & 0 & \overline{f} & -\underline{f} & \overline{k} & -\underline{k} \\ 0 & \underline{M_2} & -\underline{f} & \overline{f} & -\underline{k} & \overline{k} \end{bmatrix}$$
$$\preceq \begin{bmatrix} \overline{M_1} & 0 & \overline{f} & -\overline{f} & \overline{k} & -\overline{k} \\ 0 & \overline{M_2} & -\overline{f} & \overline{f} & -\overline{k} & \overline{k} \end{bmatrix}$$

■ Example: two mass spring system - continued

$$M(\Delta) = \begin{bmatrix} M_1 & 0 & f & -f & k & -k \\ 0 & M_2 & -f & f & -k & k \end{bmatrix}$$

● Parallelotopic model

▲ Nominal model: $M^{|0|}$ (center of the intervals)

$$M(\Delta) = M^{|0|} + \delta_{M_1} \tfrac{1}{2}(\overline{M_1} - \underline{M_1}) \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\vdots$$

$$+ \underbrace{\delta_k \frac{1}{2}(\overline{k} - \underline{k}) \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}}_{M^{|k|}}$$

▲ $M^{|k|}$ "axis" of variations along δ_k.

■ Example: two mass spring system - continued

$$M(\Delta) = \begin{bmatrix} M_1 & 0 & f & -f & k & -k \\ 0 & M_2 & -f & f & -k & k \end{bmatrix}$$

● Polytopic model: $2^4 = 16$ vertices ($\delta_i = \pm 1$ in parallelotopic model)

$$M^{[1]} = \begin{bmatrix} \overline{M_1} & 0 & \overline{f} & -\overline{f} & \overline{k} & -\overline{k} \\ 0 & \overline{M_2} & -\overline{f} & \overline{f} & -\overline{k} & \overline{k} \end{bmatrix}$$
$$M^{[2]} = \begin{bmatrix} \underline{M_1} & 0 & \overline{f} & -\overline{f} & \overline{k} & -\overline{k} \\ 0 & \overline{M_2} & -\overline{f} & \overline{f} & -\overline{k} & \overline{k} \end{bmatrix}$$
$$\vdots$$
$$M^{[16]} = \begin{bmatrix} \underline{M_1} & 0 & \underline{f} & -\underline{f} & \underline{k} & -\underline{k} \\ 0 & \underline{M_2} & -\underline{f} & \underline{f} & -\underline{k} & \underline{k} \end{bmatrix}$$

■ Some techniques to deal with affine uncertain models

● General type of analysis criteria

$$\eta^T \Theta(\Delta)\eta < 0 \quad , \quad \forall \eta \neq 0 \quad : \quad M(\Delta)\eta = 0$$

▲ Example: negative definite derivative of Lyapunov function $V = x^T P(\Delta) x$

$$\dot{V} = \eta^T \begin{bmatrix} \mathrm{O} & P(\Delta) \\ P(\Delta) & \mathrm{O} \end{bmatrix} \eta < 0, \; \forall \eta = \begin{pmatrix} \dot{x} \\ x \end{pmatrix} \neq 0 : \; \begin{bmatrix} -\mathrm{I} & A(\Delta) \end{bmatrix} \eta = 0$$

● Usual LMI type result ($M^{\perp}(\Delta)$ matrix generating the null-space of $M(\Delta)$)

$$M^{\perp T}(\Delta)\Theta(\Delta)M^{\perp}(\Delta) < \mathrm{O}$$

▲ Problem: $M^{\perp}(\Delta)$ is a rational function of Δ

▲ Solutions: SOS [HG05, Sch06], Polyá [CGTV09, OdOP08]

■ Some techniques to deal with affine uncertain models

● General type of analysis criteria

$$\eta^T \Theta(\Delta)\eta < 0 \quad , \quad \forall \eta \neq 0 \quad : \quad M(\Delta)\eta = 0$$

