

Интегральное квадратичное разделение:

описание подхода, связь с функциями Ляпунова и S-процедурой Якубовича,  
приложение к анализу устойчивости спутника с ограничением по входу

Dimitri PEAUCELLE / Дмитрий Жанович Посель-Коновалов

LAAS-CNRS - Université de Toulouse - FRANCE

Санкт-Петербург



Май 2013

# Evaluating regions of attraction of LTI systems with saturation in IQS framework

Dimitri Peaucelle

Sophie Tarbouriech

LAAS-CNRS



Martine Ganet-Schoeller



Samir Bennani



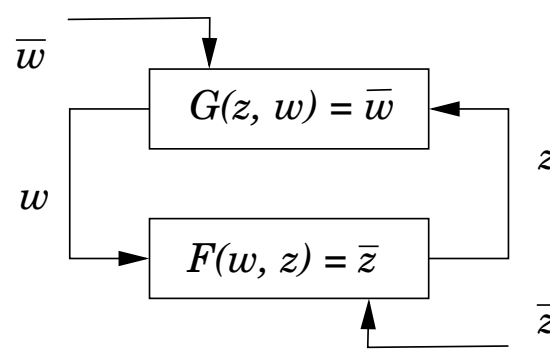
Presented first at 7th IFAC Symposium on Robust Control Design / Aalborg

■ Saturated control of a linear system

$$\dot{x} = Ax + Bu, \quad u = \text{sat}(Ky), \quad y = Cx$$

- Assume  $K$  designed for the linear system (no saturation)
  - System with saturation: Stability is (in general) only local
  - Problem: find (largest possible) set of  $x(0)$  such that  $x(\infty) = 0$
- Goal of this presentation : formalize the problem in the IQS framework
- Can "system augmentation" relaxations provide less conservative results ?

■ Well-posedness of a feedback loop

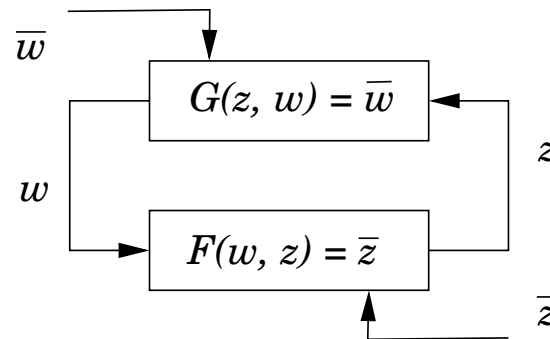


● Uniqueness and boundedness of internal signals for all bounded disturbances

$$\exists \gamma : \forall (\bar{w}, \bar{z}) \in L_2 \times L_2, \quad \left\| \begin{array}{c} w - w_0 \\ z - z_0 \end{array} \right\| \leq \gamma \left\| \begin{array}{c} \bar{w} \\ \bar{z} \end{array} \right\|,$$

● with  $\begin{cases} G(z_0, w_0) = 0 \\ F(w_0, z_0) = 0 \end{cases}$  solution to the system without perturbations

■ Well-posedness of a feedback loop



■ Theorem: Well-posed **iff** exists a topological separator  $\theta$

● ‘Negative’ on the inverse graph of one component

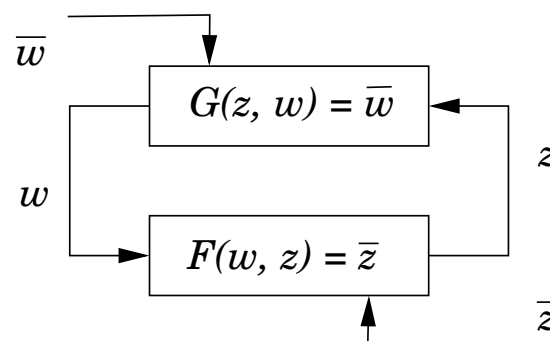
$$\mathcal{G}^I(\bar{w}) = \{(w, z) : G(z, w) = \bar{w}\} \subset \{(w, z) : \theta(w, z) \leq \phi_2(\|\bar{w}\|)\}$$

● ‘Positive definite’ on the graph of the other component of the loop

$$\mathcal{F}(\bar{z}) = \{(w, z) : F(w, z) = \bar{z}\} \subset \{(w, z) : \theta(w, z) > -\phi_1(\|\bar{z}\|)\}$$

▲ Issue 1: How to choose  $\theta$  ? Answer: S-procedure.

▲ Issue 2: How to test the separation inequalities ? Answer: LMIs.



