Discussion on S-variable based static-output feedback design heuristics

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Toulouse - 13 March 2023

Static Output Feedback

Find a gain $K \in \mathbb{R}^{m \times p}$ such that $\dot{x} = (A + BKC)x$ is stable

- ➡ Without having an initial guess
  $\neq$ Knowing $A + BK_o C$ stable, find a better gain (with some criterion)
- ➡ Without having indications of a range of admissible values
  no possibility to test on a grid
- ➡ Robust w.r.t. uncertainties in A, B and C
  eg. matrices in a polytope
- ➡ Structured : eg. K diagonal (decentralized control)

Structured Static Output Feedback - a generic problem

Assume a linear plant $\dot{x} = A(K_1, \ldots, K_{\bar{k}})x$ rational in the design parameters $K_{k=1\ldots\bar{k}}$

- ➡ Linear plant : First step before considering nonlinear dynamics
- ➡ K_k : gains of a dynamic control, filter param., decentralized control ... or plant parameters
- ➡ Rational in the parameters : or in 'gains' with one-to-one NL maps to true design parameters

It can always be reformulated (Linear Fractional Transformation) as a feedback control loop

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \qquad u = Ky = \begin{bmatrix} I_{r_1} \otimes K_1 & & 0 \\ 0 & \ddots & 0 \\ 0 & & I_{r_{\bar{k}}} \otimes K_{\bar{k}} \end{bmatrix} y$$

where K is structured (block-diagonal), parameters may be repeated: $I_{r_k} \otimes K_k$.

- ➡ with $y := y - Du$ one may consider $D = 0$
- ➡ All these design problems look 'simple': find a (structured) K s.t. $A + BKC$ is Hurwitz

Optimization based approaches

Find a (structured) K s.t. $A + BKC$ is Hurwitz $\Leftarrow$ minimization of a non-linear, non-smooth function

- ➡ [Apkarian, Noll 2003] optimize an H_∞ gain (and more) - hinfstruct - Clark's sub-gradient
- ➡ [Overton et al. 2006] optimize an H_∞, H_2 gains, spectral abscissa - Hifoo - gradient sampling
- ➡ [Peretz 2013] optimize the spectral abscissa - randomized approximation

▲ ▲ Very efficient in practice

▼ Randomized flavour (random initial conditions, randomization in the algorithm)

▼ No extensions to robust design

▲ Lyapunov based approches with matrix inequalities may handle robustness

$$P \succ 0 \quad , \quad (A + BKC)^* P + P(A + BKC) = \{P(A + BKC)\}^{\mathcal{H}} \prec 0$$

$$Q \succ 0 \quad , \quad \{(A + BKC)Q\}^{\mathcal{H}} \prec 0$$

▼ Bilinear matrix inequalities (not convex)

LMI cases

Find a (structured) K s.t. $A+BKC$ is Hurwitz is sometimes convex (up to a transformation)

- ➡ $B=C=I$ State-Injection : $\{PA+L\}^{\mathcal{H}} \prec 0$ gives $K=P^{-1}L$
- ➡ $B=I$ Output-Injection : $\{PA+LC\}^{\mathcal{H}} \prec 0$ gives $K=P^{-1}L$
- ➡ $C=I$ State-Feedback : $\{AQ+BF\}^{\mathcal{H}} \prec 0$ gives $K=FQ^{-1}$
- ➡ Almost commutative on the input : $\{PA+BLC\}^{\mathcal{H}} \prec 0,\ PB=B\hat{P}$ gives $K=\hat{P}^{-1}L$
- ➡ Almost commutative on the output : $\{AQ+BFC\}^{\mathcal{H}} \prec 0,\ CQ=\hat{Q}C$ gives $K=F\hat{Q}^{-1}$

▼ Applies only to special cases

▲ Robustness can be considered
(eg. test on all vertices with common decision variables proves stability of the polytope)

▼ Structured K cannot be considered
unless one considers structured P and Q (very conservative)

Iterative LMI method

Simple P-K-iterative algorithm

Initialization	Choose a positive definite P
K-iteration	For fixed P find $K = \arg\min \alpha$ under $\{P(A+BKC)\}^{\mathcal{H}} \prec \alpha I$
P-iteration	For fixed K find $P = \arg\min \alpha$ under $\{P(A+BKC)\}^{\mathcal{H}} \prec \alpha I$
Stop	Repeat until $\alpha < 0$ (success) or varies too slowly (failure)

- ▲ Easy to implement
- ▲ Strictly decreasing sequence of α
- ▼ Very sensitive to initialization
- ▼ Little progress after very few steps
- ▼ Not effective in practice

