

**Affine and LFT modeling of systems with uncertainty:
some benefits of descriptor forms**

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■ Outline

- ① Motivations for uncertain descriptor modeling
- ② Affine polytopic models
- ③ LFT models - frequency dependent models
- ④ Augmented descriptor models and conservatism reduction

1 Motivations for uncertain descriptor modeling

- Models issued from physics are naturally in descriptor form

$$E\dot{x} = Ax + Bu$$

- Example: mechanical system

$$M\ddot{\theta} + C\dot{\theta} + K\theta = u$$

M : inertia ; C : friction ; K : stiffness ; u external forces

$$\begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \begin{pmatrix} \ddot{\theta} \\ \dot{\theta} \end{pmatrix} = \begin{bmatrix} -C & -K \\ I & 0 \end{bmatrix} \begin{pmatrix} \dot{\theta} \\ \theta \end{pmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} u$$

- Example: robotic systems with Lagrange formulations [MG89]

E may not be invertible

- Example: networks of systems with algebraic constraints describing links

1 Motivations for uncertain descriptor modeling

■ Descriptor models can be converted to usual models

● Example: mechanical system

$$\ddot{\theta} + M^{-1}C\dot{\theta} + M^{-1}K\theta = M^{-1}u$$

● Assumes that M^{-1} is well-conditioned and known

● If some parameters are unknown: $M(\Delta)$, $C(\Delta)$, $K(\Delta)$

$$\ddot{\theta} + M^{-1}(\Delta)C(\Delta)\dot{\theta} + M^{-1}(\Delta)K(\Delta)\theta = M^{-1}(\Delta)u$$

● Increased complexity of the model

■ Descriptor models are preferable for describing systems with uncertainty

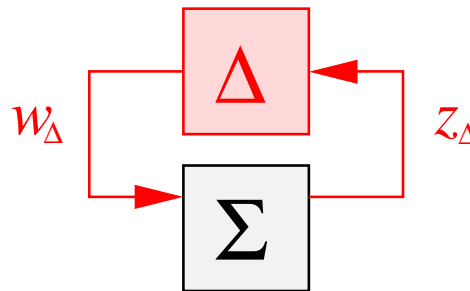
1 Motivations for uncertain descriptor modeling

■ Example: DC motor $I\dot{\omega} = bu$ regulated in speed $u = -\omega$

● Parameters are assumed uncertain $I = 1 + \delta_1, b = 1 + \delta_2$

$$(1 + \delta_1)\dot{\omega} = -(1 + \delta_2)\omega \Rightarrow \dot{\omega} = -\frac{1 + \delta_2}{1 + \delta_1}\omega$$

● Model is rational w.r.t. uncertainties: exists an LFT representation



$$-\frac{1 + \delta_2}{1 + \delta_1} = A + B_{\Delta}\Delta(I - D_{\Delta}\Delta)^{-1}C_{\Delta} = \begin{bmatrix} A & B_{\Delta} \\ C_{\Delta} & D_{\Delta} \end{bmatrix} \star \Delta$$

▲ Can be build with Robust Control toolbox of Matlab or LFRT [Mag05]

1 Motivations for uncertain descriptor modeling

● Building the LFT for $\dot{\omega} = -\frac{1+\delta_2}{1+\delta_1}\omega$

▲ 1st step: descriptor form with no denominators

$$\dot{\omega} + \delta_1 \dot{\omega} = -\omega - \delta_2 \omega$$

▲ 2nd step: all multiplications correspond to a feedback

$$\dot{\omega} + w_1 = -\omega - w_2 \quad : \quad \begin{array}{l} w_1 = \delta_1 z_1 \\ z_1 = \dot{\omega} \end{array}, \quad \begin{array}{l} w_1 = \delta_2 z_2 \\ z_2 = \omega \end{array}$$

▲ 3rd step: descriptor LFT: $\Delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}$, $z_\Delta = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, $w_\Delta = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$,

$$\left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline -1 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \begin{pmatrix} \dot{\omega} \\ z_\Delta \end{pmatrix} = \left[\begin{array}{c|cc} -1 & -1 & -1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 0 \end{array} \right] \begin{pmatrix} \omega \\ w_\Delta \end{pmatrix}, \quad w_\Delta = \Delta z_\Delta$$

