# Robust Observed-State Feedback Design

# for Discrete-Time Systems Rational in the Uncertainties

Dimitri Peaucelle

Yoshio Ebihara & Yohei Hosoe

Workshop "Recent Topics in Control"

December 21, 2015 Katsura Campus, Kyoto University

Extends results from two papers presented at 19th IFAC World Congress (Cape Town).

Results submitted to Automatica: hal.archives-ouvertes.fr/hal-01225068v1

- Discrete-time linear system with uncertainties

$$x_{k+1} = A_r(\theta)x_k + B_r(\theta)u_k, \qquad y_k = Cx_k$$

- Luenberger-like observer $\hat{x}_{k+1} = A_o\hat{x}_k + B_o u_k + L(C\hat{x}_k - y_k)$
- Observed-state feedback $u_k = K\hat{x}_k$

- Our goals:
  - Build a separation-like heuristic with first, $K$ design, then, $A_o, B_o, L$ design
  - Use up-to-date SV-LMI tools
  - For systems rational in the uncertainties $\theta$

- Closed-loop dynamics (state $x$ and error $e = x - \hat{x}$) driven by the state matrix

$$\begin{pmatrix} x_{k+1} \\ e_{k+1} \end{pmatrix} = \begin{bmatrix} A_r(\theta) + B_r(\theta)K & -B_r(\theta)K \\ \Delta_A(\theta) + \Delta_B(\theta)K & A_o + LC - \Delta_B(\theta)K \end{bmatrix} \begin{pmatrix} x_k \\ e_k \end{pmatrix}$$

where $\Delta_A(\theta) = A_r(\theta) - A_o$ and $\Delta_B(\theta) = B_r(\theta) - B_o$ .

- Separation obtained when $\Delta_A(\theta) = 0$ and $\Delta_B(\theta) = 0$

$$\begin{pmatrix} x_{k+1} \\ e_{k+1} \end{pmatrix} = \begin{bmatrix} A_r(\theta) + B_r(\theta)K & -B_r(\theta)K \\ 0 & A_o + LC \end{bmatrix} \begin{pmatrix} x_k \\ e_k \end{pmatrix}$$

- Impossible when $\theta$ are uncertainties

(Notion of observed-state not quite defined for uncertain systems)

- Closed-loop dynamics (state $x$ and error $e = x - \hat{x}$) driven by the state matrix

$$\begin{pmatrix} x_{k+1} \\ e_{k+1} \end{pmatrix} = \begin{bmatrix} A_r(\theta) + B_r(\theta)K & -B_r(\theta)K \\ \Delta_A(\theta) + \Delta_B(\theta)K & A_o + LC - \Delta_B(\theta)K \end{bmatrix} \begin{pmatrix} x_k \\ e_k \end{pmatrix}$$

where $\Delta_A(\theta) = A_r(\theta) - A_o$ and $\Delta_B(\theta) = B_r(\theta) - B_o$ .

- Choices from the literature: $A_o = A_r(\theta_{nom})$, but why?
- Possible choice $\min_{A_o} \max_{\theta} \|A_r(\theta) - A_o\|$, but what properties?

- Our choice: optimize the input/output performances of

$$e_{k+1} = (A_o + LC - \Delta_B(\theta)K)e_k + (\Delta_A(\theta) + \Delta_B(\theta)K)x_k, \quad \epsilon_k = Ke_k$$

where $x_k$ is treated as the input and $\epsilon_k$ is the output.