● Usual LMI type result ($M^\perp(\Delta)$ matrix generating the null-space of $M(\Delta)$)

$$M^{\perp T}(\Delta)\Theta(\Delta)M^\perp(\Delta) < \mathrm{O}$$

● Slack variables results (variables issued from the "creation" Finsler lemma)

$$\exists F \quad : \quad \Theta(\Delta) < FM(\Delta) + M^T(\Delta)F^T$$

▲ Conservative, but sufficient to test at the vertices

▲ $\Theta(\Delta)$ affine (parameter-dependent Lyapunov function)

▲ [OBG99, PABB00, EH04, PDSV09]

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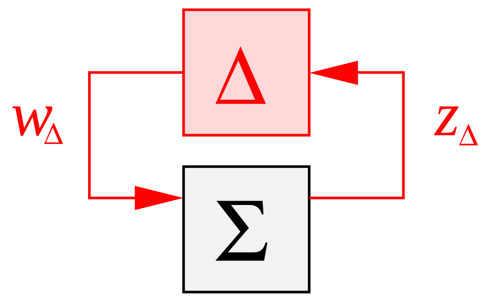
- Affine polytopic models well adapted for scalar parametric uncertainty
- LFT suitable for other types of uncertainties:
  - Frequency dependent uncertainties
  - Linear / non-linear operators
  - Time-varying parameters
  - ...



- Linear combination of repeated uncertainties [Sch06]

$$\Delta = \sum J_j(\mathbf{I}_{r_j} \otimes \Delta_j)K_j \quad , \qquad [J_1 \cdots J_{\bar{j}}] \text{ orthonormal}$$

- Example: bloc-diagonal uncertainties $\Delta = \begin{bmatrix} \Delta_1 & & \\ & \delta_2 & \\ & & \delta_2 \end{bmatrix}$

■ Dissipative structured parametric uncertainties

● An uncertain matrix Δ is said to be $\{\Psi_1, \Psi_2, \Psi_3\}$-dissipative if

$$\begin{bmatrix} I & \Delta^* \end{bmatrix} \begin{bmatrix} \Psi_1 & \Psi_2 \\ \Psi_2^* & \Psi_3 \end{bmatrix} \begin{bmatrix} I \\ \Delta \end{bmatrix} \leq 0 \, .$$

● An uncertain matrix Δ is said to be $\{\Phi_1, \Phi_2, \Phi_3\}$-structured if

$$\begin{bmatrix} I & \Delta^* \end{bmatrix} \begin{bmatrix} \Phi_1 & \Phi_2 \\ \Phi_2^* & \Phi_3 \end{bmatrix} \begin{bmatrix} I \\ \Delta \end{bmatrix} = 0 \, .$$

● Examples of $\{\Psi_1, \Psi_2, \Psi_3\}$-dissipative uncertainties

▲ Norm-bounded uncertainties are $\{-\gamma^2 \mathbf{I}, \mathbf{O}, \mathbf{I}\}$-dissipative

$$\begin{bmatrix} \mathbf{I} & \Delta^* \end{bmatrix} \begin{bmatrix} -\gamma^2 \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} \\ \Delta \end{bmatrix} = -\gamma^2 \mathbf{I} + \Delta^* \Delta \leq \mathbf{O}\,.$$

$$w_\Delta = \Delta z_\Delta \quad \Rightarrow \quad \|w_\Delta\| \leq \gamma \|z_\Delta\|$$

▲ Positive real uncertainties are $\{\mathbf{O}, -\mathbf{I}, \mathbf{O}\}$-dissipative

$$\begin{bmatrix} \mathbf{I} & \Delta^* \end{bmatrix} \begin{bmatrix} \mathbf{O} & -\mathbf{I} \\ -\mathbf{I} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{I} \\ \Delta \end{bmatrix} = -(\Delta^* + \Delta) \leq \mathbf{O}\,.$$

$$w_\Delta = \Delta z_\Delta \quad \Rightarrow \quad < w_\Delta | z_\Delta > \quad \geq 0$$