▲ last step: invert the left-hand side matrix

1 Motivations for uncertain descriptor modeling

● LFT for $\dot{\omega} = -\frac{1+\delta_2}{1+\delta_1}\omega$

$$\begin{aligned}
 -\frac{1+\delta_2}{1+\delta_1} &= \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline -1 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]^{-1} \left[\begin{array}{c|cc} -1 & -1 & -1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 0 \end{array} \right] * \left[\begin{array}{cc} \delta_1 & 0 \\ 0 & \delta_2 \end{array} \right] \\
 &= \left[\begin{array}{c|cc} -1 & -1 & -1 \\ \hline -1 & -1 & -1 \\ 1 & 0 & 0 \end{array} \right] * \left[\begin{array}{cc} \delta_1 & 0 \\ 0 & \delta_2 \end{array} \right] \\
 &= -1 + \left[\begin{array}{cc} -1 & -1 \end{array} \right] \left[\begin{array}{cc} \delta_1 & 0 \\ 0 & \delta_2 \end{array} \right] \left(\mathbb{I} - \left[\begin{array}{cc} -1 & -1 \\ 0 & 0 \end{array} \right] \left[\begin{array}{cc} \delta_1 & 0 \\ 0 & \delta_2 \end{array} \right] \right)^{-1} \left[\begin{array}{c} -1 \\ 1 \end{array} \right] \\
 &= -1 - \left[\begin{array}{cc} \delta_1 & \delta_2 \end{array} \right] \left[\begin{array}{cc} 1 + \delta_1 & \delta_2 \\ 0 & 1 \end{array} \right]^{-1} \left[\begin{array}{c} -1 \\ 1 \end{array} \right] \\
 &= -1 - \left[\begin{array}{cc} \delta_1 & \delta_2 \end{array} \right] \left[\begin{array}{c} -\frac{1+\delta_2}{1+\delta_1} \\ 1 \end{array} \right] = -1 + \frac{\delta_1 - \delta_2}{1 + \delta_1} = -\frac{1 + \delta_2}{1 + \delta_1}
 \end{aligned}$$

■ The example shows the interest of descriptor models,
even if only for technical manipulations

1 Motivations for uncertain descriptor modeling

■ All fractional models have affine descriptor representations [MAS03]

▲ Proof: All fractional models can be converted to an LFT

$$\begin{pmatrix} \dot{x} \\ z_{\Delta} \end{pmatrix} = \begin{bmatrix} A & B_{\Delta} \\ C_{\Delta} & D_{\Delta} \end{bmatrix} \begin{pmatrix} x \\ w_{\Delta} \end{pmatrix}, \quad w_{\Delta} = \Delta z_{\Delta}$$

▲ the LFT gives the affine descriptor form:

$$\begin{bmatrix} I & -B_{\Delta}\Delta \\ 0 & I - D_{\Delta}\Delta \end{bmatrix} \begin{pmatrix} \dot{x} \\ z_{\Delta} \end{pmatrix} = \begin{bmatrix} A \\ C_{\Delta} \end{bmatrix} x$$

● Can give representations of smaller dimensions

▲ Example

$$\begin{bmatrix} 1 & \delta_1 & \delta_2 \\ 0 & 1 + \delta_1 & \delta_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \dot{\omega} \\ z_1 \\ z_2 \end{pmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \omega \Leftrightarrow (1 + \delta_1)\dot{\omega} = -(1 + \delta_2)\omega$$

■ General descriptor models

$$E_{xx}\dot{x} + E_{x\pi}\pi = Ax + Bu$$

$$y + E_{xy}\dot{x} + E_{y\pi}\pi = Cx + Du$$

- x : state ; u : inputs
- π : linearly constrained signals
- E_{xx} and A may not be square

▲ Can be converted to $\hat{E}\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u$ with \hat{E} and \hat{A} square and $\dot{\hat{x}} = \begin{pmatrix} \dot{x} \\ \pi \\ \lambda \end{pmatrix}$

▲ Not recommend: increased size of the model

- ① Motivations for uncertain descriptor modeling
- ② Affine polytopic models
- ③ LFT models - frequency dependent models
- ④ Augmented descriptor models and conservatism reduction

2 Affine polytopic models

- All fractional models have affine descriptor representations

$$E_{xx}(\Delta)\dot{x} + E_{x\pi}(\Delta)\pi = A(\Delta)x + B(\Delta)u$$

$$y + E_{xy}(\Delta)\dot{x} + E_{y\pi}(\Delta)\pi = C(\Delta)x + D(\Delta)u$$

- Models also used for polynomial non-linear models [CTF02]

- General affine descriptor data

$$\begin{bmatrix} -E_{xx}(\Delta) & -E_{x\pi}(\Delta) & A(\Delta) & B(\Delta) \\ -E_{yx}(\Delta) & -E_{y\pi}(\Delta) & C(\Delta) & D(\Delta) \end{bmatrix} = M(\Delta)$$

- Different classes of affine models

▲ intervals: $\underline{M} \preceq M(\Delta) \preceq \overline{M}$ (element-wise $\underline{m}_{ij} \leq m_{ij}(\Delta) \leq \overline{m}_{ij}$)