■ Well-posedness of a feedback loop

● In case of causal  $G(z, w) : w = \Delta z$ ,  $\Delta \in \mathcal{RH}_{\infty}^{m \times l}$   
and stable proper LTI  $F(w, z) : z = H(s)w$

● Necessary and sufficient (lossless) choice of separator

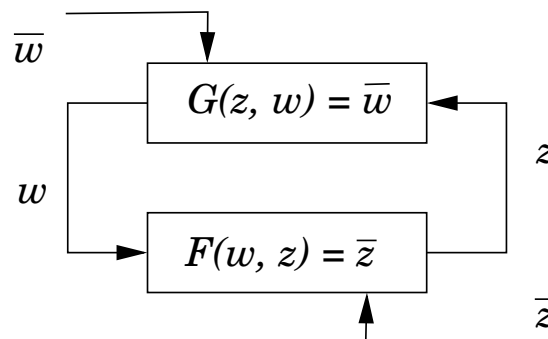
$$\theta(w, z) = \|w\|^2 - \gamma^2 \|z\|^2$$

● Separation inequalities:

$$\theta(w, z) = \|w\|^2 - \gamma^2 \|z\|^2 \leq 0, \forall w = \Delta z \Leftrightarrow \|\Delta\|_{\infty}^2 \leq \gamma^2$$

$$\theta(w, z) = \|w\|^2 - \gamma^2 \|z\|^2 > 0, \forall z = H(s)w \Leftrightarrow \|H\|_{\infty}^2 < \frac{1}{\gamma^2}$$

■ Well-posedness of a feedback loop



● In case of passive  $G(z, w) : w = \Delta z$

and stable LTI  $F(w, z) : z = H(s)w$

● Necessary and sufficient (lossless) choice of separator

$$\theta(w, z) = - \langle w | z \rangle$$

● Separation inequalities:

$$\theta(w, z) = - \langle w | z \rangle \leq 0, \quad \forall w = \Delta z \Leftrightarrow \int_0^{\infty} w^T(t) z(t) dt \geq 0$$

$$\theta(w, z) = - \langle w | z \rangle > 0, \quad \forall z = H(s)w \Leftrightarrow H^*(j\omega) + H(j\omega) < 0, \quad \forall \omega$$

- From topological separation to IQS: Choice of an Integral Quadratic Separator

$$\theta(w, z) = \left\langle \begin{pmatrix} z \\ w \end{pmatrix} \middle| \ominus \begin{pmatrix} z \\ w \end{pmatrix} \right\rangle = \int_0^{\infty} \begin{pmatrix} z^T(t) & w^T(t) \end{pmatrix} \ominus(t) \begin{pmatrix} z(t) \\ w(t) \end{pmatrix} dt$$

- Identical choice to IQC framework [Megretski, Rantzer, Jönsson]

$$\theta(w, z) = \int_{-\infty}^{+\infty} \begin{pmatrix} z^T(j\omega) & w^T(j\omega) \end{pmatrix} \Pi(j\omega) \begin{pmatrix} z(j\omega) \\ w(j\omega) \end{pmatrix} d\omega$$

- $\Pi$  is called a multiplier.  $\theta(w, z) \leq 0$  is called an IQC.

- Conservatism reduction in IQC framework :  $\omega$ -dependent multipliers:

$$\Pi(j\omega) = \begin{bmatrix} \mathbf{1} & \Psi_1(j\omega)^* & \dots & \Psi_r(j\omega)^* \end{bmatrix} \hat{\Pi} \begin{bmatrix} \mathbf{1} \\ \Psi_1(j\omega) \\ \vdots \\ \Psi_r(j\omega) \end{bmatrix}$$



- Main IQS result (both for  $\omega$  or  $t$  or  $k$  dependent signals)
- IQS is **necessary and sufficient** under assumptions (proof based on [Iwasaki 2001])
- One component is a linear application, can be descriptor form  $F(w, z) = \mathcal{A}w - \mathcal{E}z$
- ▲ can be time-varying  $\mathcal{A}(t)w(t) - \mathcal{E}(t)z(t)$  or frequency dep.  $\hat{\mathcal{A}}(\omega)\hat{w}(\omega) - \hat{\mathcal{E}}(\omega)\hat{z}(\omega)$
- ▲  $\mathcal{A}(t), \mathcal{E}(t)$  are bounded and  $\mathcal{E}(t) = \mathcal{E}_1(t)\mathcal{E}_2$  where  $\mathcal{E}_1(t)$  is full column rank
- The other component can be defined in a set