- Examples of $\{\Psi_1, \Psi_2, \Psi_3\}$-dissipative uncertainties - continued

▲ Uncertainties in an "ellipsoid" are $\{\Delta_0^* Z \Delta_0 - R, -\Delta_0^* Z, Z\}$-dissipative

$$\begin{bmatrix} \mathrm{I} & \Delta^* \end{bmatrix} \begin{bmatrix} \Delta_0^* Z \Delta_0 - R & -\Delta_0^* Z \\ -Z\Delta_0 & Z \end{bmatrix} \begin{bmatrix} \mathrm{I} \\ \Delta \end{bmatrix} = (\Delta - \Delta_0)^* Z (\Delta - \Delta_0) - R \leq \mathrm{O}\,.$$

(discs of the complex plane for scalar $\Delta = \delta$)

▲ Uncertainties in a "half space" $\{-\Delta_0^* N + N\Delta_0, N^*, \mathrm{O}\}$-dissipative

$$\begin{bmatrix} \mathrm{I} & \Delta^* \end{bmatrix} \begin{bmatrix} -\Delta_0^* N + N\Delta_0 & N^* \\ N & \mathrm{O} \end{bmatrix} \begin{bmatrix} \mathrm{I} \\ \Delta \end{bmatrix} = (\Delta - \Delta_0)^* N + N^* (\Delta - \Delta_0) \leq \mathrm{O}\,.$$

(half of the complex plane limited by a line for scalar $\Delta = \delta$)

- $\{\Phi_1, \Phi_2, \Phi_3\}$-structured uncertainties: border of "ellipsoids" or of "half spaces"

- $\{\Psi_1, \Psi_2, \Psi_3\}$-dissipative $\{\Phi_1, \Phi_2, \Phi_3\}$-structured, examples [IH05]

- $\delta = (\mathrm{j}\omega)^{-1}$ with bounded frequencies $\omega \in [\,\underline{\omega}\ \ \overline{\omega}\,]$:

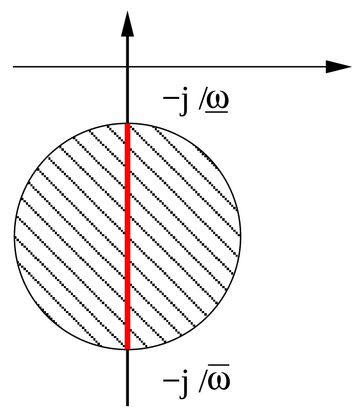
  $\{2, -\mathrm{j}(\underline{\omega} + \overline{\omega}), 2\underline{\omega}\overline{\omega}\}$-dissipative $\{0, 1, 0\}$-structured

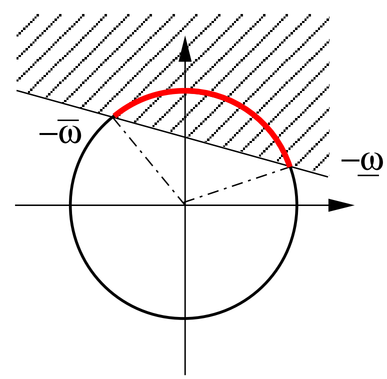
- $\delta = (\mathrm{j}\omega)^{-1}$ with high frequencies $\omega \geq \underline{\omega} \geq 0$:

  $\{0, -\mathrm{j}\underline{\omega}, 2\underline{\omega}^2\}$-dissipative $\{0, 1, 0\}$-structured

- $\delta = e^{-\mathrm{j}\omega}$ with bounded frequencies $\omega \in [\,\underline{\omega}\ \ \overline{\omega}\,]$:

  $\{2\cos\frac{\overline{\omega}-\underline{\omega}}{2}, -e^{\mathrm{j}\frac{\overline{\omega}+\underline{\omega}}{2}}, 0\}$-dissipative $\{-1, 0, 1\}$-structured