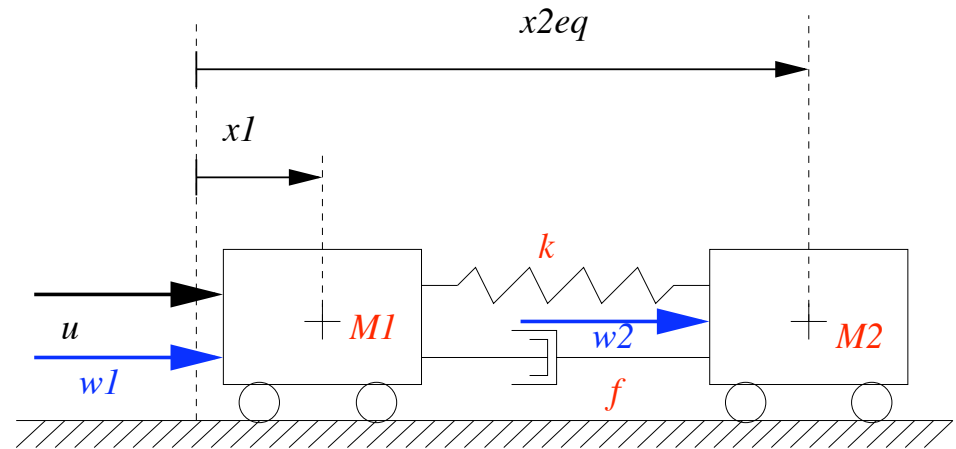
▲ parallelotopes: $M(\Delta) = M^{|0|} + \sum_{p=1}^{\bar{p}} \delta_p M^{|p|} \quad : \quad |\delta_p| \leq 1$

▲ polytopes: $M(\Delta) \in \text{co} \{ M^{[v=1 \dots \bar{v}]} \}$

- intervals \subset parallelotopes \subset polytopes

② Affine polytopic models

■ Example: two mass spring system



$$M_1 \ddot{x}_1 + f(\dot{x}_1 - \dot{x}_2) + k(x_1 - x_2) = u + w_1$$

$$M_2 \ddot{x}_2 + f(\dot{x}_2 - \dot{x}_1) + k(x_2 - x_1) = w_2$$

● General affine data model

$$\underbrace{\begin{bmatrix} M_1 & 0 & f & -f & k & -k \\ 0 & M_2 & -f & f & -k & k \end{bmatrix}}_{=M(\Delta)} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \dot{x}_1 \\ \dot{x}_2 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} u + w_1 \\ w_2 \end{pmatrix}$$

2 Affine polytopic models

■ Example: two mass spring system - continued

● 4 uncertain parameters $\underline{M}_i \leq M_i \leq \overline{M}_i$, $\underline{f} \leq f \leq \overline{f}$, $\underline{k} \leq k \leq \overline{k}$

$$M(\Delta) = \begin{bmatrix} M_1 & 0 & f & -f & k & -k \\ 0 & M_2 & -f & f & -k & k \end{bmatrix}$$

● Interval model generates conservatism (elements assumed independent)

$$\begin{bmatrix} \underline{M}_1 & 0 & \underline{f} & -\underline{f} & \underline{k} & -\underline{k} \\ 0 & \underline{M}_2 & -\underline{f} & \underline{f} & -\underline{k} & \underline{k} \end{bmatrix} \begin{matrix} \supset \\ \supset \end{matrix} \begin{bmatrix} \underline{M}_1 & 0 & \overline{f} & -\overline{f} & \overline{k} & -\overline{k} \\ 0 & \underline{M}_2 & -\underline{f} & \overline{f} & -\underline{k} & \overline{k} \end{bmatrix} \begin{matrix} \supset \\ \supset \end{matrix} \begin{bmatrix} \overline{M}_1 & 0 & \overline{f} & -\overline{f} & \overline{k} & -\overline{k} \\ 0 & \overline{M}_2 & -\overline{f} & \overline{f} & -\overline{k} & \overline{k} \end{bmatrix}$$

■ Example: two mass spring system - continued

$$M(\Delta) = \begin{bmatrix} M_1 & 0 & f & -f & k & -k \\ 0 & M_2 & -f & f & -k & k \end{bmatrix}$$

● Parallelotopic model

▲ Nominal model: $M^{|0|}$ (center of the intervals)

$$M(\Delta) = M^{|0|} + \delta_{M_1} \frac{1}{2} (\overline{M_1} - \underline{M_1}) \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & \vdots & \\ & & & & & \end{bmatrix} + \delta_k \frac{1}{2} (\overline{k} - \underline{k}) \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}}_{M^{|k|}}$$

▲ $M^{|k|}$ "axis" of variations along δ_k .