1. Descriptor multi-affine modeling of rational systems
2. LMI results for robust design and robust analysis
3. Observed-state feedback design heuristic
4. Example

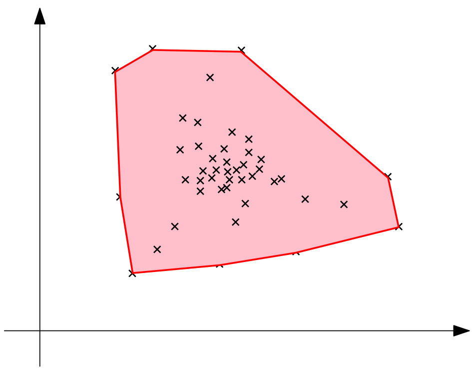
- $\bar{p}$ independent uncertain vectors $\theta_p \in \mathbb{R}^{m_p}$ indexed by $p = 1 \cdots \bar{p}$

$$\theta \in \Theta = \{(\theta_1, \ldots, \theta_{\bar{p}}) \in \Theta_1 \times \ldots \times \Theta_{\bar{p}}\}.$$

- Each $\theta_p$ in a polytope with $\bar{v}_p$ vertices $\mathcal{V}_p = \left\{\theta_p^{[1]}, \ldots, \theta_p^{[\bar{v}_p]}\right\}$

$$\Theta_p = Co(\mathcal{V}_p) = \left\{\theta_p = \sum_{v=1}^{\bar{v}_p} \xi_{p,v}\theta_p^{[v]} \ : \ \xi_{p,v} \geq 0, \ \sum_{v=1}^{\bar{v}_p} \xi_{p,v} = 1\right\}.$$

- Example: scalar uncertainty in an interval: $\theta_p \in [\,\theta_p^{[1]}, \ \theta_p^{[2]}\,]$.
- Example: 2D vector in convex hull of points issued from identification process



■ Multi-affine matrices: affine in each $\theta_p$

● Example for two scalar uncertainties $\theta_1 \in [\theta_1^{[1]},\ \theta_1^{[2]}]$, $\theta_2 \in [\theta_2^{[1]},\ \theta_2^{[2]}]$

$$1+\theta_1+\theta_1\theta_2 = \begin{array}{l} \xi_{1,1}\xi_{2,1}(1+\theta_1^{[1]}+\theta_1^{[1]}\theta_2^{[1]}) \\ +\xi_{1,1}\xi_{2,2}(1+\theta_1^{[1]}+\theta_1^{[1]}\theta_2^{[2]}) \\ +\xi_{1,2}\xi_{2,1}(1+\theta_1^{[2]}+\theta_1^{[2]}\theta_2^{[1]}) \\ +\xi_{1,2}\xi_{2,2}(1+\theta_1^{[2]}+\theta_1^{[2]}\theta_2^{[2]}). \end{array}$$

▲ Not the same as the convex hull of all possible vertices

● Example: $\left[\begin{array}{ccc} \theta_1 & \theta_1\theta_2 & \theta_2 \end{array}\right]$ with $\theta_1 \in [1,\ 2]$ and $\theta_2 \in [1,\ 2]$.

$$\frac{1}{2}\left[\begin{array}{ccc} 1 & 1 & 1 \end{array}\right]+\frac{1}{2}\left[\begin{array}{ccc} 2 & 4 & 2 \end{array}\right]=\left[\begin{array}{ccc} \frac{3}{2} & \frac{5}{2} & \frac{3}{2} \end{array}\right]\neq\left[\begin{array}{ccc} \frac{3}{2} & \frac{9}{4} & \frac{3}{2} \end{array}\right]$$

■ Any matrix rational in $\theta$ admits a descriptor multi-affine representation (DMAR)

$$R(\theta) = M_1(\theta) M_2^{-1}(\theta) M_3(\theta)$$

where $M_1(\theta)$, $M_2(\theta)$, $M_3(\theta)$ are multi-affine in $\theta$.

