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Ellipsoidal Sets for Static Output-Feedback

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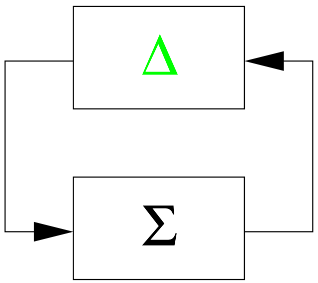
Motivation

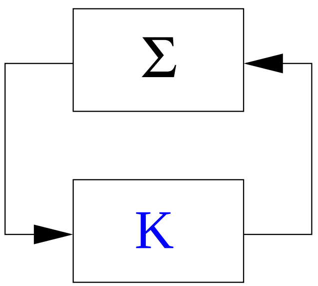


$$\exists \Theta : \begin{cases} \begin{bmatrix} \mathbb{1} & \Delta' \end{bmatrix} \Theta \begin{bmatrix} \mathbb{1} \\ \Delta \end{bmatrix} \leq \mathbb{0} & \forall \Delta \in \boldsymbol{\Delta} \\ \begin{bmatrix} \Sigma^*(j\omega) & \mathbb{1} \end{bmatrix} \Theta \begin{bmatrix} \Sigma(j\omega) \\ \mathbb{1} \end{bmatrix} > \mathbb{0} & \forall \omega \in \mathbb{R} \end{cases}$$

Topological separation for robust analysis $\Uparrow$

Motivation

for synthesis $\Downarrow$:



$$\exists \Theta : \left\{ \begin{array}{ll} \begin{bmatrix} \Sigma^*(j\omega) & \mathbb{1} \end{bmatrix} \Theta \begin{bmatrix} \Sigma(j\omega) \\ \mathbb{1} \end{bmatrix} > \mathbb{0} & \forall \omega \in \mathbb{R} \\ \begin{bmatrix} \mathbb{1} & K' \end{bmatrix} \Theta \begin{bmatrix} \mathbb{1} \\ K \end{bmatrix} \leq \mathbb{0} & \text{non-empty set} \end{array} \right.$$

Matrix ellipsoids

Ellipsoids of the vector space $\mathbb{R}^n$

Centre $x_o \in \mathbb{R}^n$, radius r and geometry $Z \in \mathbb{R}^{n\times n}$ s.t. $Z > \mathbb{0}$, $\|Z\| = 1$.

$$\{ \; x \in \mathbb{R}^n \quad : \quad (x - x_o)' Z (x - x_o) \leq r \; \}$$

Matrix ellipsoids of $\mathbb{R}^{m\times p}$

Centre $K_o \in \mathbb{R}^{m\times p}$, radius $R \in \mathbb{R}^{p\times p}$ and geometry $Z \in \mathbb{R}^{m\times m}$ s.t. $Z > \mathbb{0}$, $\|Z\| = 1$.

$$\{ \; K \in \mathbb{R}^{m\times p} \quad : \quad (K - K_o)' Z (K - K_o) \leq R \; \}$$

$\{X, Y, Z\}$-ellipsoid : definition

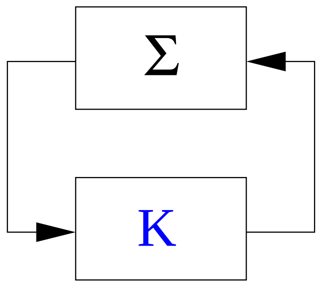
$$\left\{ K \in \mathbb{R}^{m\times p} \quad : \quad \begin{bmatrix} \mathbb{1} & K' \end{bmatrix} \begin{bmatrix} X & Y \\ Y' & Z \end{bmatrix} \begin{bmatrix} \mathbb{1} \\ K \end{bmatrix} \leq \mathbb{0} \quad , \quad Z > \mathbb{0} \right\}$$

