Robust LFR-based technique

for stability analysis of periodic solutions

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Robustness and Non-Linear problems

✪ Robustness: prove stability whatever uncertainties in bounded sets

✪ Lyapunov theory: prove stability whatever initial conditions in bounded sets

➙ Model non-linearities on the states as uncertainties on an LTI model

Stability of periodic solutions and periodic systems

✪ Dynamics of relative motion: stability of the origin of periodic system

✪ Numerical methods for periodic systems: sampling

➙ Modeling of sampled uncertainties

Outline

① Discrete-time Periodic Uncertain models for stability of periodic solutions

② Linear Matrix Inequality (LMI) results

③ Illustrative example

Linear-Fractional Representation (LFR)

Non-linear, uncertain, system $\dot{\eta}(t) = f(\eta(t), t, \delta)$

with periodic solution:

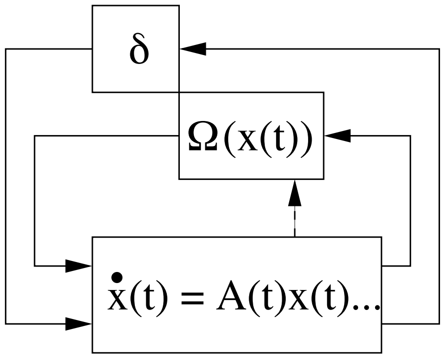
$$\dot{\eta}_s(t) = f(\eta_s(t), t, \delta) \ , \ \forall \delta \in [\underline{\delta}, \overline{\delta}] \ : \ \eta_s(t+T) = \eta_s(t)$$

Quasi-linearization around periodic solution ($x^c = \eta_s - \eta$)

$$\dot{x}^c(t) = A^c(t)x^c(t) + B^c_\delta(t)w_\delta(t) + B^c_\Omega(t)w_\Omega(t)$$
$$z^c_\delta(t) = C^c_\delta(t)x^c(t) + D^c_{\delta\delta}(t)w_\delta(t) + D^c_{\delta\Omega}(t)w_\Omega(t)$$
$$z^c_\Omega(t) = C^c_\Omega(t)x^c(t) + D^c_{\Omega\delta}(t)w_\delta(t) + D^c_{\Omega\Omega}(t)w_\Omega(t)$$

where uncertainties and non-linearities are feedback connected

$$w^c_\delta(t) = \delta z^c_\delta(t)$$
$$w^c_\Omega(t) = \Omega(x^c(t))z^c_\Omega(t)$$



LFR Hypothesis

LMI results for robust stability need quadratic separators

$$\exists \Theta \quad : \quad \Upsilon_\Theta(\delta 1) = \begin{bmatrix} 1 \\ \delta 1 \end{bmatrix}^T \Theta \begin{bmatrix} 1 \\ \delta 1 \end{bmatrix} \leq 0 \quad , \quad \forall \delta \in [\underline{\delta}, \overline{\delta}]$$

➙ Let $\boldsymbol{\Theta}$ the set of all Θ such that this holds.

Bounded $\Omega(x)$ for x bounded

✪ <u>HYP 0:</u> Exists a positive definite matrix Q and a matrix Ξ such that

$$x^T Q x \leq \gamma \quad \Rightarrow \quad \Upsilon_\Xi(\Omega(x)) = \begin{bmatrix} 1 \\ \Omega(x) \end{bmatrix}^T \Xi \begin{bmatrix} 1 \\ \Omega(x) \end{bmatrix} \leq 0$$

➙ For given Q and γ, let $\boldsymbol{\Xi}_\gamma$ the set of all Ξ such that HYP 0 holds.

➙ Based on HYP 0, non-linearities may be treated as if uncertainties.