- Alternative to linear-fractional representations
- Usually of smaller size, and easier to build
- Example:

$$\begin{bmatrix} \frac{\theta_1}{1+\theta_2} & {\theta_1}^2\theta_2 \\ \frac{1}{\theta_1} & 0 \end{bmatrix} = \begin{bmatrix} \theta_1 & \theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1+\theta_2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \theta_1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \theta_1\theta_2 \\ 0 & 1 \end{bmatrix}.$$

- Discrete-time linear system, with performance I/O, rational in the uncertainties

$$\begin{aligned} x_{k+1} &= A_r(\theta)x_k + B_{rw}(\theta)w_k \\ z_k &= C_{rz}(\theta)x_k + D_{rzw}(\theta)w_k \end{aligned}$$

- The DMAR

$$\begin{bmatrix} A_r(\theta) & B_{rw}(\theta) \\ C_{rz}(\theta) & D_{rzw}(\theta) \end{bmatrix} = \begin{bmatrix} E_x(\theta) \\ E_z(\theta) \end{bmatrix} E_\pi^{-1}(\theta) \begin{bmatrix} A(\theta) & B_w(\theta) \end{bmatrix}$$

- gives the following descriptor multi-affine representation of the system

$$\begin{bmatrix} I & 0 & E_x(\theta) & 0 & 0 \\ 0 & I & E_z(\theta) & 0 & 0 \\ 0 & 0 & E_\pi(\theta) & A(\theta) & B_w(\theta) \end{bmatrix} \begin{pmatrix} x_{k+1} \\ z_k \\ \pi_k \\ x_k \\ w_k \end{pmatrix} = E(\theta)\eta_k = 0$$

- $\pi_k$: exogenous vector $= E_\pi^{-1}(\theta)(A(\theta)x_k + B_w(\theta)w_k)$

■ If there exists $P^{[v]} = P^{[v]T} \succ 0$, $S$ and $\mu^2$ such that for all vertices $\theta^{[v]} \in \mathcal{V}$

$$\text{diag}\left[\begin{array}{ccccc} P^{[v]} & I & 0 & -P^{[v]} & -\mu^2 I \end{array}\right] \prec (SE(\theta^{[v]})) + (SE(\theta^{[v]}))^T$$

then the system is robustly stable (i.e. $\forall\theta \in \Theta$) with robust $H_\infty$ performance $\mu$.

- Proof - step 1 - By convexity the condition holds for all $\theta \in \Theta$:

$$\text{diag}\left[\begin{array}{ccccc} P(\theta) & I & 0 & -P(\theta) & -\mu^2 I \end{array}\right] \prec (SE(\theta)) + (SE(\theta))^T$$

  with multi-affine Lyapunov matrix $P(\theta) \succ 0$.

- Proof - step 2 - Since $E(\theta)\eta_k = 0$ one gets

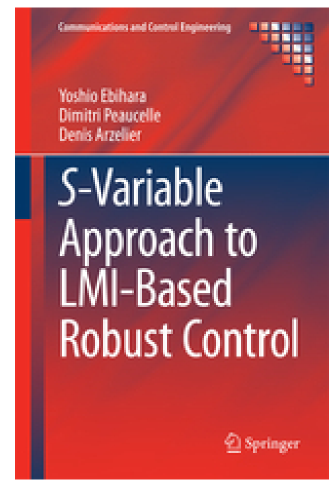
$$\eta_k^T \text{diag}\left[\begin{array}{ccccc} P(\theta) & I & 0 & -P(\theta) & -\mu^2 I \end{array}\right] \eta_k$$
$$= x_{k+1}^T P(\theta) x_{k+1} + z_k^T z_k - x_k^T P(\theta) x_k - \mu^2 w_k^T w_k < 0$$

■ If there exists $P^{[v]} = P^{[v]T} \succ 0$, $S$ and $\mu^2$ such that for all vertices $\theta^{[v]} \in \mathcal{V}$

$$\text{diag}\left[\begin{array}{ccccc} P^{[v]} & I & 0 & -P^{[v]} & -\mu^2 I \end{array}\right] \prec (SE(\theta^{[v]})) + (SE(\theta^{[v]}))^T$$

then the system is robustly stable (i.e. $\forall \theta \in \Theta$) with robust $H_\infty$ performance $\mu$.