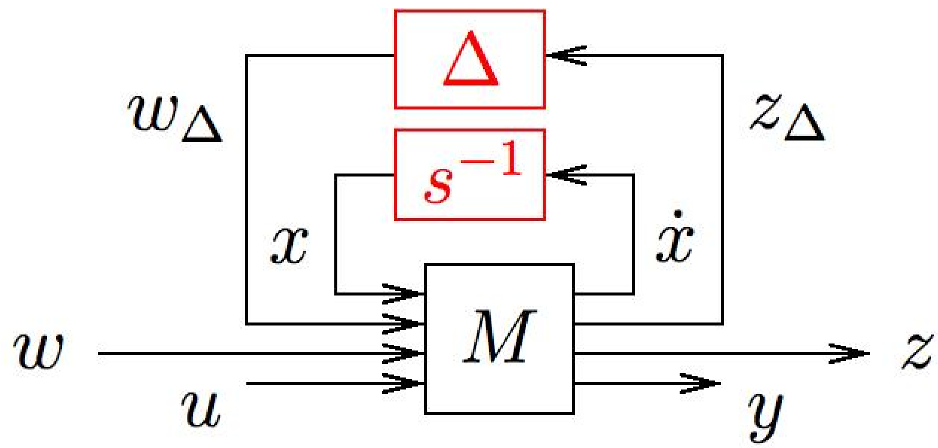




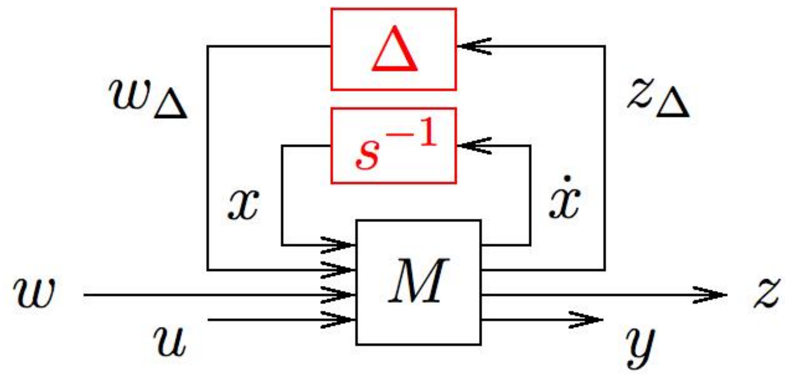
- $\delta \in \mathbb{R}$ bounded $\delta \in [\,\underline{\delta}\ \ \overline{\delta}\,]$:

  $\{2\underline{\delta}\overline{\delta}, -(\underline{\delta} + \overline{\delta}), 2\}$-dissipative $\{0, \mathrm{j}, 0\}$-structured

- System analysis in frequency intervals: $s^{-1} = (\mathrm{j}\omega)^{-1}$, $\omega \in [\,\underline{\omega}\ \ \overline{\omega}\,]$



- Allows loop-shaping type specifications
- Have interpretations in time domain [IHF05]



■ Performances?

● Stability: no poles in $\mathbb{C}_+$ $\Leftrightarrow$ well-posedness w.r.t. $s^{-1} \in \mathbb{C}_+$