2 Affine polytopic models

■ Example: two mass spring system - continued

$$M(\Delta) = \begin{bmatrix} M_1 & 0 & f & -f & k & -k \\ 0 & M_2 & -f & f & -k & k \end{bmatrix}$$

● Polytopic model: $2^4 = 16$ vertices ($\delta_i = \pm 1$ in parallelotopic model)

$$\begin{aligned} M^{[1]} &= \begin{bmatrix} \overline{M}_1 & 0 & \overline{f} & -\overline{f} & \overline{k} & -\overline{k} \\ 0 & \overline{M}_2 & -\overline{f} & \overline{f} & -\overline{k} & \overline{k} \end{bmatrix} \\ M^{[2]} &= \begin{bmatrix} \underline{M}_1 & 0 & \underline{f} & -\underline{f} & \underline{k} & -\underline{k} \\ 0 & \underline{M}_2 & -\underline{f} & \underline{f} & -\underline{k} & \underline{k} \end{bmatrix} \\ &\quad \vdots \\ M^{[16]} &= \begin{bmatrix} \underline{M}_1 & 0 & \underline{f} & -\underline{f} & \underline{k} & -\underline{k} \\ 0 & \underline{M}_2 & -\underline{f} & \underline{f} & -\underline{k} & \underline{k} \end{bmatrix} \end{aligned}$$

- Some techniques to deal with affine uncertain models
- General type of analysis criteria

$$\eta^T \Theta(\Delta) \eta < 0 \quad , \quad \forall \eta \neq 0 \quad : \quad M(\Delta) \eta = 0$$

- ▲ Example: negative definite derivative of Lyapunov function $V = x^T P(\Delta) x$

$$\dot{V} = \eta^T \begin{bmatrix} 0 & P(\Delta) \\ P(\Delta) & 0 \end{bmatrix} \eta < 0, \quad \forall \eta = \begin{pmatrix} \dot{x} \\ x \end{pmatrix} \neq 0 : \quad \begin{bmatrix} -I & A(\Delta) \end{bmatrix} \eta = 0$$

- Usual LMI type result $(M^\perp(\Delta))$ matrix generating the null-space of $M(\Delta)$

$$M^{\perp T}(\Delta) \Theta(\Delta) M^\perp(\Delta) < 0$$

- ▲ Problem: $M^\perp(\Delta)$ is a rational function of Δ

- ▲ Solutions: SOS [HG05, Sch06], Polyá [CGTV09, OdOP08]

2 Affine polytopic models

- Some techniques to deal with affine uncertain models
- General type of analysis criteria

$$\eta^T \Theta(\Delta) \eta < 0 \quad , \quad \forall \eta \neq 0 \quad : \quad M(\Delta) \eta = 0$$

- Usual LMI type result $(M^\perp(\Delta))$ matrix generating the null-space of $M(\Delta)$

$$M^{\perp T}(\Delta) \Theta(\Delta) M^\perp(\Delta) < 0$$

- Slack variables results (variables issued from the "creation" Finsler lemma)

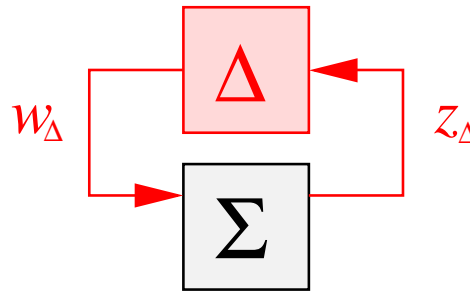
$$\exists F \quad : \quad \Theta(\Delta) < FM(\Delta) + M^T(\Delta)F^T$$

- ▲ Conservative, but sufficient to test at the vertices
- ▲ $\Theta(\Delta)$ affine (parameter-dependent Lyapunov function)
- ▲ [OBG99, PABB00, EH04, PDSV09]

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③ LFT models

- Affine polytopic models well adapted for scalar parametric uncertainty
- LFT suitable for other types of uncertainties:
 - Frequency dependent uncertainties
 - Linear / non-linear operators
 - Time-varying parameters
 - ...



- Linear combination of repeated uncertainties [Sch06]

$$\Delta = \sum J_j (\mathbf{I}_{r_j} \otimes \Delta_j) K_j, \quad [J_1 \cdots J_j] \text{ orthonormal}$$

▲ Example: bloc-diagonal uncertainties $\Delta = \begin{bmatrix} \Delta_1 & & \\ & \delta_2 & \\ & & \delta_2 \end{bmatrix}$

■ Dissipative structured parametric uncertainties

- An uncertain matrix Δ is said to be $\{\Psi_1, \Psi_2, \Psi_3\}$ -dissipative if

$$\begin{bmatrix} I & \Delta^* \end{bmatrix} \begin{bmatrix} \Psi_1 & \Psi_2 \\ \Psi_2^* & \Psi_3 \end{bmatrix} \begin{bmatrix} I \\ \Delta \end{bmatrix} \leq 0.$$

- An uncertain matrix Δ is said to be $\{\Phi_1, \Phi_2, \Phi_3\}$ -structured if

$$\begin{bmatrix} I & \Delta^* \end{bmatrix} \begin{bmatrix} \Phi_1 & \Phi_2 \\ \Phi_2^* & \Phi_3 \end{bmatrix} \begin{bmatrix} I \\ \Delta \end{bmatrix} = 0.$$