Matrix ellipsoids

- Algebraic rules: $K_o = -Z^{-1}Y' \quad , \quad R = K_o{}'ZK_o - X.$
- Non-emptiness condition : $R \geq \mathbb{0} \Leftrightarrow \boxed{X \leq YZ^{-1}Y'}$
- LMI description : $\begin{bmatrix} X + YK + K'Y' & K'Z \\ ZK & -Z \end{bmatrix} \leq \mathbb{0}$
- $\{X, Y, Z\}$-ellipsoid : a compact convex set.
- $VOL(\{X, Y, Z\}\text{-ellipsoid}) = \sqrt{\frac{det(R)^m}{det(Z)^p}} \cdot VOL(\{-\mathbb{1}, \mathbb{0}, \mathbb{1}\}\text{-ellipsoid}).$

Static output-feedback

Notations



$$: \ \Sigma \star K \ : \ \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \\ u(t) = Ky(t) \end{cases}$$

Σ is stabilisable via static output-feedback

iff $\exists K$ s.t. $\Sigma \star K$ is stable.

iff there exist a Lyapunov matrix P and a non-empty $\{X, Y, Z\}$-ellipsoid s.t.:

$$\begin{bmatrix} \mathbb{1} & \mathbb{0} \\ A & B \end{bmatrix}' \begin{bmatrix} \mathbb{0} & P \\ P & \mathbb{0} \end{bmatrix} \begin{bmatrix} \mathbb{1} & \mathbb{0} \\ A & B \end{bmatrix} < \begin{bmatrix} C & D \\ \mathbb{0} & \mathbb{1} \end{bmatrix}' \begin{bmatrix} X & Y \\ Y' & Z \end{bmatrix} \begin{bmatrix} C & D \\ \mathbb{0} & \mathbb{1} \end{bmatrix}$$

Remarks

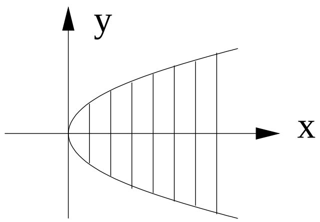
A set of control laws:

$V(x) = x'Px$ proves $\Sigma \star K$ stability for any gain K in the $\{X, Y, Z\}$-ellipsoid.

The non-convex constraint

SOF stabilisability $\Longleftrightarrow$ LMIs $\oplus$ a non-linear constraint ($X \leq YZ^{-1}Y'$).

Example for $K \in \mathbb{R}^{1\times 1}$: $z = 1$ et $x \leq y^2$



Fragility and resilience

Definition : Let K_o be a stabilising gain and $\mathbb{\Delta}_K$ an additive uncertainty.

- **Fragile :** $\exists \Delta_K \in \mathbb{\Delta}_K$ s.t. $\Sigma\star(K_o + \Delta_K)$ is unstable.
- **Resilient :** $\forall \Delta_K \in \mathbb{\Delta}_K \;:\; \Sigma\star(K_o + \Delta_K)$ is stable.
- **Quadratically resilient :**
  A unique quadratic Lyapunov function ($V(x) = x'Px$) proves the resiliency.

Fragility and resilience

Corollaries

- $\mathbb{\Delta}_K$: ellipsoidal matrix set centred at the origin.

LMI constraint $\oplus$ non-linear constraint $\boxed{X \leq YZ^{-1}Y'}$ $\Longleftrightarrow$
centre K_o of the $\{X, Y, Z\}$-ellipsoid is quadratically resilient to $\Delta_K' Z \Delta_K \leq R$.

- $\mathbb{\Delta}_K$: norm-bounded uncertainty.

LMI constraint $\oplus$ $Z = \mathbb{1}$ $\oplus$ non-linear constraint $\boxed{\mathbb{0} < \rho\mathbb{1} \leq YY' - X}$ $\Longleftrightarrow$
centre K_o of the $\{X, Y, Z\}$-ellipsoid is quadratically resilient to $\Delta_K' \Delta_K \leq \rho\mathbb{1}$.