- S-variable result
- Extended in present work to multi-affine representations
- Exist tools to reduce numerical burden (sometimes lossless)
- ⚠ Example: no $S$ if plant is multi-affine in $\theta$ & common $P = P(\theta)$
- Extensions to mixed constant/time-varying uncertainties



## SV-LMI for robust state-feedback design

If there exist $P_d^{[v]} \succ 0$, $S_{dx}$, $S_{dy}$, $S_{d\pi}$ such that LMIs $\mathcal{L}_{sf}(\theta^{[v]})$ hold for all $\theta^{[v]} \in \mathcal{V}$ then $K = {S_{dy}}^T({S_{dx}}^T)^{-1}$ is a robustly stabilizing state-feedback gain s.t.

$$\begin{array}{rcl} x_{k+1} & = & A_r(\theta)x_k + B_r(\theta)u_k + B_{rw}(\theta)w_k \\ z_k & = & C_{rz}(\theta)x_k + D_{rzu}(\theta)u_k + D_{rzw}(\theta)w_k \end{array} \quad , \quad u_k = K x_k$$

has an $H_\infty$ performance smaller than $\mu_d$ whatever $\theta \in \Theta$.

- Linearizing change of variables on $S$-variables
- Proof uses equivalence with dual system $x_{d,k+1} = A_r^T(\theta)x_{d,k} + \ldots$.
- Result is new because for rational systems
- Easy extensions for regional pole location, $H_2$ performance, etc.

**SV-LMI for analysis of state trajectories under fixed state-feedback $K = K$**

If there exist $P^{[v]} \succ 0$, $Q$ and $S$ such that LMIs $\mathcal{L}_{sf,a}(\theta^{[v]})$ hold for all $\theta^{[v]} \in \mathcal{V}$, then

$$x_{k+1} = A_r(\theta)x_k + B_r(\theta)u_k \ , \ u_k = Kx_k + \epsilon_k$$

is robustly stable and $x_k$ is bounded for bounded control errors $\epsilon_k$:

$\|Wx\|_2 \leq \|\epsilon\|_2$ where $W = Q^{1/2}$.

- Allow to estimate the state trajectories in case of corrupted state-feedback

(inevitable when feedback is with observed-state)

■ **SV-LMI for robust observer design under fixed $K = K$**

**and expected state trajectories $W = W$**

If there exist $P_\infty^{[v]} \succ 0$, $P_p^{[v]} \succeq K^T K$, $S_x$, $S_a$, $S_b$, $S_l$, $S_{2\pi}$, $S_{p\pi}$ such that LMIs $\mathcal{L}_{ob}(\theta^{[v]})$ hold for all $\theta^{[v]} \in \mathcal{V}$, then $A_o = S_x{}^{-1} S_a$, $B_o = S_x{}^{-1} S_b$, $L = S_x{}^{-1} S_l$ define an observer that guarantees:

$$\|\epsilon\|_2 \leq \gamma_2 \|Wx\|_2 \ , \ \|\epsilon\|_p \leq \gamma_p \|Wx\|_2$$

where $\epsilon_k = K e_k$. The properties hold whatever bounded $x$ and whatever $\theta \in \Theta$.

- Norm-to-norm perf: asymptotic coupling of observation error on system dynamics
- Norm-to-peak perf: avoid waterbed effects of transient peaks

- Small gain theorem: if $\gamma_2 < 1$ observed-state feedback robustly stabilizes

■ **SV-LMI for observed-state feedback analysis under fixed $K = K$, $A_o = A_o$ etc.**

If there exist $P_c^{[v]} \succ 0$, $S_c$ such that LMIs $\mathcal{L}_{ob,a}(\theta^{[v]})$ hold for all $\theta^{[v]} \in \mathcal{V}$, then

$$\begin{aligned} x_{k+1} &= A_r(\theta)x_k + B_r(\theta)u_k + B_{rw}(\theta)w_k \\ z_k &= C_{rz}(\theta)x_k + D_{rzu}(\theta)u_k + D_{rzw}(\theta)w_k \end{aligned}$$

$$\hat{x}_{k+1} = A_o\hat{x}_k + B_o u_k + L(C\hat{x}_k - y_k), \;\; u_k = K\hat{x}_k$$

has an $H_\infty$ performance smaller than $\mu_c$ whatever $\theta \in \Theta$.