● Examples of $\{\Psi_1, \Psi_2, \Psi_3\}$ -dissipative uncertainties

▲ Norm-bounded uncertainties are $\{-\gamma^2 I, 0, I\}$ -dissipative

$$\begin{bmatrix} I & \Delta^* \\ & \end{bmatrix} \begin{bmatrix} -\gamma^2 I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I \\ \Delta \end{bmatrix} = -\gamma^2 I + \Delta^* \Delta \leq 0.$$

$$w_\Delta = \Delta z_\Delta \Rightarrow \|w_\Delta\| \leq \gamma \|z_\Delta\|$$

▲ Positive real uncertainties are $\{0, -I, 0\}$ -dissipative

$$\begin{bmatrix} I & \Delta^* \\ & \end{bmatrix} \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix} \begin{bmatrix} I \\ \Delta \end{bmatrix} = -(\Delta^* + \Delta) \leq 0.$$

$$w_\Delta = \Delta z_\Delta \Rightarrow \langle w_\Delta | z_\Delta \rangle \geq 0$$

● Examples of $\{\Psi_1, \Psi_2, \Psi_3\}$ -dissipative uncertainties - continued

▲ Uncertainties in an "ellipsoid" are $\{\Delta_0^* Z \Delta_0 - R, -\Delta_0^* Z, Z\}$ -dissipative

$$\begin{bmatrix} 1 & \Delta^* \\ & \Delta \end{bmatrix} \begin{bmatrix} \Delta_0^* Z \Delta_0 - R & -\Delta_0^* Z \\ -Z \Delta_0 & Z \end{bmatrix} \begin{bmatrix} 1 \\ \Delta \end{bmatrix} = (\Delta - \Delta_0)^* Z (\Delta - \Delta_0) - R \leq 0.$$

(discs of the complex plane for scalar $\Delta = \delta$)

▲ Uncertainties in a "half space" $\{-\Delta_0^* N + N \Delta_0, N^*, 0\}$ -dissipative

$$\begin{bmatrix} 1 & \Delta^* \\ & \Delta \end{bmatrix} \begin{bmatrix} -\Delta_0^* N + N \Delta_0 & N^* \\ N & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \Delta \end{bmatrix} = (\Delta - \Delta_0)^* N + N^* (\Delta - \Delta_0) \leq 0.$$

(half of the complex plane limited by a line for scalar $\Delta = \delta$)

● $\{\Phi_1, \Phi_2, \Phi_3\}$ -structured uncertainties: border of "ellipsoids" or of "half spaces"

③ LFT models

● $\{\Psi_1, \Psi_2, \Psi_3\}$ -dissipative $\{\Phi_1, \Phi_2, \Phi_3\}$ -structured, examples [IH05]

▲ $\delta = (j\omega)^{-1}$ with bounded frequencies $\omega \in [\underline{\omega}, \bar{\omega}]$:

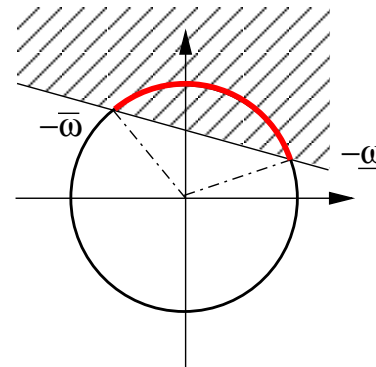
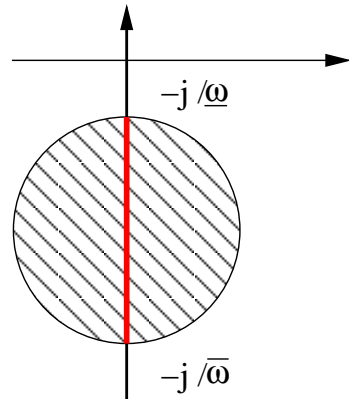
$\{2, -j(\underline{\omega} + \bar{\omega}), 2\underline{\omega}\bar{\omega}\}$ -dissipative $\{0, 1, 0\}$ -structured

▲ $\delta = (j\omega)^{-1}$ with high frequencies $\omega \geq \underline{\omega} \geq 0$:

$\{0, -j\underline{\omega}, 2\underline{\omega}^2\}$ -dissipative $\{0, 1, 0\}$ -structured

▲ $\delta = e^{-j\omega}$ with bounded frequencies $\omega \in [\underline{\omega}, \bar{\omega}]$:

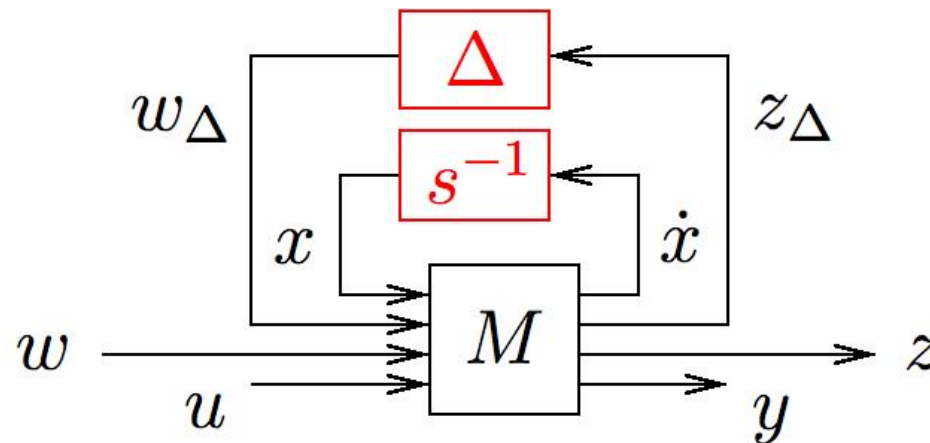
$\{2 \cos \frac{\bar{\omega} - \underline{\omega}}{2}, -e^{j\frac{\bar{\omega} + \underline{\omega}}{2}}, 0\}$ -dissipative $\{-1, 0, 1\}$ -structured



▲ $\delta \in \mathbb{R}$ bounded $\delta \in [\underline{\delta}, \bar{\delta}]$:

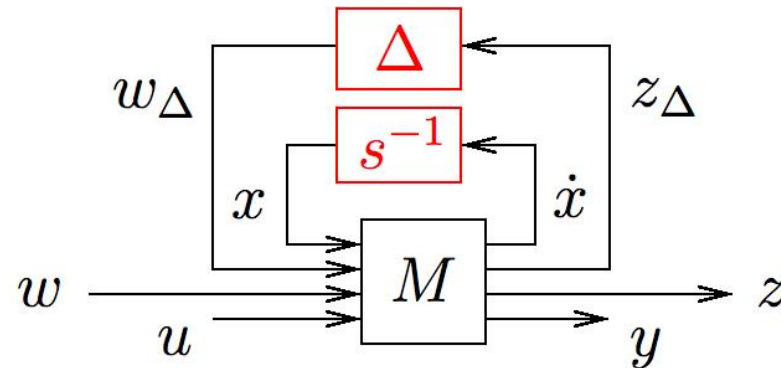
$\{2\underline{\delta}\bar{\delta}, -(\underline{\delta} + \bar{\delta}), 2\}$ -dissipative $\{0, j, 0\}$ -structured

- System analysis in frequency intervals: $s^{-1} = (j\omega)^{-1}$, $\omega \in [\underline{\omega} \ \bar{\omega}]$



- Allows loop-shaping type specifications
- Have interpretations in time domain [IHF05]

③ LFT models



■ Performances?

● Stability: no poles in \mathbb{C}_+ \Leftrightarrow well-posedness w.r.t. $s^{-1} \in \mathbb{C}_+$

● L_2 perf: $\|z\|_2 \leq \gamma \|w\|_2 \Leftrightarrow$ well-posedness w.r.t. $w = \nabla z : \|\nabla\|_2 \leq \frac{1}{\gamma}$

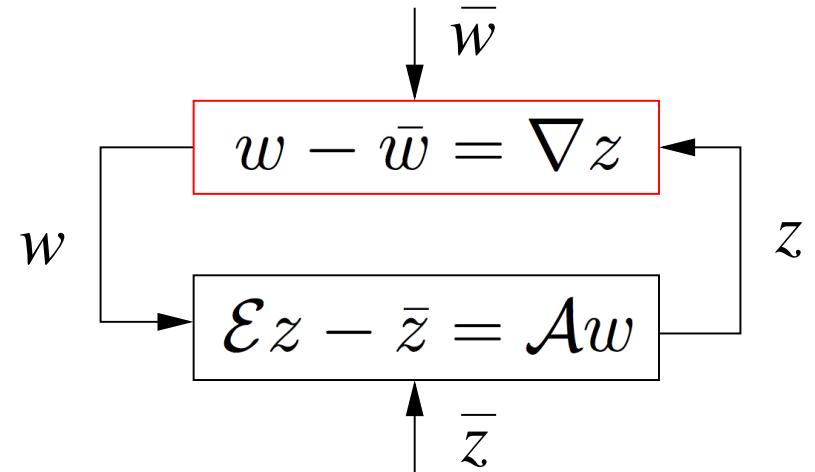
▲ Can be tested for given frequencies or for frequency intervals

● Dissipativity: $\int_0^\infty \begin{pmatrix} z(t) \\ w(t) \end{pmatrix}^* \begin{bmatrix} \Psi_1 & \Psi_2 \\ \Psi_2^* & \Psi_3 \end{bmatrix} \begin{pmatrix} z(t) \\ w(t) \end{pmatrix} dt \geq 0$

\Leftrightarrow well-posedness w.r.t. $w = \nabla z : \nabla \{\Psi_1, \Psi_2, \Psi_3\}$ -dissipative