Non-linear system $\ddot{\eta}_1 + (\eta_1^2 + \dot{\eta}_1^2 - 1)\dot{\eta}_1 + \eta_1 = u$

with uncertain control input $u(t) = (\kappa+\delta)(\cos t - \eta_1(t)) \Rightarrow \eta_{s1}(t) = \cos t$

LFR around periodic solution $x_1^c = \eta_{s1} - \eta_1 \ , \ x_2^c = \dot{x}_1^c$

$$A^c(t) = \begin{bmatrix} 0 & 1 \\ -1-\kappa+\sin 2t & \cos 2t - 1 \end{bmatrix} \quad B_\delta^c = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$B_\Omega^c(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\sin t & 2\cos t & -3\sin t & -1 \end{bmatrix}$$

$$C_\delta^c = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad D_{\delta\delta}^c = 0 \quad D_{\delta\Omega}^c = 0$$

$$C_\Omega^c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad D_{\Omega\delta}^c = 0 \quad D_{\Omega\Omega}^c = 0$$

$$\Omega(x^c) = \begin{bmatrix} x_1^c & 0 \\ x_2^c & 0 \\ 0 & x_2^c \\ 0 & x_1^{c2} + x_2^{c2} \end{bmatrix}$$

Sampling of the LFT model

Periodic sampling strategy

- Sampling sequence $\{T_s(k)\}_{k\geq 0}$: $T_s(N) = T$, $T_s(k+N) = T_s(k)$

$$x(k) = x^c(T_s(k)) \ , \ w_\delta(k) = w_\delta^c(T_s(k)) \quad \ldots \quad \Omega(k) = \Omega(x^c(T_s(k)))$$

- HYP 1: median approximation of LTI model

$$\forall t \in [T_s(k), T_s(k+1)] \ : \ A^c(t) \simeq \tilde{A}(k) = A^c(\frac{1}{2}(T_s(k)+T_s(k+1))) \ldots$$

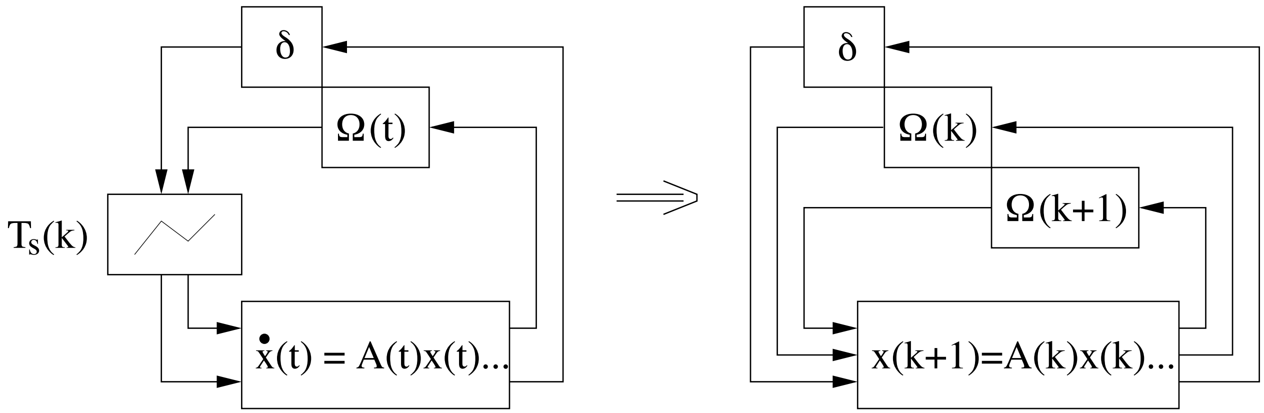
- HYP 2: first-order hold approximation of exogenous signals [Imbert 2001]

$$w_\delta^c(t) \simeq w_\delta(k) + \frac{t - T_s(k)}{T_s(k+1) - T_s(k)}(w_\delta(k+1) - w_\delta(k)) \quad \ldots$$