- Exists also SV-LMI conditions $\mathcal{L}_{ob,da}$ for the dual system: $\mu_{dc}$ upper-bound
- No apriori relation between upper-bounds $\mu_d$ (ideal state-feedback), $\mu_c$ and $\mu_{dc}$

■ 4 steps

- 1- Design stabilizing state-feedback $K$ (for example using LMI $\mathcal{L}_{sf}$)
- 2- Get estimate of state trajectories represented by $W$ (using LMIs $\mathcal{L}_{sf,a}$)

  $(\max \lambda_{\min}(W^T W)$ leads to tight estimates)
- 3- Design observer (using LMIs $\mathcal{L}_{ob}$)

  $(\min \beta_2 {\gamma_2}^2 + \beta_p {\gamma_p}^2$ to adjust tradeoff between norm and peak performances)
- 4- Analyze observed-state closed-loop (using LMIs $\mathcal{L}_{ob,a}$ or $\mathcal{L}_{ob,da}$)

△ No guarantee that next step would be feasible

△ No guarantee to find a robustly stabilizing control when exists

△ All step are purely LMI with clear control theory justification

△ Each step based on new LMI conditions

■ Academic example for illustration

$$x_{k+1} = \begin{bmatrix} -\theta_1{}^2/\theta_2 & -\theta_1 \\ 1 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ \theta_2 \end{bmatrix} u_k + \begin{bmatrix} \theta_1 \\ 0 \end{bmatrix} w_k$$

$$z_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k + \theta_2 u_k, \qquad y_k = \begin{bmatrix} 0 & 1 \end{bmatrix} x_k$$

● DMAR

$$E_x = \begin{bmatrix} \theta_1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_z = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad E_\pi = \begin{bmatrix} \theta_2 & 0 \\ 0 & 1 \end{bmatrix},$$

$$A = \begin{bmatrix} -\theta_1 & -\theta_2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \theta_2 \end{bmatrix}, \quad B_w = \begin{bmatrix} \theta_2 \\ 0 \end{bmatrix}.$$

● $\theta_1 \in [\,1-\delta_1,\ 1+\delta_1\,]$, $\theta_2 \in [\,1-\delta_2,\ 1+\delta_2\,]$ at limit of stability when $\theta_1 = \theta_2 = 1$

- Results when $\mu_d = 10$ at first step

| $(\delta_1, \delta_2)$ | $(\beta_2, \beta_p)$ | $(\gamma_2, \gamma_p)$ | $\mu_c$ | $\mu_{dc}$ |
|---|---|---|---|---|
| $(0, 0)$ | $(1, 1)$ | $(10^{-4}, 10^{-4})$ | 2.8152 | 2.8152 |
| $(0.1, 0)$ | $(1, 1)$ | $(1.0747, 1.0410)$ | 2.6672 | 2.6672 |
| $(0, 0.1)$ | $(1, 1)$ | $(0.3947, 0.3524)$ | 2.4546 | 2.4546 |
| $(0.1, 0.1)$ | $(10, 1)$ | $(1.2736, 1.1765)$ | 6.5901 | 6.5901 |
| $(0.1, 0.1)$ | $(1, 1)$ | $(1.2962, 1.0985)$ | 6.1252 | 6.1252 |
| $(0.1, 0.1)$ | $(1, 10)$ | $(1.3339, 1.0809)$ | 5.3226 | 5.3226 |
| $(0.2, 0.1)$ | $(1, 1)$ | $(1.3006, 1.2181)$ | 11.2285 | 11.2285 |
| $(0.1, 0.2)$ | $(1, 1)$ | $(1.3242, 1.1553)$ | 6.8505 | 6.8505 |
| $(0.2, 0.2)$ | $(1, 1)$ | $(3.5392, 3.0228)$ | $\infty$ | 14.3142 |