Finsler (elimination) lemma

$$\exists S \ : \ M \prec \{X_1^* S X_2\}^{\mathcal{H}} \quad \Leftrightarrow \quad \left\{ \begin{array}{l} N_{X_1}^* M N_{X_1} \prec 0 \\ N_{X_2}^* M N_{X_2} \prec 0 \end{array} \right. \ : \ X N_X = 0 \ , \quad \mathrm{Rank}(N_X) = \dim(\mathrm{Ker}(X))$$

Applied to the SOF problem [Scherer, Iwasaki...] it gives

$$\exists K \ : \ \left\{ \begin{array}{l} \{P(A+BKC)\}^{\mathcal{H}} \prec 0 \\ \{(A+BKC)Q\}^{\mathcal{H}} \prec 0 \\ PQ = I \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{l} N_C^* \{PA\}^{\mathcal{H}} N_C \prec 0 \\ N_{B^*}^* \{AQ\}^{\mathcal{H}} N_{B^*} \prec 0 \\ PQ = I \end{array} \right.$$

- ▲ May be converted to pure LMIs (of small dimensions) ... ▼ with a rank constraint
- ▲ Many dedicated iterative algorithms
- ▼ Sensitive to initial guesses, not very effective in practice
- ▼ Cannot take into account structured K

Converse Finsler lemma : S-variable approach

$$\exists S \ : \ M \prec \{X_1^* S X_2\}^{\mathcal{H}} \quad \Leftrightarrow \quad \left\{ \begin{array}{l} N_{X_1}^* M N_{X_1} \prec 0 \\ N_{X_2}^* M N_{X_2} \prec 0 \end{array} \right. \quad : \quad X N_X = 0 \ \ \mathrm{Rank}(N_X) = \dim(\mathrm{Ker}(X))$$

Assume P proves stability for both an output-feedback gain K_{of} and a state-feedback gain K_{sf} (always true with $K_{sf} = K_{of} C$)

$$P(A + B K_{of} C)\}^{\mathcal{H}} = \begin{bmatrix} I \\ K_{of} C \end{bmatrix}^* \begin{bmatrix} \{PA\}^{\mathcal{H}} & PB \\ B^* P & 0 \end{bmatrix} \begin{bmatrix} I \\ K_{of} C \end{bmatrix} \prec 0$$

$$P(A + B K_{sf})\}^{\mathcal{H}} = \begin{bmatrix} I \\ K_{sf} \end{bmatrix}^* \begin{bmatrix} \{PA\}^{\mathcal{H}} & PB \\ B^* P & 0 \end{bmatrix} \begin{bmatrix} I \\ K_{sf} \end{bmatrix} \prec 0$$

Equivalent to the existence of S such that

$$\begin{bmatrix} \{PA\}^{\mathcal{H}} & PB \\ B^* P & 0 \end{bmatrix} \prec \left\{ \begin{bmatrix} K_{sf}^{\ *} \\ -I \end{bmatrix} S \begin{bmatrix} K_{of} C & -I \end{bmatrix} \right\}^{\mathcal{H}}$$

S-variable approach

$$\exists P \succ 0, S, K_{sf}, K_{of} \quad : \quad \begin{bmatrix} \{PA\}^{\mathcal{H}} & PB \\ B^* P & 0 \end{bmatrix} \prec \left\{ \begin{bmatrix} K_{sf}{}^* \\ -I \end{bmatrix} S \begin{bmatrix} K_{of} C & -I \end{bmatrix} \right\}^{\mathcal{H}}$$

- ▼ Still not LMI, matrix inequalities of larger size, more decision variables
- ▲ More degrees of freedom, P and K_{of} separated one for the other
- ▲ Simple to code K_{sf}-K_{of}-iterative algorithm

$$K_{sf}\text{-iteration} \quad K_{of} = S^{-1} L_{of} \quad \arg\min \alpha \quad : \quad \begin{bmatrix} \{PA\}^{\mathcal{H}} - \alpha I & PB \\ B^* P & 0 \end{bmatrix} \prec \left\{ \begin{bmatrix} K_{sf}{}^* \\ -I \end{bmatrix} \begin{bmatrix} L_{of} C & -S \end{bmatrix} \right\}^{\mathcal{H}}$$

$$K_{of}\text{-iteration} \quad K_{sf} = S^{-1} L_{sf} \quad \arg\min \alpha \quad : \quad \begin{bmatrix} \{PA\}^{\mathcal{H}} - \alpha I & PB \\ B^* P & 0 \end{bmatrix} \prec \left\{ \begin{bmatrix} L_{sf}{}^* \\ -S \end{bmatrix} \begin{bmatrix} K_{of} C & -I \end{bmatrix} \right\}^{\mathcal{H}}$$

- ▲ Smart initial guess of K_{sf} (finding K_{sf} is a convex problem)
- ▲ Strictly decreasing sequence of α
- ▲ Much more efficient than the P-K-iterative algorithm (P is free at each step)

  Implicitly the algorithm searches for $K_{sf} \rightarrow K_{of} C$.