$$G(z, w) = \nabla(z) - w, \quad \nabla \in \mathbb{W}$$

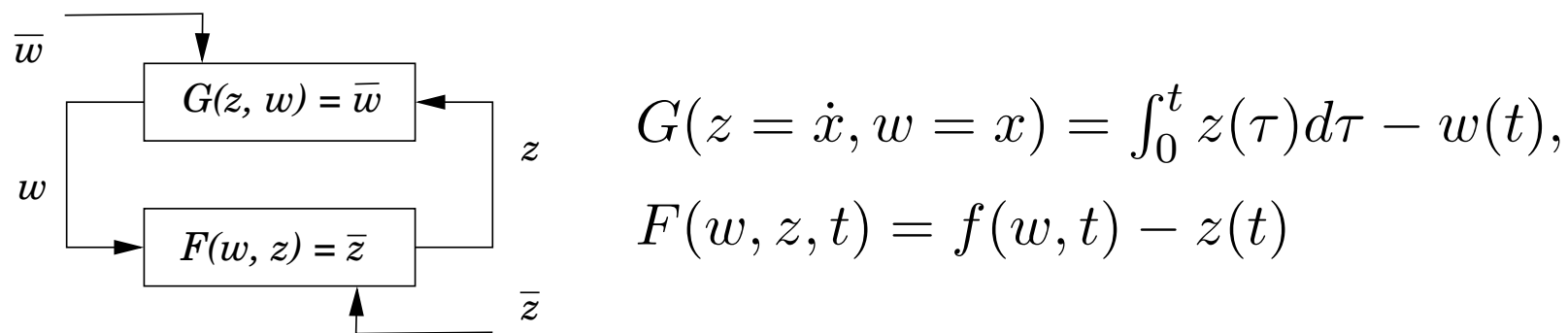
- ▲  $\mathbb{W}$  must have a linear-like property

$$\forall (z_1, z_2), \quad \forall \nabla \in \mathbb{W}, \quad \exists \tilde{\nabla} \in \mathbb{W} : \nabla(z_1) - \nabla(z_2) = \tilde{\nabla}(z_1 - z_2)$$

- ▲  $\mathbb{W}$  does not need to be causal

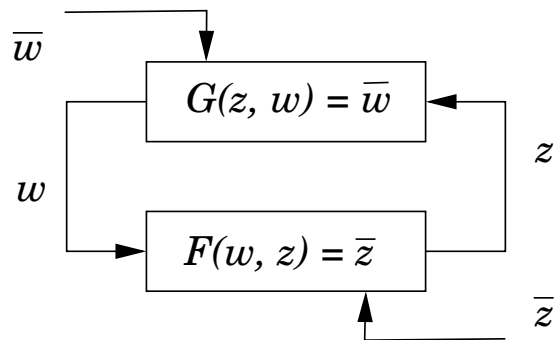
- The matrix  $\Theta$  must satisfy an IQC over  $\mathbb{W}$  + an LMI involving  $(\mathcal{E}, \mathcal{A})$

■ Global stability of a non-linear system  $\dot{x} = f(x, t)$



- $\bar{w}$  plays the role of the initial conditions,  $\bar{z}$  are external disturbances
- Well-posedness: for all bounded initial conditions and all bounded disturbances, the state remains bounded around the equilibrium  $\equiv$  global stability

■ Global stability of a linear TV system  $\dot{x} = A(t)x$



$$G(z = \dot{x}, w = x) = \int_0^t z(\tau) d\tau - w(t) = s^{-1}z - w,$$

$$F(w, z, t) = A(t)w(t) - z(t)$$

● IQS:  $\theta(w, z) = \int_0^\infty \begin{pmatrix} z^T(t) & w^T(t) \end{pmatrix} \begin{bmatrix} \mathbf{0} & -P(t) \\ -P(t) & -\dot{P}(t) \end{bmatrix} \begin{pmatrix} z(t) \\ w(t) \end{pmatrix} dt$

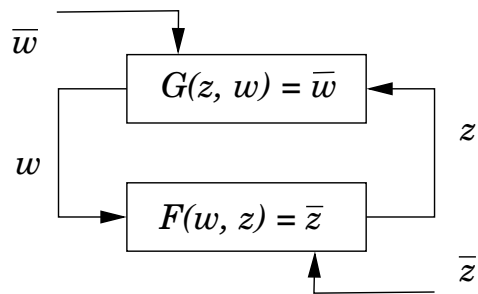
▲  $\theta(w, z) \leq 0$  for all  $G(z, w) = 0$  iff  $P(t) \geq \mathbf{0}$

$$\left( x(0) = 0, \int_0^t (\dot{x}^T P x + x^T \dot{P} x + x^T P \dot{x}) d\tau = x^T(t) P(t) x(t) \right)$$