- Results when $\mu_d$ is minimized at first step and $(\beta_2, \beta_p) = (1, 1)$.

| $(\delta_1, \delta_2)$ | $\mu_d$ | $\tilde{\mu}_d$ | $(\gamma_2, \gamma_p)$ | $\mu_c$ | $\mu_{dc}$ |
|---|---|---|---|---|---|
| $(0, 0)$ | 1 | 1 | $(10^{-4}, 10^{-4})$ | 3.3731 | 3.3731 |
| $(0.1, 0)$ | 1.5303 | 1.5303 | $(0.2349, 0.2349)$ | 2.8013 | 2.8013 |
| $(0, 0.1)$ | 1.2497 | 1.2502 | $(0.3061, 0.2777)$ | 2.5644 | 2.5644 |
| $(0.1, 0.1)$ | 2.0284 | 2.0284 | $(0.5611, 0.5181)$ | 4.3398 | 4.3398 |
| $(0.2, 0.1)$ | 3.8506 | 3.8506 | $(1.1902, 1.0970)$ | 10.1302 | 10.1302 |
| $(0.1, 0.2)$ | 3.1161 | 3.1161 | $(1.3334, 1.1567)$ | 7.7934 | 7.7934 |
| $(0.2, 0.2)$ | 8.7776 | 8.7776 | $(3.0445, 2.5166)$ | $\infty$ | $\infty$ |

- $\tilde{\mu}_d$ computed on SV-LMI with reduced size

- Size of LMI conditions (applying non-conservative size reduction procedures)

| $(\delta_1, \delta_2)$ | $\mathcal{L}_{sf}$ | $\tilde{\mathcal{L}}_{sf}$ | $\mathcal{L}_{sf,a}$ | $\mathcal{L}_{ob}$ | $\mathcal{L}_{ob,a}$ | $\mathcal{L}_{ob,da}$ |
|---|---|---|---|---|---|---|
| $(0,0)$ | $\{5,0\}$ | $\{5,0\}$ | $\{3,0\}$ | $\{6,0\}$ | $\{5,0\}$ | $\{5,0\}$ |
| $(1,0)$ | $\{6,1\}$ | $\{6,1\}$ | $\{5,2\}$ | $\{7,1\}$ | $\{7,2\}$ | $\{7,2\}$ |
| $(0,1)$ | $\{7,2\}$ | $\{6,1\}$ | $\{5,2\}$ | $\{8,2\}$ | $\{7,2\}$ | $\{10,5\}$ |
| $(1,1)$ | $\{7,2\}$ | $\{6,1\}$ | $\{6,3\}$ | $\{8,2\}$ | $\{8,3\}$ | $\{11,6\}$ |

- $\{N_r, N_c\}$ with $N_r$ rows in each LMI (to be multiplied by nb of vertices)
- $\{N_r, N_c\}$ with $N_c$ columns of S-variables

■ System model

$$x_{k+1} = \begin{bmatrix} -\theta_1{}^2/\theta_2 & -\theta_1 \\ 1 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ \theta_2 \end{bmatrix} u_k + \begin{bmatrix} \theta_1 \\ 0 \end{bmatrix} w_k$$

● Observer model (design for $\delta_1 = \delta_2 = 0.2$)

$$A_o = \begin{bmatrix} -0.7832 & -1.1081 \\ 1.1373 & 0.9274 \end{bmatrix}, \quad B_o = \begin{bmatrix} 0.1425 \\ 0.2195 \end{bmatrix}$$

- Revisited Luenberger observer design in case of uncertain systems
  - Separation principle replaced by a mixed norm/peak performance measure
  - LMI design of state-feedback and observer gains one after the other
  - Heuristic with no guarantee of success
  - Surprisingly, $K$ design for fixed observer is more complex (prospective work)
  - Extensions for continuous-time systems:

    raises issues about S-variable conditions for design (tuning parameters)

- Promising descriptor multi-affine representation combined to SV-LMIs