- $\mathbb{\Delta}_K$: multiplicative uncertainty with radius $\bar{\delta}$

LMI constraint $\oplus$ non-linear constraint $\boxed{X \leq (1 - \bar{\delta}^2) YZ^{-1}Y'}$ $\Longleftrightarrow$
centre K_o of the $\{X, Y, Z\}$-ellipsoid is quadratically resilient to $\Delta_K' = \delta K_o$ with $|\delta| \leq \bar{\delta}$.

Bounded or dissipative specifications on K

Definition

Design a stabilising control law K that belongs to a given $\{X_K, Y_K, Z_K\}$-ellipsoid.

Example 1 : Find a control law with bounded gain $(K'K \leq \rho_K \mathbb{1})$

Exemple 2 : Find a passive control law $(K' + K \leq \mathbb{0})$

LMI constraint

$$\left\{ \begin{array}{l} \nu > 0 \\ \nu \begin{bmatrix} X_K & Y_K \\ Y_K' & Z_K \end{bmatrix} \leq \begin{bmatrix} X & Y \\ Y' & Z \end{bmatrix} \end{array} \right.$$

Pole location

$\{X_R, Y_R, Z_R\}$-stability

The poles of $\Sigma \star K$ belong to an ellipsoidal region of the complex plane:

$$\{ \; s \in \mathbb{C} \;\; : \;\; X_R + sY_R + s^* Y_R^* + ss^* Z_R < \mathbb{0} \; \}$$

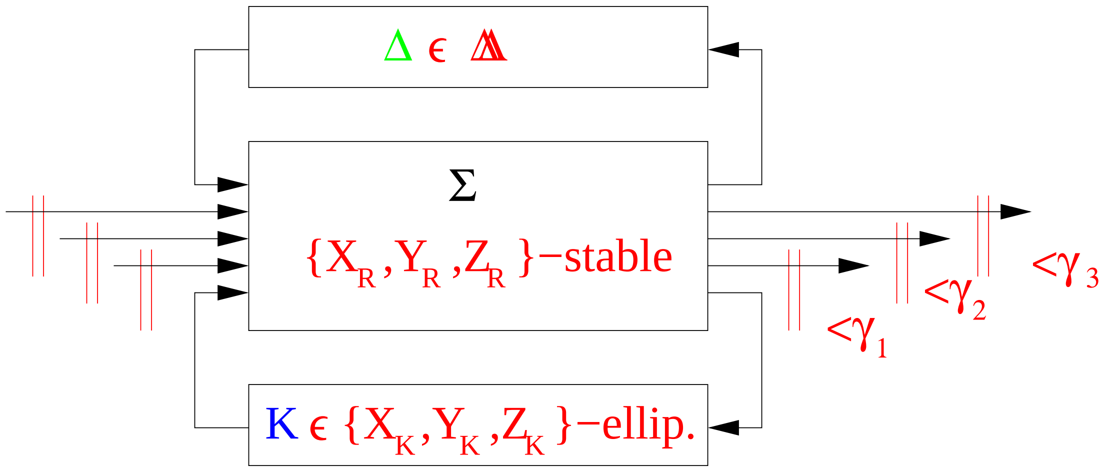
Examples: half-planes, discs, sectors, parabolas...

Static output-feedback $\{X_R, Y_R, Z_R\}$-stabilisability

$$\begin{bmatrix} * \end{bmatrix}' \begin{bmatrix} X_R \otimes P & Y_R \otimes P \\ Y_R^* \otimes P & Z_R \otimes P \end{bmatrix} \begin{bmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{1} \otimes A & \mathbb{1} \otimes B \end{bmatrix} < \begin{bmatrix} * \end{bmatrix}' \begin{bmatrix} \mathbb{1} \otimes X & \mathbb{1} \otimes Y \\ \mathbb{1} \otimes Y' & \mathbb{1} \otimes Z \end{bmatrix} \begin{bmatrix} \mathbb{1} \otimes C & \mathbb{1} \otimes D \\ \mathbb{0} & \mathbb{1} \end{bmatrix}$$

$\oplus$ non-linear constraint

Extensions to multi-objective synthesis



- An LMI constraint for each specification.
- A Lyapunov function for each specification.
- A unique non-linear constraint ($X \leq YZ^{-1}Y'$)

Algorithms to solve non-linear matrix inequalities

- SOF design has no convex formulation in the general case.
- Elimination based approach $\Rightarrow$ LMIs $\oplus$ $(XY = \mathbb{1})$ proved to be NP-hard [Fu & Luo 1997].
- Heuristic algorithms such as coordinated-descent iterative resolutions of BMIs.
- Efficient sub-optimal first order algorithms:
  - ⋆ Cone complementarity algorithm [El Ghaoui & al 1997].
  - ⋆ Alternation projection algorithm [Grigoriadis & Skelton 1996].