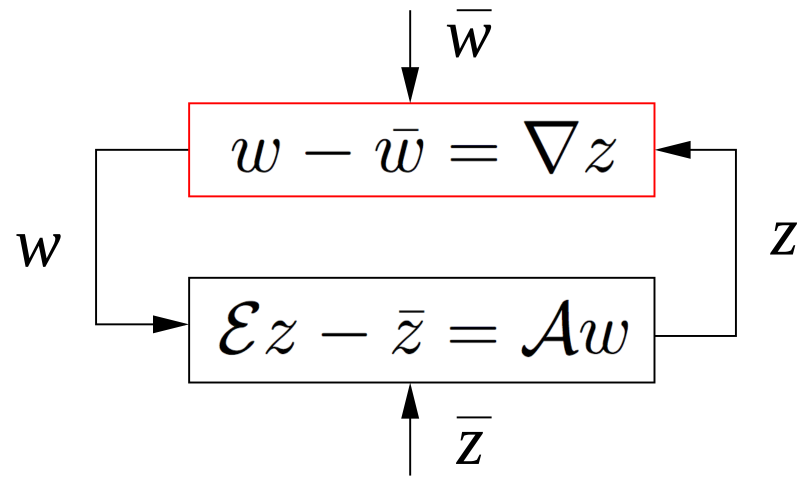
● L_2 perf: $\|z\|_2 \leq \gamma \|w\|_2$ $\Leftrightarrow$ well-posedness w.r.t. $w = \nabla z$: $\|\nabla\|_2 \leq \frac{1}{\gamma}$

▲ Can be tested for given frequencies or for frequency intervals

● Dissipativity: $\int_0^\infty \begin{pmatrix} z(t) \\ w(t) \end{pmatrix}^* \begin{bmatrix} \Psi_1 & \Psi_2 \\ \Psi_2^* & \Psi_3 \end{bmatrix} \begin{pmatrix} z(t) \\ w(t) \end{pmatrix} dt \geq 0$

$\Leftrightarrow$ well-posedness w.r.t. $w = \nabla z$: ∇ $\{\Psi_1, \Psi_2, \Psi_3\}$-dissipative

■ Performances $\Leftrightarrow$ well-posedness w.r.t. uncertain operators



■ Well-posedness [Saf80]

● For all bounded external signals $(\bar{w}, \bar{z})$

the internal signals $(w, \mathcal{E}z)$ are unique and bounded

▲ Example: $z = \dot{x}$, $w = \nabla\dot{x} = x$ (*i.e.* ∇ integrator).

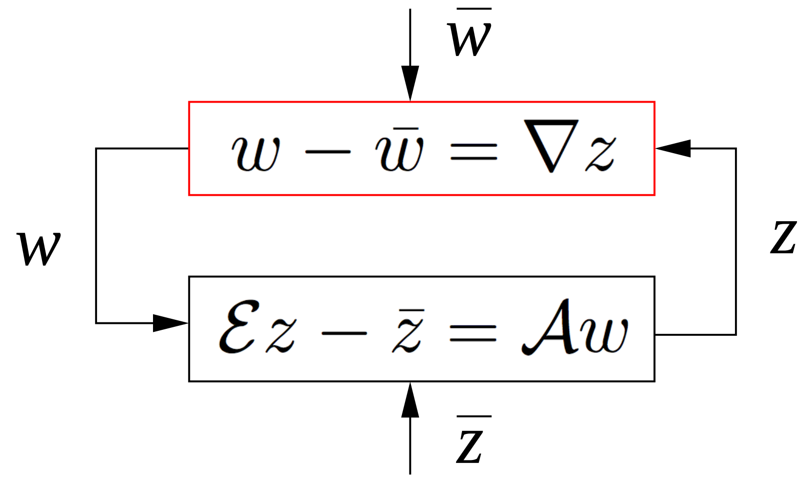
Well posedness $\Leftrightarrow$

$\forall$ initial conditions $\bar{w} = x(0)$ and bounded disturbances $\bar{z}$

x is unique, no impulsive modes, no divergence (stability)

❸ LFT models

- Well-posedness [PAHG07]



- Integral quadratic separation (IQS) result
  - Assume $\mathcal{E} = \mathcal{E}_1 \mathcal{E}_2$ with $\mathcal{E}_1$ full column rank and let $M = \begin{bmatrix} -\mathcal{E}_1 & \mathcal{A} \end{bmatrix}$
  - The loop is well-posed **iff** there exists Θ such that

$$\eta^T \Theta \eta < 0 \quad , \quad \forall \eta \neq 0 \quad : \quad M\eta = 0 \tag{1}$$

and for all “uncertainties"

$$\int_0^\infty \begin{pmatrix} \mathcal{E}_2 z(t) \\ [\nabla z](t) \end{pmatrix}^* \Theta \begin{pmatrix} \mathcal{E}_2 z(t) \\ [\nabla z](t) \end{pmatrix} dt \geq 0 \tag{2}$$