■ Performances \Leftrightarrow well-posedness w.r.t. uncertain operators

■ Well-posedness [Saf80]



● For all bounded external signals (\bar{w}, \bar{z})

the internal signals $(w, \mathcal{E}z)$ are unique and bounded

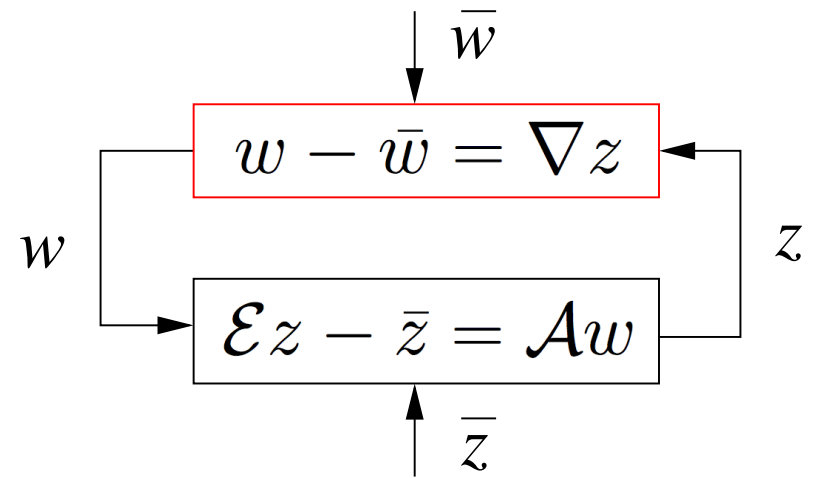
▲ Example: $z = \dot{x}$, $w = \nabla \dot{x} = x$ (i.e. ∇ integrator).

Well posedness \Leftrightarrow

\forall initial conditions $\bar{w} = x(0)$ and bounded disturbances \bar{z}

x is unique, no impulsive modes, no divergence (stability)

■ Well-posedness [PAHG07]



● Integral quadratic separation (IQS) result

▲ Assume $\mathcal{E} = \mathcal{E}_1 \mathcal{E}_2$ with \mathcal{E}_1 full column rank and let $M = \begin{bmatrix} -\mathcal{E}_1 & \mathcal{A} \end{bmatrix}$

▲ The loop is well-posed **iff** there exists Θ such that

$$\eta^T \Theta \eta < 0 \quad , \quad \forall \eta \neq 0 \quad : \quad M \eta = 0 \quad (1)$$

and for all “uncertainties”

$$\int_0^\infty \begin{pmatrix} \mathcal{E}_2 z(t) \\ [\nabla z](t) \end{pmatrix}^* \Theta \begin{pmatrix} \mathcal{E}_2 z(t) \\ [\nabla z](t) \end{pmatrix} dt \geq 0 \quad (2)$$

- IQS result

- ▲ (1): LMI-type constraint ($M^{\perp T} \Theta M^{\perp} < 0$)

- ▲ (2): Integral quadratic constraint (IQC)

- IQS applies easily to uncertainties defined, or included in IQCs

- ▲ Example: $\nabla = I_r \otimes \Delta$,

Δ is $\{\Psi_1, \Psi_2, \Psi_3\}$ -dissipative $\{\Phi_1, \Phi_2, \Phi_3\}$ -structured

$$\Theta = \begin{bmatrix} P \otimes \Psi_1 & P \otimes \Psi_2 \\ P \otimes \Psi_2^* & P \otimes \Psi_3 \end{bmatrix} + \begin{bmatrix} R \otimes \Psi_1 & R \otimes \Psi_2 \\ R \otimes \Psi_2^* & R \otimes \Psi_3 \end{bmatrix} \quad \begin{array}{l} P > 0 \\ R = R^* \end{array}$$

(generalized version of D/G-scaling)

■ IQC also defined in the frequency domain (Parseval)

$$\int_0^\infty \begin{pmatrix} \mathcal{E}_2 z(j\omega) \\ [\nabla z](j\omega) \end{pmatrix}^* \ominus \begin{pmatrix} \mathcal{E}_2 z(j\omega) \\ [\nabla z](j\omega) \end{pmatrix} d\omega \geq 0 \quad (3)$$

● IQC formulas [MR97, JM99] apply for building IQS results

● ∇ can contain

▲ Saturations [FLR06, PTGSB12]

▲ Delays [JS01, GP07, PAHG07]

▲ ... to be continued

■ Parametrizing \ominus solutions to the IQC is in general conservative [MSF97, IH05]

▲ Lossless formulas **iff** $\Delta = \sum_{j=1}^{\bar{j}} J_j (I_{r_j} \otimes \Delta_j) K_j \begin{cases} \bar{j} = 2, r_1 > 1, r_2 = 1 \\ \bar{j} = 3, r_1 = r_2 = r_3 = 1 \end{cases}$