S-variable approach - variant (dual)

Variant of the same result based on output-injection gain K_{oi}

$$\exists Q \succ 0, S, K_{oi}, K_{of} \quad : \quad \begin{bmatrix} \{AQ\}^{\mathcal{H}} & QC^* \\ CQ & 0 \end{bmatrix} \prec \left\{ \begin{bmatrix} BK_{of} \\ -I \end{bmatrix} S \begin{bmatrix} K_{oi}{}^* & -I \end{bmatrix} \right\}^{\mathcal{H}}$$

- ▲ Simple to code K_{oi}-K_{of}-iterative algorithm
- ▲ Smart initial guess of K_{oi} (finding K_{oi} is a convex problem)
- ▲ Strictly decreasing sequence of α
- ▲ Much more efficient than the P-K-iterative algorithm (P is free at each step)
  Implicitly the algorithm searches for $K_{oi} \rightarrow BK_{of}$.

- ▲ Robustness can be dealt with easily (eg. solve the constraints for all vertices of a polytope)
  - ▼ yet conservative (common Lyapunov certificate P or Q for all uncertainties)
- ▲ Structured SOF : achievable with constraints on S, not on P of Q
  - ▼ yet conservative

S-variable approach - variant (Lyapunov certificate)

Variant [Peres et al. 2020] assuming two Lyapunov certificates P_1 and P_2 for $A+BKC$

$$\exists P_1 \succ 0, P_2 \succ 0, S = S^*, K \quad : \quad \begin{bmatrix} 0 & (A+BKC)^* \\ A+BKC & 0 \end{bmatrix} \prec \left\{ \begin{bmatrix} P_2 \\ -I \end{bmatrix} S \begin{bmatrix} P_1 & -I \end{bmatrix} \right\}^{\mathcal{H}}$$

- ▲ Matrix inequalities of larger size
- ▲ Simple to code P_1-P_2-iterative algorithm
- ▼ No smart initial guess of P
- ▲ Robustness can be dealt with easily (eg. solve the constraints for all vertices of a polytope)
- ▲ No difficulty to include structure constraints on K
- ▼ Seems less efficient than the K_{sf}-K_{of}-iterative algorithm

S-variable approach - In Honor of Roberto Tempo

$$\begin{bmatrix} 0 & 0 & Q \\ 0 & 0 & 0 \\ Q & 0 & 0 \end{bmatrix} \prec \left\{ \begin{bmatrix} -I \\ LC+M \\ A \end{bmatrix} S_1 + \begin{bmatrix} 0 \\ -S_2 \\ BF \end{bmatrix} \begin{bmatrix} 0 & -I & H^* \end{bmatrix} \right\}^{\mathcal{H}}$$

- ➡ If $L=0$ then $K_{si}=HM$ is a stabilizing state-injection gain ($A+K_{si}$ is stable)
- ➡ If $L=0$ then $K_{sf}=FS_2^{-1}M$ is a stabilizing state-feedback gain ($A+BK_{sf}$ is stable)
- ➡ If $M=0$ then $K_{oi}=HL$ is a stabilizing output-injection gain ($A+K_{oi}C$ is stable)
- ➡ If $M=0$ then $K_{of}=FS_2^{-1}L$ is a stabilizing output-feedback gain ($A+BK_{of}C$ is stable)

- ▲ Stabilizing state-injection property : Good for initialization
- ▲ All matrices Q, A, B and C decoupled:
  OK for robustness with parameter-dependent Lyapunov certificates
- ▲ Results are new (and efficient) even for robust K_{sf} and K_{oi} design
- ▲ No need to structure the Lyapunov certificate for structured SOF
- ▼ Algorithm is less trivial than the previous ones (contains a line search)
- ➡ (S_1, K_{sf})-K_{si}-iterative algorithm shows to be efficient with low number of iterations
- ➡ During the algorithm $K_{si} \to K_{oi}C$ and $K_{sf} \to K_{of}C$

Conclusions

➡ Several results for SOF using the S-variable approach

▲ Rather efficient to deal with robust structured SOF

▼ No guarantee of success

▲ ▼ Results with more or less intuitive initialization

➡ Latest result - In Honor of Roberto Tempo

▲ Promising numerical experiments

▲ Elegant combination of the SI, OI, SF, OF problems

▲ Variations of the algorithms for

Deterministic robust design

Probabilistic robust design

With comparisons of the two

"Robust static output feedback design with deterministic and probabilistic certificates",
D. Arzelier, F. Dabbene, S. Formentin, D. Peaucelle, L. Zaccarian
Birkhäuser Mathematics - Springer Nature Chapter for "Uncertainty in Networked Systems"