- IQS result
  - (1): LMI-type constraint ($M^{\perp T}\Theta M^{\perp} < 0$)
  - (2): Integral quadratic constraint (IQC)

- IQS applies easily to uncertainties defined, or included in IQCs
  - Example: $\nabla = \mathrm{I}_r \otimes \Delta$,

Δ is $\{\Psi_1, \Psi_2, \Psi_3\}$-dissipative $\{\Phi_1, \Phi_2, \Phi_3\}$-structured

$$\Theta = \begin{bmatrix} P \otimes \Psi_1 & P \otimes \Psi_2 \\ P \otimes \Psi_2^* & P \otimes \Psi_3 \end{bmatrix} + \begin{bmatrix} R \otimes \Psi_1 & R \otimes \Psi_2 \\ R \otimes \Psi_2^* & R \otimes \Psi_3 \end{bmatrix} \quad \begin{matrix} P > 0 \\ R = R^* \end{matrix}$$

(generalized version of D/G-scaling)

- IQC also defined in the frequency domain (Parseval)

$$\int_0^{\infty} \begin{pmatrix} \mathcal{E}_2 z(\mathrm{j}\omega) \\ [\nabla z](\mathrm{j}\omega) \end{pmatrix}^* \Theta \begin{pmatrix} \mathcal{E}_2 z(\mathrm{j}\omega) \\ [\nabla z](\mathrm{j}\omega) \end{pmatrix} d\omega \geq 0 \qquad (3)$$

- IQC formulas [MR97, JM99] apply for building IQS results
- ∇ can contain
  - Saturations [FLR06, PTGSB12]
  - Delays [JS01, GP07, PAHG07]
  - ... to be continued

- Parametrizing Θ solutions to the IQC is in general conservative [MSF97, IH05]
  - Lossless formulas **iff** $\Delta = \sum_{j=1}^{\bar{j}} J_j (\mathrm{I}_{r_j} \otimes \Delta_j) K_j$ $\left\{ \begin{array}{l} \bar{j} = 2\ ,\ r_1 > 1\ ,\ r_2 = 1 \\ \bar{j} = 3\ ,\ r_1 = r_2 = r_3 = 1 \end{array} \right.$

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- Except in special cases finding Θ solution to the IQCs is conservative
- Existing parameterizations of Θ apply both for constant and time-varying Δ
- But $\mathbb{\Delta}_\infty \subset \mathbb{\Delta}_d \subset \mathbb{\Delta}_0$ where
  - $\mathbb{\Delta}_0$ set of constant uncertainties such that $\Sigma(\Delta)$ is stable
  - $\mathbb{\Delta}_d$ set of uncertainties with $\|\dot{\Delta}\| \leq d$ such that $\Sigma(\Delta)$ is stable
  - $\mathbb{\Delta}_\infty$ set of TV uncertainties with switches such that $\Sigma(\Delta)$ is stable
- Reducing conservatism: can be achieved by introducing information about $\dot{\Delta}$

Descriptor augmentation method - example

- Assume system in LFT form

$$\left\{\begin{array}{l} \dot{x} = Ax + Bw_\Delta \\ z_\Delta = Cx + Dw_\Delta \end{array}\right. \star \left\{\begin{array}{l} x = s^{-1}\dot{x} \\ w_\Delta = \Delta z_\Delta \end{array}\right.$$

- Add information about derivative of uncertainty:

$$\dot{w}_\Delta = \underbrace{\Delta \dot{z}_\Delta}_{=w_{\Delta 1}} + \underbrace{\dot{\Delta} z_\Delta}_{=w_{\Delta 2}}$$