▲  $\theta(w, z) > 0$  for all  $F(w, z) = 0$  iff  $A^T(t)P(t) + P(t)A(t) + \dot{P}(t) < \mathbf{0}$

$$\left( z^T P w + w^T \dot{P} w + w^T P w = w^T (A^T P + P A + \dot{P}) w \right)$$

■ Global stability of a system with a dead-zone

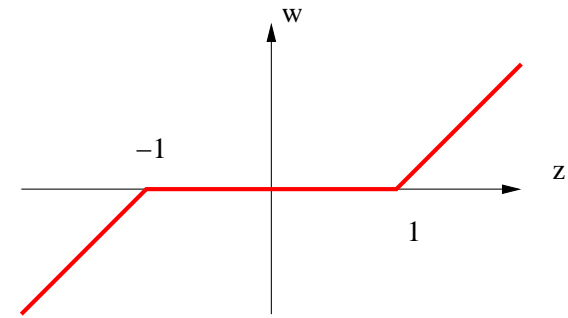


$$G_1(\dot{x}, x) = \int_0^t \dot{x}(\tau) d\tau - x(t),$$

$$G_2(g, v) = dz(g(t)) - v(t),$$

$$F_1(x, v, \dot{x}, t) = f_1(x, v, t) - \dot{x}(t),$$

$$F_2(x, v, g, t) = f_2(x, v, t) - g(t)$$



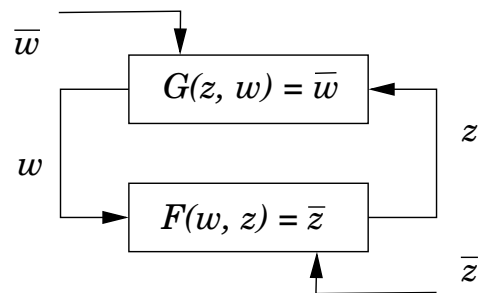
● Dead-zone embedded in a sector uncertainty  $\mathbb{W}_\infty = \{\nabla_\infty : 0 \leq \nabla_\infty(g) \leq g\}$

$$\mathcal{G}_2^I = \{(v, g) : G_2(g, v) = 0\} \subset \{(v, g) : v = \nabla_\infty(g), \nabla_\infty \in \mathbb{W}_\infty\}$$

▲ Choosing  $\theta$  IQS w.r.t.  $\mathbb{W}_\infty$  rather than w.r.t  $\mathcal{G}_2^I$ , is a source of conservatism

■ IQS applies for linear  $f_1, f_2$

■ Global stability of a system with a dead-zone



$$G_1(\dot{x}, x) = \int_0^t \dot{x}(\tau) d\tau - x(t),$$

$$G_2(g, v) = dz(g(t)) - v(t),$$

$$F_1(x, v, \dot{x}, t) = Ax(t) + Bv(t) - \dot{x}(t),$$

$$F_2(x, v, g, t) = Cx(t) + Dv(t) - g(t)$$

● LMI conditions obtained for the IQS defined by

$$\Theta = \left[ \begin{array}{cc|cc} 0 & 0 & -P & 0 \\ 0 & 0 & 0 & -p_1 \\ \hline -P & 0 & 0 & 0 \\ 0 & -p_1 & 0 & 2p_1 \end{array} \right], \quad \begin{array}{l} P > 0, \\ p_1 > 0. \end{array}$$

● Result is exactly identical to circle theorem result

■ Launcher in ballistic phase : attitude control

- Neglected atmospheric friction, sloshing modes, ext. perturbation, axes coupling:  $I\ddot{\theta} = T$
- Saturated actuator:  $T = \text{sat}_{\bar{T}}(u) = u - \bar{T} \text{dz}(\frac{1}{\bar{T}}u)$
- PD control  $u = -K_P\theta - K_D\dot{\theta}$

