

Integral Quadratic Separation Framework

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The logo for LAAS-CNRS, featuring the text "LAAS-CNRS" in a blue, sans-serif font. The text is centered between two horizontal lines: a red line above and a yellow line below.

Seminar at UFSC, Florianópolis

October 2009

① Topological separation & related theory

- Well-posedness definition and main result
- Relations with Lyapunov theory
- The case of linear uncertain systems : quadratic separation
- The Lur'e problem
- Relations with IQC framework & μ -theory
- A \mathcal{S} -procedure like result

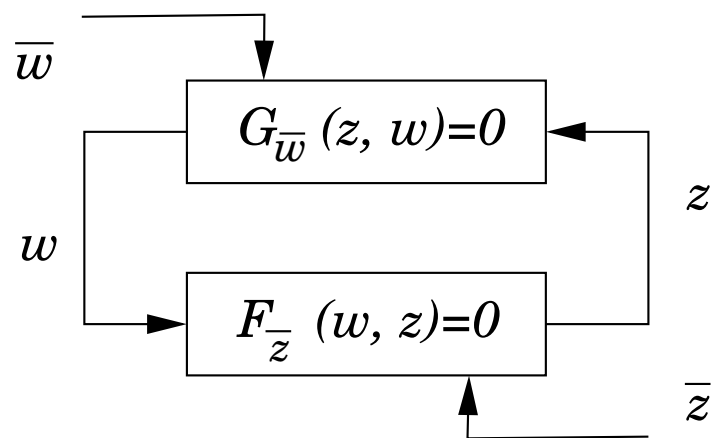
② Integral Quadratic Separation (IQS) for the descriptor case

③ Performances in the IQS framework

④ System augmentation : a way towards conservatism reduction

⑤ The Romuald toolbox

Well-posedness



Well-Posedness:

Bounded $(\bar{w}, \bar{z}) \Rightarrow$ unique bounded (w, z)

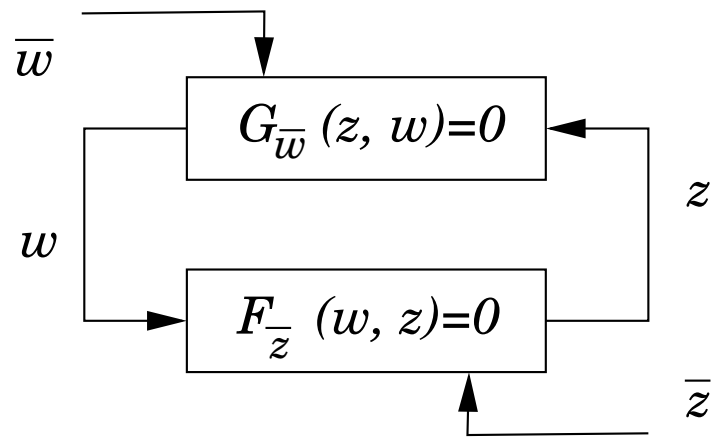
- In case $\underbrace{z = Aw + \bar{z}}_F$ and $\underbrace{w = \Delta z + \bar{w}}_G$ are linear applications

Well-posedness : $(1 - A\Delta)$ non-singular

- ▲ What if $\Delta = \triangle \in \triangle$ is uncertain ?
- ▲ If $A = T(j\omega)$ is an LTI system ?
- ▲ If G is non-linear ?

...

Well-posedness & topological separation



Well-Posedness:

Bounded (\bar{w}, \bar{z})

$\Rightarrow \exists!(w, z)$, $\exists \gamma$:

$$\left\| \begin{array}{c} w \\ z \end{array} \right\| \leq \gamma \left\| \begin{array}{c} \bar{w} \\ \bar{z} \end{array} \right\|$$

● [Safonov 80] $\exists \theta$ topological separator:

$$\mathcal{G}^I(\bar{w}) = \{(w, z) : G_{\bar{w}}(z, w) = 0\} \subset \{(w, z) : \theta(w, z) \leq \phi_2(\|\bar{w}\|)\}$$

$$\mathcal{F}(\bar{z}) = \{(w, z) : F_{\bar{z}}(w, z) = 0\} \subset \{(w, z) : \theta(w, z) > -\phi_1(\|\bar{z}\|)\}$$

- For dynamic systems $\dot{x} = f(x)$, topological separation \equiv Lyapunov theory

$$\overbrace{z(t) = f(w(t)) + \bar{z}(t)}^F, \quad \overbrace{w(t) = \int_0^t \underbrace{z(\tau)}_{\dot{x}(t)} d\tau + \bar{w}(t)}^G$$

- ▲ \bar{w} : contains information on initial conditions ($x(0) = 0$ by convention)
- Well-posedness \Rightarrow for zero initial conditions and zero perturbations :
 $w = z = 0$ (equilibrium point).
- Well-posedness (global stability)
 \Rightarrow whatever bounded perturbations the state remains close to equilibrium

