Robust stability analysis of discrete-time systems with parametric and switching uncertainties

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- Study of LMIs for stability analysis of discrete-time polytopic systems

$$x_{k+1} = A(\theta_k)x_k \ , \quad A(\theta) = \sum_{v=1}^{\bar{v}} \theta_v A^{[v]} \ : \ \theta \in \Xi_{\bar{v}} = \left\{ \theta_{v=1\dots\bar{v}} \geq 0 \ , \ \sum_{v=1}^{\bar{v}} \theta_v = 1 \right\}$$

- Classical “quadratic stability" result [Bar85]

$$\exists P \succ 0 \ : \ A^{[v]T} P A^{[v]} - P \prec 0 \quad \forall v = 1 \dots \bar{v}$$

- PDLF result for “switching" uncertainties $\theta_k \neq \theta_{k+1}, \forall k \geq 0$ [DB01, DRI02]

$$\exists P^{[v]} \succ 0 \ : \ A^{[v]T} P^{[w]} A^{[v]} - P^{[v]} \prec 0 \ , \quad \begin{array}{l} \forall v = 1 \dots \bar{v} \\ \forall w = 1 \dots \bar{v} \end{array}$$

- PDLF result for “parametric" uncertainties $\theta_k = \phi, \forall k \geq 0$ [PABB00]

$$\begin{array}{l} \exists P^{[v]} \succ 0 \\ \exists G \end{array} \ : \ \begin{bmatrix} P^{[v]} & 0 \\ 0 & -P^{[v]} \end{bmatrix} \prec \left\{ G \begin{bmatrix} I & -A^{[v]} \end{bmatrix} \right\}^{\mathcal{S}} \ , \quad \forall v = 1 \dots \bar{v}$$

- Difference and links between the two PDLF results?
- The PDLF in both cases is $P(\theta) = \sum_{v=1}^{\bar{v}} \theta_v P^{[v]}$.

- PDLF result for "switching" descriptor systems
- Non-conservative reduction of the numerical burden
- Robustness w.r.t. parametric and switching uncertainties
- Numerical example
- Conclusions

- General descriptor models, affine in the uncertainties [CTF02, MAS03]

$$E_x(\theta_k)x_{k+1} + E_\pi(\theta_k)\pi_k = F(\theta_k)x_k$$

$$\begin{bmatrix} E_x(\theta) & E_\pi(\theta) & -F(\theta) \end{bmatrix} = \sum_{v=1}^{\bar{v}} \theta_v \begin{bmatrix} E_x^{[v]} & E_\pi^{[v]} & -F^{[v]} \end{bmatrix} \;:\; \theta \in \Xi_{\bar{v}}$$

- In this paper $\begin{bmatrix} E_x(\theta) & E_\pi(\theta) \end{bmatrix}$ is assumed square invertible $\forall \theta \in \Xi_{\bar{v}}$

- This modeling is an alternative to LFTs:

  Any rationally dependent non descriptor state-space model can be reformulated as such.

- Example: $x_{k+1} = \begin{bmatrix} -{b_k}^2/a_k & -b_k \\ 1 & 0 \end{bmatrix} x_k$ writes also as

$$\begin{bmatrix} a_k & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} x_{k+1} + \begin{bmatrix} b_k \\ 0 \\ 1 \end{bmatrix} \pi_k = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ b_k & a_k \end{bmatrix} x_k$$

PDLF result for "switching" descriptor systems

- Stability of the descriptor system with “switching" uncertainties $\theta_k \neq \theta_{k+1}, \forall k \geq 0$ if

$$\begin{array}{l}\exists P^{[v]} \succ 0 \\ \exists G^{[v]}\end{array} : \left[\begin{array}{ccc} P^{[w]} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -P^{[v]} \end{array}\right] \prec \left\{ G^{[w]} \left[\begin{array}{ccc} E_x^{[v]} & E_\pi^{[v]} & -F^{[v]} \end{array}\right] \right\}^{\mathcal{S}}, \quad \begin{array}{l}\forall v = 1 \dots \bar{v} \\ \forall w = 1 \dots \bar{v}\end{array}$$

- The proof combines characteristics of the both previously cited PDLF methods
- $G^{[v]}$ are S-variables with many interesting properties, see

The S-Variable Approach to LMI-Based Robust Control

Springer, Y. Ebihara, D. Peaucelle, D. Arzelier, 2015

- Major drawback: many large decision variables and many large LMI constraints

- Assume there exists a basis in which the descriptor matrix has θ independent columns

$$\exists T \ : \ \left[\begin{array}{cc} E_x^{[v]} & E_\pi^{[v]} \end{array} \right] T = \left[\begin{array}{cc} E_1 & E_2^{[v]} \end{array} \right] \ , \ \forall v = 1 \dots \bar{v}$$

- then the LMIs can be replaced losslessly by an LMI of the type (formulas given in the paper)

$$\begin{array}{l} \exists P^{[v]} \succ 0 \\ \exists \hat{G}^{[v]} \end{array} : N_1^{[v]T} \hat{M}(P^{[w]}, P^{[v]}) N_1^{[v]} \prec \left\{ \hat{G}^{[w]} N_2^{[v]} \right\}^{\mathcal{S}} , \quad \begin{array}{l} \forall v = 1 \dots \bar{v} \\ \forall w = 1 \dots \bar{v} \end{array}$$