$$G_1(\dot{x}, x) = \int_0^t \dot{x}(\tau) d\tau - x(t),$$

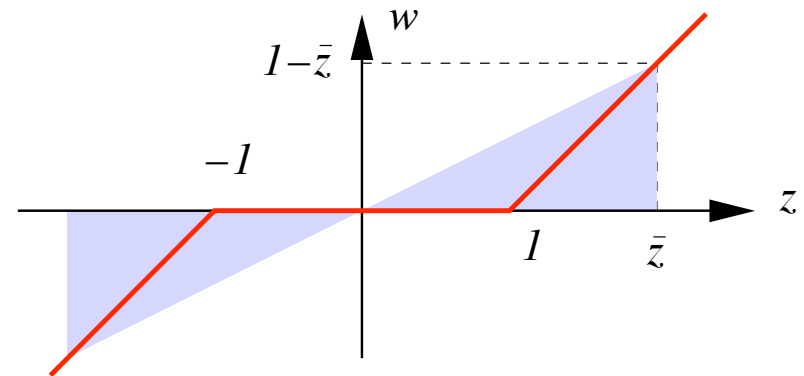
$$G_2(g, v) = \text{dz}(g(t)) - v(t),$$

$$F_1(x, v, \dot{x}, t) = \begin{bmatrix} 0 & 1 \\ -K_P & -K_D \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ -\bar{T} \end{bmatrix} v(t) - \dot{x}(t),$$

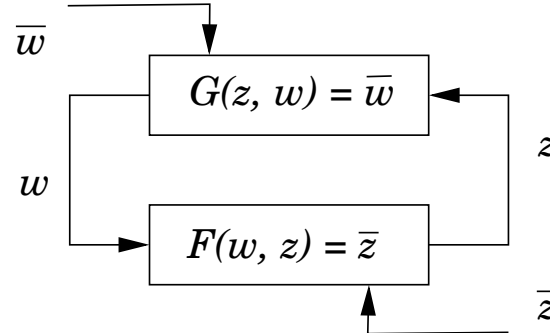
$$F_2(x, v, g, t) = \begin{bmatrix} -\frac{K_P}{\bar{T}} & -\frac{K_D}{\bar{T}} \end{bmatrix} x(t) - g(t)$$

- Global stability LMI test fails
- ▲ Sector uncertainty  $\mathbb{W}_\infty$  includes  $\nabla_\infty = 1$  for which the system is  $I\ddot{\theta} = 0$  (unstable)
- LMI test succeeds (whatever  $\bar{g} < \infty$ ) if dead-zone is restricted to belong to

$$\mathbb{W}_{\bar{g}} = \left\{ \nabla_{\bar{g}} : 0 \leq \nabla_{\bar{g}}(g) \leq \frac{1-\bar{g}}{\bar{g}} g \right\}$$



- ▲ Useful if one can prove for constrained  $x(0)$  that  $|g(\theta)| \leq \bar{g}$  holds  $\forall \theta \geq 0$ .
- How can one prove local properties in IQS framework ?



■ Well-posedness of a feedback loop

● Uniqueness and boundedness of internal signals for all bounded disturbances

$$\exists \gamma : \forall (\bar{w}, \bar{z}) \in L_2 \times L_2, \quad \left\| \begin{array}{c} w - w_0 \\ z - z_0 \end{array} \right\| \leq \gamma \left\| \begin{array}{c} \bar{w} \\ \bar{z} \end{array} \right\|, \quad \begin{array}{l} G(z_0, w_0) = 0 \\ F(w_0, z_0) = 0 \end{array}$$

▲ How to introduce initial conditions  $x(0)$  and “final” conditions  $g(\theta)$  in IQS framework?

■ Square-root of the Dirac operator: linear operator such that

$$\begin{aligned} x \mapsto \varphi_\theta x : \quad & \langle \varphi_\theta x | M \varphi_\theta x \rangle = \int_0^\infty \varphi_\theta x^T(t) M \varphi_\theta x(t) dt = x^T(\theta) M x(\theta) \\ & \langle \varphi_{\theta_1} x | M \varphi_{\theta_2} x \rangle = 0 \text{ if } \theta_1 \neq \theta_2 \end{aligned}$$

● Such operator is also used for PDE to describe states on the boundary



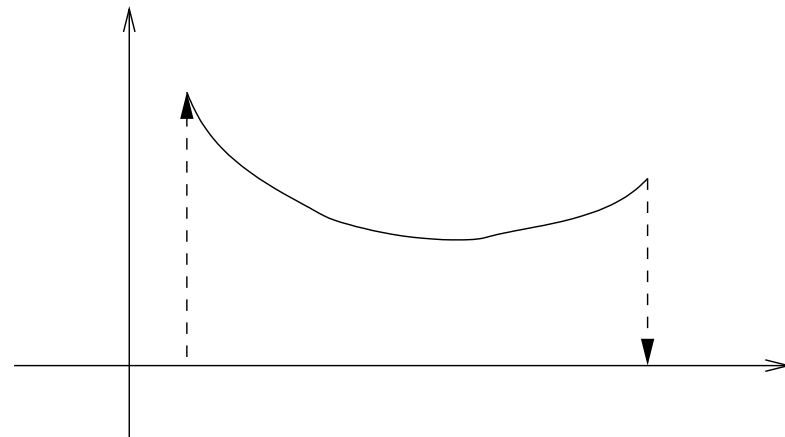
- System with initial and final conditions writes as