Numerical experiments with cone complementarity algorithm

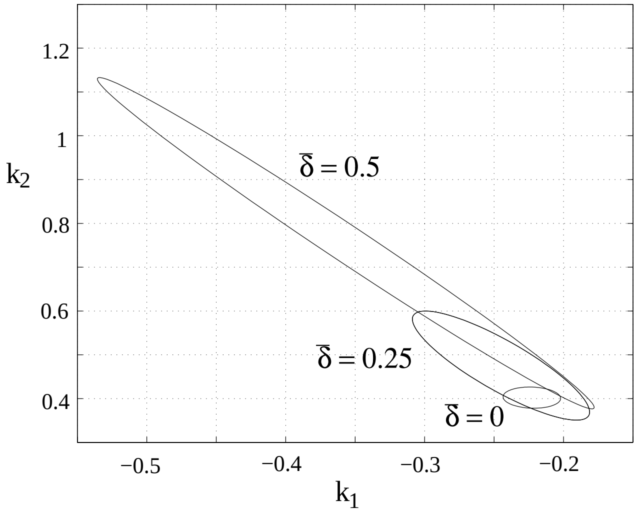
- Considered non-linear constraint: $X \leq (1-\bar{\delta}^2) Y Z^{-1} Y'$.
- Linear relaxation: $X \leq (1-\bar{\delta}^2)\hat{X}$ $\quad$ $Y Z^{-1} Y' \leq \hat{X}$
- Algorithm designed for: $\hat{X} \longrightarrow Y Z^{-1} Y'$.
- Stopping criterium: $X \leq (1-\bar{\delta}^2) Y Z^{-1} Y'$.

Algorithms to solve non-linear matrix inequalities

SOF design such that $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}' \in \mathbb{R}^{2\times 1}$.

Stability

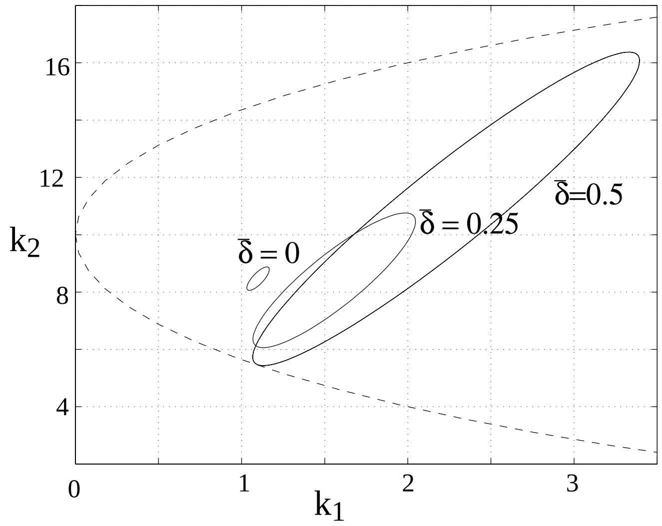
and resiliency



Poles location ($Re(\text{pôles}) < -0.15$)

bounded K (radius 10, center $\begin{bmatrix} 10 & 10 \end{bmatrix}'$).

and resiliency



Conclusions and prospectives

- ✪ New static output-feedback design based on the topological separation theory.
- ✪ No conservatism when compared to LMI analysis techniques.
- ✪ Encouraging numerical results.

- ⋆ Develop new adapted algorithms.
- ⋆ Extensions to other design problems.