- Define derivative of exogenous signals:

$$\begin{array}{rcl} \dot{z}_\Delta & = & C\dot{x} + D\dot{w}_\Delta \\ & = & C\dot{x} + Dw_{\Delta 1} + Dw_{\Delta 2} \end{array} , \quad w_\Delta = s^{-1}\dot{w}_\Delta$$

■ Descriptor augmentation method - example - continued

$$\left\{\begin{array}{l}\dot{x} = Ax + Bw_\Delta \\ z_\Delta = Cx + Dw_\Delta\end{array}\right. \star \left\{\begin{array}{l}x = s^{-1}\dot{x} \\ w_\Delta = \Delta z_\Delta\end{array}\right.$$

● All equations gathered separating linear constrains and "uncertainties"

$$\left\{\begin{array}{l}\dot{x} = Ax + Bw_\Delta \\ z_\Delta = Cx + Dw_\Delta \\ \dot{z}_\Delta - C\dot{x} = Dw_{\Delta 1} + Dw_{\Delta 2} \\ \dot{w}_\Delta = w_{\Delta 1} + w_{\Delta 2} \\ 0 = w_\Delta - w_\Delta\end{array}\right. \star \left\{\begin{array}{l}x = s^{-1}\dot{x} \\ w_\Delta = s^{-1}\dot{w}_\Delta \\ w_\Delta = \Delta z_\Delta \\ w_{\Delta 1} = \Delta \dot{z}_\Delta \\ w_{\Delta 2} = \dot{\Delta} z_\Delta\end{array}\right.$$

❹ Augmented descriptor models and conservatism reduction

■ Descriptor augmentation method - example - continued

● Descriptor LFT form of system augmented with information on derivatives

$$z = \begin{pmatrix} \dot{x} & \dot{w}_\Delta & z_\Delta & \dot{z}_\Delta \end{pmatrix}^T, \; w = \begin{pmatrix} x & w_\Delta & w_\Delta & w_{\Delta 1} & w_{\Delta 2} \end{pmatrix}^T$$

$$\begin{bmatrix} \mathsf{I} & 0 & 0 & 0 \\ 0 & 0 & \mathsf{I} & 0 \\ -C & 0 & 0 & \mathsf{I} \\ 0 & \mathsf{I} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} z = \begin{bmatrix} A & B & 0 & 0 & 0 \\ C & D & 0 & 0 & 0 \\ 0 & 0 & 0 & D & D \\ 0 & 0 & 0 & \mathsf{I} & \mathsf{I} \\ 0 & \mathsf{I} & -\mathsf{I} & 0 & 0 \end{bmatrix} w$$

$$\star \; w = \begin{bmatrix} s^{-1}\mathsf{I}_{n+m} & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \\ 0 & \dot{\Delta} & 0 \end{bmatrix} z$$

❹ Augmented descriptor models and conservatism reduction

■ IQS results applicable to the original & the augmented descriptor model

● LMI formulas for augmented model are <u>less conservative</u>

● IQS formulas implicitly contain Lyapunov certificates

▲ Lyapunov certificate in case of LMI for original model

$$V = x^T P x$$

▲ Parameter-dependent Lyapunov certificate in case of augmented system

$$V = \begin{pmatrix} x \\ w_\Delta \end{pmatrix}^T P \begin{pmatrix} x \\ w_\Delta \end{pmatrix} \quad , \quad w_\Delta = \Delta(\mathbf{I} - D\Delta)^{-1} C x$$

● The system augmentation technique is also proved efficient for systems with delays [PAHG07], with saturations [PTGSB12]...

Conclusions

- Some advantages of descriptor models

- Natural models from physics

- More flexibility for system description

- Affine descriptor models are alternatives to LFT models

- Well-posedness of descriptor LFT loops

  = general framework for robust analysis problems

  (includes frequency dependent specifications/uncertainties)

- System augmentation with derivatives = less conservative results

  (made possible thanks to descriptor modeling)

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