$$\begin{pmatrix} \varphi_0 x \\ \mathcal{T}_\theta \dot{x} \\ \mathcal{T}_\theta g \\ \varphi_\theta g \end{pmatrix} = \left[ \begin{array}{cc|cc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ A & \mathbf{0} & B & \mathbf{0} \\ \hline C & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & C & \mathbf{0} & \mathbf{0} \end{array} \right] \begin{pmatrix} \mathcal{T}_\theta x \\ \varphi_\theta x \\ \mathcal{T}_\theta v \\ \varphi_0 x \end{pmatrix}$$

- ▲  $\mathcal{T}_\theta x$  is the truncated signal such that  $\mathcal{T}_\theta x(t) = x(t)$  for  $t \leq \theta$  and  $= 0$  for  $t > \theta$ .

- The integration operator is redefined as a mapping

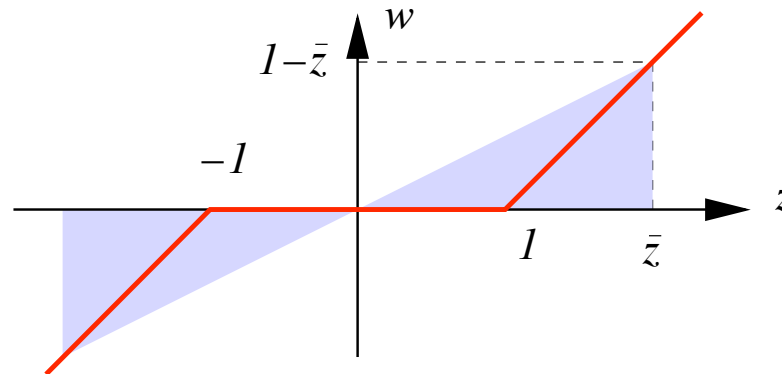
$$\begin{pmatrix} \mathcal{T}_\theta x \\ \varphi_\theta x \end{pmatrix} = \mathcal{I} \begin{pmatrix} \varphi_0 x \\ \mathcal{T}_\theta \dot{x} \end{pmatrix}$$



$$\begin{pmatrix} \varphi_0 x \\ \mathcal{T}_\theta \dot{x} \\ \mathcal{T}_\theta g \\ \varphi_\theta g \end{pmatrix} = \left[ \begin{array}{cc|cc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ A & \mathbf{0} & B & \mathbf{0} \\ \hline C & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & C & \mathbf{0} & \mathbf{0} \end{array} \right] \begin{pmatrix} \mathcal{T}_\theta x \\ \varphi_\theta x \\ \mathcal{T}_\theta v \\ \varphi_0 x \end{pmatrix}$$

- Restricted sector constraint assumed to hold up to  $t = \theta$ :

$$\mathcal{T}_\theta v = \nabla_{\bar{g}} \mathcal{T}_\theta g$$



$$\begin{pmatrix} \varphi_0 x \\ \mathcal{T}_\theta \dot{x} \\ \mathcal{T}_\theta g \\ \varphi_\theta g \end{pmatrix} = \left[ \begin{array}{cc|cc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ A & \mathbf{0} & B & \mathbf{0} \\ \hline C & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & C & \mathbf{0} & \mathbf{0} \end{array} \right] \begin{pmatrix} \mathcal{T}_\theta x \\ \varphi_\theta x \\ \mathcal{T}_\theta v \\ \varphi_0 x \end{pmatrix}$$

● Goal is to prove the restricted sector condition holds strictly at time  $\theta$  (whatever  $\theta$ )

▲ i.e. find sets  $1 \geq x^T(0)Qx(0) = \langle \varphi_0 x | Q \varphi_0 x \rangle$  s.t.  $|g(\theta)| = \|\varphi_\theta g\| < \bar{g}$

▲ reformulated as well posedness problem where  $\varphi_0 x = \nabla_{ci} \varphi_\theta g$  defined by

$$w_{ci} = \nabla_{ci} z_{zi} \quad : \quad \bar{g}^2 < w_{ci} | Q w_{ci} \rangle \leq \|z_{ci}\|^2$$