- HYP 3: β-bounded growth of the state x over sampling period ($\beta \geq 1$)

$$x^T(k)Qx(k) \leq q \ \Rightarrow \ x^T(k+1)Qx(k+1) \leq \beta q$$

HYP 0-3 : T-periodic continuous $\Rightarrow$ N-periodic discrete



$$x(k+1) = A(k)x(k) + B_\delta(k)\tilde{w}_\delta(k) + B_\Omega(k)\tilde{w}_\Omega(k)$$
$$\tilde{z}_\delta(k) = C_\delta(k)x(k) + D_{\delta\delta}\tilde{w}_\delta(k) + D_{\delta\Omega}(k)\tilde{w}_\Omega(k)$$
$$\tilde{z}_\Omega(k) = C_\Omega(k)x(k) + D_{\Omega\delta}\tilde{w}_\delta(k) + D_{\Omega\Omega}(k)\tilde{w}_\Omega(k)$$

$$\tilde{w}_\delta(k) = \delta\tilde{z}_\delta(k)$$
$$\tilde{w}_\omega(k) = \tilde{\Omega}(k)\tilde{z}_\Omega(k)$$

$$\tilde{\Omega} = \begin{bmatrix} \Omega(k) & 0 \\ 0 & \Omega(k+1) \end{bmatrix} \text{ with } \begin{array}{l} \Upsilon_{\Xi\in\boldsymbol{\Xi}_q}(\Omega(k)) \leq 0 \\ \Upsilon_{\hat{\Xi}\in\boldsymbol{\Xi}_{\beta q}}(\Omega(k+1)) \leq 0 \end{array} \text{ if } x^T(k)Qx(k) \leq q.$$

Invariant sets $x^T Q x \leq q$:

If there exist $\Xi(k) \in \mathbf{\Xi}_q$, $\hat{\Xi}(k) \in \mathbf{\Xi}_{\beta q}$, $\Theta(k) \in \mathbf{\Theta}$ that satisfy for all $k = 1 \ldots N$ the LMIs

$$N_x^T(k) \begin{bmatrix} Q & 0 \\ 0 & -Q \end{bmatrix} N_x(k) < \begin{array}{r} N_\delta^T(k)\Theta(k)N_\delta(k) \\ +N_\Omega^T(k)\Xi(k)N_\Omega(k) \\ +\hat{N}_\Omega^T(k)\hat{\Xi}(k)\hat{N}_\Omega(k) \end{array}$$

where N_x, N_δ, N_Ω and $\hat{N}_\Omega$ matrices are functions of model matrices $A, B_\delta \ldots$

then for any bounded initial conditions such that $x^T(0)Qx(0) \leq q$

the trajectory remains bounded such that $x^T(k)Qx(k) \leq q$.

Convergence to periodic solution for initial conditions $x^T\hat{Q}x \leq q$:

If there exist $P(k) \geq 0$, $\Xi(k) \in \mathbf{\Xi}_q$, $\hat{\Xi}(k) \in \mathbf{\Xi}_{\beta q}$, $\Theta(k) \in \mathbf{\Theta}$ that satisfy for all $k = 1 \ldots N$ the LMIs

$$\hat{Q}(k) = Q + P(k) > 0$$

$$N_x^T(k) \begin{bmatrix} \hat{Q}(k) & 0 \\ 0 & -\hat{Q}(k) \end{bmatrix} N_x(k) < \begin{array}{r} N_\delta^T(k)\Theta(k)N_\delta(k) \\ +N_\Omega^T(k)\Xi(k)N_\Omega(k) \\ +\hat{N}_\Omega^T(k)\hat{\Xi}(k)\hat{N}_\Omega(k) \end{array}$$

then for any bounded initial conditions such that $x^T(0)\hat{Q}(0)x(0) \leq q$
the trajectory satisfies invariance $x^T(k)Qx(k) \leq x^T(k)\hat{Q}(k)x(k) \leq q$
decreasing Lyapunov function $x^T(k+1)\hat{Q}(k+1)x(k+1) \leq x^T(k)\hat{Q}(k)x(k)$
and convergence to the periodic solution $x(k) = \eta_s(k) - \eta(k) \rightarrow 0$.