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4 Augmented descriptor models and conservatism reduction

- Except in special cases finding Θ solution to the IQCs is conservative
- Existing parameterizations of Θ apply both for constant and time-varying Δ
- But $\Delta_\infty \subset \Delta_d \subset \Delta_0$ where
 - ▲ Δ_0 set of constant uncertainties such that $\Sigma(\Delta)$ is stable
 - ▲ Δ_d set of uncertainties with $\|\dot{\Delta}\| \leq d$ such that $\Sigma(\Delta)$ is stable
 - ▲ Δ_∞ set of TV uncertainties with switches such that $\Sigma(\Delta)$ is stable
- Reducing conservatism: can be achieved by introducing information about $\dot{\Delta}$

4 Augmented descriptor models and conservatism reduction

■ Descriptor augmentation method - example

● Assume system in LFT form

$$\begin{cases} \dot{x} = Ax + Bw_{\Delta} \\ z_{\Delta} = Cx + Dw_{\Delta} \end{cases} \star \begin{cases} x = s^{-1}\dot{x} \\ w_{\Delta} = \Delta z_{\Delta} \end{cases}$$

● Add information about derivative of uncertainty:

$$\dot{w}_{\Delta} = \underbrace{\Delta \dot{z}_{\Delta}}_{=w_{\Delta 1}} + \underbrace{\dot{\Delta} z_{\Delta}}_{=w_{\Delta 2}}$$

● Define derivative of exogenous signals:

$$\begin{aligned} \dot{z}_{\Delta} &= C\dot{x} + D\dot{w}_{\Delta} \\ &= C\dot{x} + Dw_{\Delta 1} + Dw_{\Delta 2} \end{aligned}, \quad w_{\Delta} = s^{-1}\dot{w}_{\Delta}$$

■ Descriptor augmentation method - example - continued

$$\begin{cases} \dot{x} = Ax + Bw_{\Delta} \\ z_{\Delta} = Cx + Dw_{\Delta} \end{cases} \star \begin{cases} x = s^{-1}\dot{x} \\ w_{\Delta} = \Delta z_{\Delta} \end{cases}$$

● All equations gathered separating linear constrains and "uncertainties"

$$\begin{cases} \dot{x} = Ax + Bw_{\Delta} \\ z_{\Delta} = Cx + Dw_{\Delta} \\ \dot{z}_{\Delta} - C\dot{x} = Dw_{\Delta 1} + Dw_{\Delta 2} \\ \dot{w}_{\Delta} = w_{\Delta 1} + w_{\Delta 2} \\ 0 = w_{\Delta} - w_{\Delta} \end{cases} \star \begin{cases} x = s^{-1}\dot{x} \\ w_{\Delta} = s^{-1}\dot{w}_{\Delta} \\ w_{\Delta} = \Delta z_{\Delta} \\ w_{\Delta 1} = \Delta \dot{z}_{\Delta} \\ w_{\Delta 2} = \dot{\Delta} z_{\Delta} \end{cases}$$

4 Augmented descriptor models and conservatism reduction

■ Descriptor augmentation method - example - continued

● Descriptor LFT form of system augmented with information on derivatives

$$z = \begin{pmatrix} \dot{x} & \dot{w}_\Delta & z_\Delta & \dot{z}_\Delta \end{pmatrix}^T, \quad w = \begin{pmatrix} x & w_\Delta & w_\Delta & w_{\Delta 1} & w_{\Delta 2} \end{pmatrix}^T$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -C & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} z = \begin{bmatrix} A & B & 0 & 0 & 0 \\ C & D & 0 & 0 & 0 \\ 0 & 0 & 0 & D & D \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix} w$$

$$\star w = \begin{bmatrix} s^{-1} I_{n+m} & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \\ 0 & \dot{\Delta} & 0 \end{bmatrix} z$$

4 Augmented descriptor models and conservatism reduction

- IQS results applicable to the original & the augmented descriptor model
- LMI formulas for augmented model are less conservative
- IQS formulas implicitly contain Lyapunov certificates
- ▲ Lyapunov certificate in case of LMI for original model

$$V = x^T P x$$

- ▲ Parameter-dependent Lyapunov certificate in case of augmented system

$$V = \begin{pmatrix} x \\ w_\Delta \end{pmatrix}^T P \begin{pmatrix} x \\ w_\Delta \end{pmatrix}, \quad w_\Delta = \Delta (I - D\Delta)^{-1} C x$$

- The system augmentation technique is also proved efficient for systems with delays [PAHG07], with saturations [PTGSB12]...

- Some advantages of descriptor models
 - Natural models from physics
 - More flexibility for system description
 - Affine descriptor models are alternatives to LFT models
 - Well-posedness of descriptor LFT loops
 - = general framework for robust analysis problems
 - (includes frequency dependent specifications/uncertainties)
 - System augmentation with derivatives = less conservative results
 - (made possible thanks to descriptor modeling)

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