$\nabla_{ci}$  is a non-causal, virtual, operator, used to define the problem in IQS framework

$$\begin{pmatrix} \varphi_{0x} \\ \mathcal{T}_{\theta\dot{x}} \\ \mathcal{T}_{\theta g} \\ \varphi_{\theta g} \end{pmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ A & \mathbf{0} & B & \mathbf{0} \\ C & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathcal{T}_{\theta x} \\ \varphi_{\theta x} \\ \mathcal{T}_{\theta v} \\ \varphi_{0x} \end{pmatrix}$$

■ Problem defined in this way is a well-posedness problem with  $\nabla$  composed of 3 blocs

$$\nabla = \begin{bmatrix} \mathcal{I} & & \mathbf{0} \\ & \nabla_{\bar{g}} & \\ \mathbf{0} & & \nabla_{ci} \end{bmatrix}$$

- IQS framework applies and gives conservative LMI conditions
- Equivalent to LaSalle invariance principle with  $V(x) = x^T Q x$  (ellipsoidal sets of IC)

- How to reduce conservatism ?
- Needed a description of the dead-zone better than sector uncertainty
- Needed to have dead-zone dependent sets of initial conditions
- Both features derived via descriptor modeling of system augmented with  $\dot{v}$  and  $\dot{g}$

$$v = dz(g) : \begin{cases} \text{if } g > 1 & v = g - 1 & \dot{v} = \dot{g} \\ \text{if } |g| \geq 1 & v = 0 & \dot{v} = 0 \\ \text{if } g < -1 & v = g + 1 & \dot{v} = \dot{g} \end{cases}$$

- For IQS, link between  $\dot{v}$  and  $\dot{g}$  is embedded in  $\dot{v} = \nabla_{\{0,1\}} \dot{g}$ , with  $\nabla_{\{0,1\}} \in \{0, 1\}$ .
- Also needed to specify that  $v$  is the integral of  $\dot{v}$  (thus descriptor form)

- All system equations:

$$\begin{cases} \mathcal{T}_\theta \dot{x} = A\mathcal{T}_\theta x + B\mathcal{T}_\theta v \\ \mathcal{T}_\theta g = C\mathcal{T}_\theta x \\ \mathcal{T}_\theta \dot{g} = C\mathcal{T}_\theta \dot{x} \\ \varphi_{\theta g} = C\varphi_{\theta x} \end{cases}, \quad \begin{pmatrix} \mathcal{T}_\theta x \\ \mathcal{T}_\theta v \\ \varphi_{\theta x} \\ \varphi_{\theta v} \end{pmatrix} = \mathcal{I} \begin{pmatrix} \varphi_{0x} \\ \varphi_{0v} \\ \mathcal{T}_\theta \dot{x} \\ \mathcal{T}_\theta \dot{v} \end{pmatrix}, \quad \begin{aligned} \mathcal{T}_\theta v &= \nabla_{\bar{g}} \mathcal{T}_\theta g \\ \varphi_{\theta v} &= \nabla_{\bar{g}} \varphi_{\theta g} \\ \mathcal{T}_\theta \dot{v} &= \nabla_{\{0,1\}} \mathcal{T}_\theta \dot{g} \\ \begin{pmatrix} \varphi_{0x} \\ \varphi_{0v} \end{pmatrix} &= \nabla_{ci} \varphi_{\theta g} \end{aligned}$$

- Gives a descriptor matrix linear transformation

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -C & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \varphi_{0x} \\ \varphi_{0v} \\ \mathcal{T}_\theta \dot{x} \\ \mathcal{T}_\theta \dot{v} \\ \mathcal{T}_\theta g \\ \varphi_{\theta g} \\ \mathcal{T}_\theta \dot{g} \\ \varphi_{\theta g} \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ A & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \mathcal{T}_\theta x \\ \mathcal{T}_\theta v \\ \varphi_{\theta x} \\ \varphi_{\theta v} \\ \mathcal{T}_\theta v \\ \varphi_{\theta v} \\ \mathcal{T}_\theta \dot{v} \\ \varphi_{0x} \\ \varphi_{0v} \end{pmatrix}$$