- Let n be the order of the system,

  q the size of the exogenous π vector

  and p the number of θ independent columns E_1

  then :

  - the number of decision variables is reduced by $\bar{v}(3n + 2q - p)p$
  - the number of rows of the LMI problem is reduced by $\bar{v}^2 p$

- In the case of non-descriptor systems $x_{k+1} = A(\theta_k)x_k$ the two equivalent LMIs read as

$$\left[\begin{array}{cc} P^{[w]} & 0 \\ 0 & -P^{[v]} \end{array}\right] \prec \left\{ G^{[w]} \left[\begin{array}{cc} I & -A^{[v]} \end{array}\right] \right\}^{\mathcal{S}}$$

$$A^{[v]T} P^{[w]} A^{[v]} - P^{[v]} \prec 0$$

- In such cases, the S-variables are useless (known result [DRI02]).

■ In the paper we also provide a reduced lossless LMI condition for the case when there are vertex independent rows in the system representation:

$$\exists S \ : \ S \left[\begin{array}{ccc} E_x^{[v]} & E_\pi^{[v]} & -F^{[v]} \end{array} \right] = \left[\begin{array}{c} F_1 \\ F_2^{[v]} \end{array} \right] , \ \forall v = 1 \dots \bar{v}$$

● The two results can be combined for further reducing the numerical burden.

- General descriptor models, affine in both “switching" and “parametric" uncertainties

$$E_x(\theta_k,\phi)x_{k+1} + E_\pi(\theta_k,\phi)\pi_k = F(\theta_k,\phi)x_k \ , \ \ \theta \in \Xi_{\bar{v}} \ , \ \phi \in \Xi_{\bar{\mu}}$$

$$\left[\begin{array}{ccc} E_x(\theta,\phi) & E_\pi(\theta,\phi) & -F(\theta,\phi) \end{array} \right] = \sum_{v=1}^{\bar{v}} \sum_{\mu=1}^{\bar{\mu}} \theta_v \phi_\mu \left[\begin{array}{ccc} E_x^{[v,\mu]} & E_\pi^{[v,\mu]} & -F^{[v,\mu]} \end{array} \right]$$

- Stability assessed by :

$$\begin{array}{l} \exists P^{[v,\mu]} \succ 0 \\ \exists G^{[v]} \end{array} \ : \left[\begin{array}{ccc} P^{[w,\mu]} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -P^{[v,\mu]} \end{array} \right] \prec \left\{ G^{[w]} \left[\begin{array}{ccc} E_x^{[v,\mu]} & E_\pi^{[v,\mu]} & -F^{[v,\mu]} \end{array} \right] \right\}^{\mathcal{S}} , \ \begin{array}{l} \forall v = 1 \ldots \bar{v} \\ \forall w = 1 \ldots \bar{v} \\ \forall \mu = 1 \ldots \bar{\mu} \end{array}$$

- Similar size reduction methods apply for these LMIs

- The two LMI conditions expressed in the introduction are special cases of this general result.

- Considered system: $a_k y_{k+2} + {b_k}^2 y_{k+1} + a_k b_k y_k = 0$ with affine descriptor model

$$\begin{bmatrix} a_k & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} x_{k+1} + \begin{bmatrix} b_k \\ 0 \\ 1 \end{bmatrix} \pi_k = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ b_k & a_k \end{bmatrix} x_k$$

- Uncertainties bounded by $a \in [1,\ 2]$ and $b \in [-0.5,\ \beta]$
- Aim: find maximal β that preserves robust stability in the four cases
  - a_k and b_k are both time-varying (“switching")
  - a_k is switching and b is constant (“parametric")
  - a is parametric and b is switching
  - a and b are parametric

- By adding one step ahead information, the system also reads as

$$\begin{bmatrix} 0 \\ a_{k+1} \end{bmatrix} y_{k+3} + \begin{bmatrix} a_k \\ {b_{k+1}}^2 \end{bmatrix} y_{k+2} + \begin{bmatrix} {b_k}^2 \\ a_{k+1}b_{k+1} \end{bmatrix} y_{k+1} + \begin{bmatrix} a_k b_k \\ 0 \end{bmatrix} y_k = 0$$

and admits an affine descriptor representation to which the LMI conditions can be applied.

- The LMI conditions for the augmented system are less conservative (see [EPAH05, PAHG07]), but with increased numerical burden.

β (nb vars/nb rows)	original syst.	augmented syst.	true bound
a_k, b_k	0.81094 (44/64)	0.84677 (480/1536)	?
a, b_k	0.89027 (28/32)	0.90293 (144/192)	?
a_k, b	0.82658 (28/32)	0.85375 (144/192)	?
a, b	0.98059 (20/16)	0.99519 (48/24)	1

- Conclusions
  - New general result for both time varying and parametric uncertainties
  - Methodology that allows systematic reduction of numerical burden
  - Conservatism reduction achieved by system augmentation

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