LMI conservative representations of separators

Need for tools to choose $\Theta \in \boldsymbol{\Theta}$ and $\Xi \in \boldsymbol{\Xi}_\gamma$

Vertex separator: If Θ satisfies the LMIs

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \Theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} \geq 0\,,\ \Upsilon_\Theta(\underline{\delta}1) \leq 0\,,\ \Upsilon_\Theta(\overline{\delta}1) \leq 0$$

then $\Theta \in \boldsymbol{\Theta}$ (*i.e.* $\Upsilon_\Theta(\delta 1) \leq 0 \quad \forall \delta \in [\underline{\delta}, \overline{\delta}]$)

LMIs conditions of Ξ for $\Upsilon_\Xi(\Omega(x)) \leq 0$ whatever $x^T Q x \leq \gamma$?

Example: $\Omega(x) = \begin{bmatrix} x_1 & 0 \\ x_2 & 0 \\ 0 & x_2 \\ 0 & x_1^2 + x_2^2 \end{bmatrix}$ with $x^T Q x = x^T x = x_1^2 + x_2^2$.

Take Ξ structured as

$$\Xi = \left[\begin{array}{cc|cccc} -\gamma(\alpha_1+\alpha_4) & -\gamma(\alpha_2+\alpha_5) & 0 & 0 & 0 & \alpha_2 \\ -\gamma(\alpha_2+\alpha_5) & -\gamma(2\alpha_3+\alpha_6+\gamma\alpha_7) & 0 & 0 & 0 & \alpha_3 \\ \hline 0 & 0 & \alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_1+\alpha_4 & \alpha_5 & 0 \\ 0 & 0 & 0 & \alpha_5 & \alpha_6 & 0 \\ \alpha_2 & \alpha_3 & 0 & 0 & 0 & \alpha_7 \end{array}\right]$$

$$\Xi_1 = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & 2\alpha_3 \end{bmatrix} \geq 0, \quad \Xi_2 = \begin{bmatrix} \alpha_4 & \alpha_5 \\ \alpha_5 & \alpha_6 \end{bmatrix} \geq 0, \quad \Xi_3 = \begin{bmatrix} 0 & 0 \\ 0 & \alpha_7 \end{bmatrix} \geq 0$$

then

$$\Upsilon_\Xi(\Omega(x)) = (x_1^2+x_2^2-\gamma)\Xi_1 + (x_2^2-\gamma)\Xi_2 + ((x_1^2+x_2^2)^2-\gamma^2)\Xi_3 \leq 0$$

for all $x^TQx = x^Tx = x_1^2+x_2^2 \leq \gamma$.

Results for the example

Non-linear system $\ddot{\eta}_1 + (\eta_1^2 + \dot{\eta}_1^2 - 1)\dot{\eta}_1 + \eta_1 = u$

with uncertain control input $u(t) = (\kappa + \delta)(\cos t - \eta_1(t)) \;\Rightarrow\; \eta_{s1}(t) = \cos t$

Tests made for control gain $\kappa = 4.5$ and $\overline{\delta} = -\underline{\delta} = 0.5$ uncertainty.