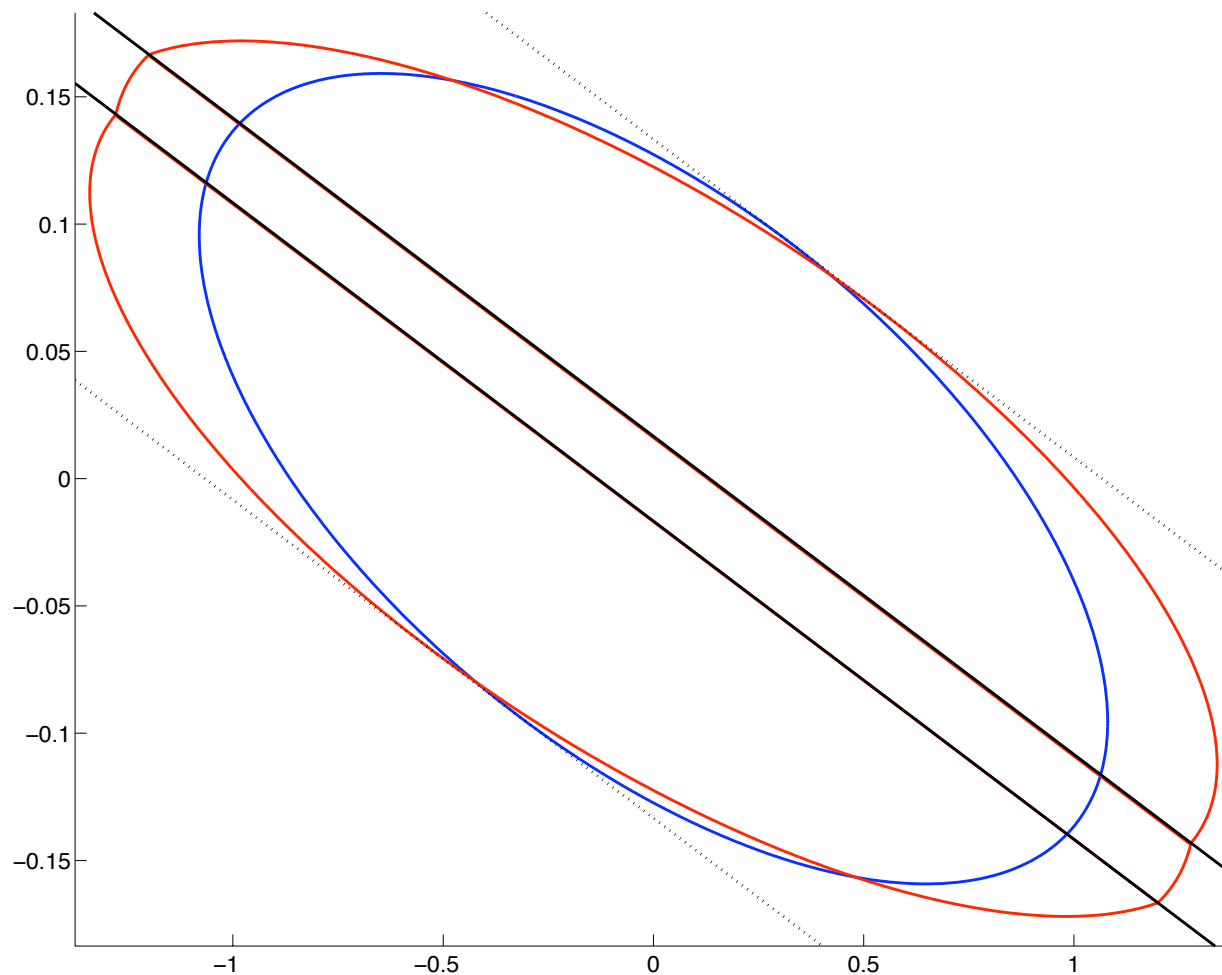
- Problem defined in this way is a well-posedness problem with  $\nabla$  composed of 5 blocs

- IQS framework applies and gives less conservative LMI conditions
- Equivalent to LaSalle invariance principle with

$$V(x) = \begin{pmatrix} x \\ v \end{pmatrix}^T Q_a \begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} x \\ dz(Cx) \end{pmatrix}^T Q_a \begin{pmatrix} x \\ dz(Cx) \end{pmatrix}$$

- ▲ Such result cannot be obtained when applying classical IQC results

## ■ LMIs tested on the launcher example



- Sets of initial conditions for which  $|g(\theta)| \leq \delta$  is guaranteed
- Improvement thanks to piecewise quadratic sets of initial conditions



■ IQS framework can handle local stability issues

● Provides LMI tests - conservative

● System augmentation + descriptor modeling = reduction of conservatism

■ Prospectives

● Improved construction of the IQS  $\equiv$  "generalized sector conditions"

● Further system augmentation with higher derivatives (?)

● Simultaneous handling of saturation, uncertainties, delays...

● Hybrid systems ?