LMI tests

LMIs solved for $N = 20$ samples per period (uniformly spaced),

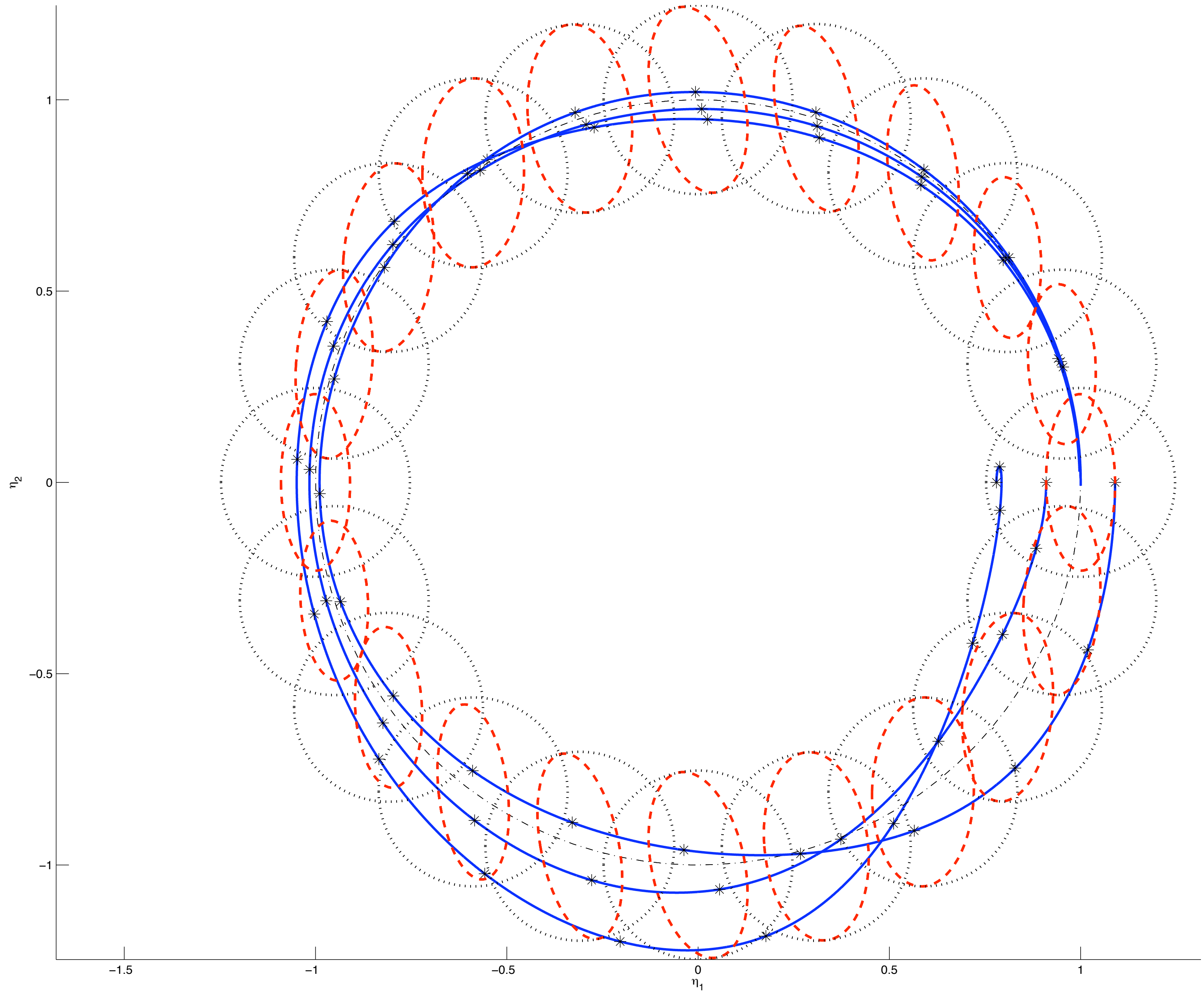
$\beta = 1.5$ (assumed bound on state growth over one sample)

and various values of q: feasible for all $q \leq q_{max} = 0.056$.

When taking $\beta = 1.1$ then $q_{max} = 0.061$.

Computation time (YALMIP+SeDuMi, 3GHz+1GB) $\simeq 1.4$sec.

Simulations for $\eta_1(0) = 0.78$, 0.91 and 1.09



Conclusions

➙ Methodology to analyze robust stability of periodic trajectories

➙ HYP 3 on bounded growth of the state embeds errors due to sampling

↗ LMI type results provide access to efficient numerical tools

↘ Results are conservative

↗ State-feedback design and others may be considered as well

↘ Difficulties to describe the sets Ξ via LMIs.