1 Topological separation & related theory

■ For dynamic systems $\dot{x} = f(x)$, topological separation \equiv Lyapunov theory

$$\overbrace{z(t) = f(w(t)) + \bar{z}(t)}^F, \quad \overbrace{w(t) = \int_0^t \underbrace{z(\tau)}_{\dot{x}(t)} d\tau + \bar{w}(t)}^G$$

● Assume a Lyapunov function $V(0) = 0$, $V(x) > 0$, $\dot{V}(x) < 0$

▲ Topological separation w.r.t. $\mathcal{G}^I(\bar{w})$ is obtained with

$$\theta(w = x, z = \dot{x}) = \int_0^\infty -\frac{\partial V}{\partial x}(x(\tau)) \dot{x}(\tau) d\tau = \lim_{t \rightarrow \infty} -V(x(t)) < \gamma_1 \|\bar{w}\|$$

▲ Topological separation w.r.t. $\mathcal{F}(\bar{z})$ does hold as well

$$\theta(w, z = f(w)) = \int_0^\infty -\dot{V}(w(\tau)) d\tau > -\gamma_2 \|\bar{z}\|$$

- For linear systems : quadratic Lyapunov function, *i.e. quadratic separator*

$$\overbrace{z(t) = Aw(t) + \bar{z}(t)}^F, \quad \overbrace{w(t) = \int_0^t \underbrace{z(\tau)}_{\dot{x}(t)} d\tau + \bar{w}(t)}^G$$

- A possible separator based on quadratic Lyapunov function $V(x) = x^T P x$

$$\theta(w, z) = \int_0^\infty \begin{pmatrix} z^T(\tau) & w^T(\tau) \end{pmatrix} \begin{bmatrix} 0 & -P \\ -P & 0 \end{bmatrix} \begin{pmatrix} z(\tau) \\ w(\tau) \end{pmatrix} d\tau$$

- ▲ Quadratic separation w.r.t. $\mathcal{G}^I(\bar{w})$:

$$\lim_{t \rightarrow \infty} -x^T(t) P x(t) \leq \gamma_1 \|\bar{w}\|, \quad \text{i.e. } P > 0$$

- ▲ Quadratic separation w.r.t. $\mathcal{F}(\bar{z})$ guaranteed if

$$\forall t > 0, \quad -2w^T(t) P A w(t) > -\gamma_2 \|\bar{z}(t)\|, \quad \text{i.e. } A^T P + P A < 0$$

■ Topological separation : alternative to Lyapunov theory

▲ Needs to manipulate systems in a new form

● Suited for systems described as feedback connected blocs

Any linear system with rational dependence w.r.t. parameters writes as such

$$\dot{x} = (A + B_{\Delta} \Delta (1 - D_{\Delta} \Delta)^{-1} C_{\Delta}) x \quad \xleftrightarrow{\text{LFT}} \quad \begin{cases} \dot{x} = Ax + B_{\Delta} w_{\Delta} \\ z_{\Delta} = C_{\Delta} x + D_{\Delta} w_{\Delta} \\ w_{\Delta} = \Delta z_{\Delta} \end{cases}$$

▲ Finding a topological separator is *a priori*

as complicated as finding a Lyapunov function

● Allows to deal with several features simultaneously in a unified way

■ Quadratic separation [Iwasaki & Hara 1998]

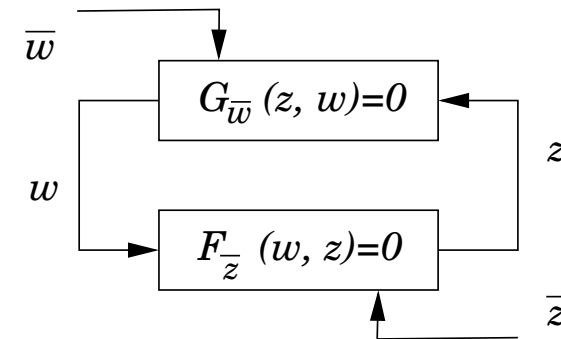
- If $F(w) = Aw$ is a linear transformation and $G = \Delta$ is an uncertain operator defined as $\Delta \in \mathbb{\Delta}$ convex set it is necessary and sufficient to look for a quadratic separator

$$\theta(z, w) = \int_0^\infty \begin{pmatrix} z^T & w^T \end{pmatrix} \Theta \begin{pmatrix} z \\ w \end{pmatrix} d\tau$$

- If $F(w) = A(\omega)w$ is a linear parameter dependent transformation and $G = \Delta$ is an uncertain operator defined as $\Delta \in \mathbb{\Delta}$ convex set necessary and sufficient to look for a parameter-dependent quadratic separator

$$\theta(z, w) = \int_0^\infty \begin{pmatrix} z^T & w^T \end{pmatrix} \Theta(\omega) \begin{pmatrix} z \\ w \end{pmatrix} d\tau$$

■ A well-known example : the Lur'e problem

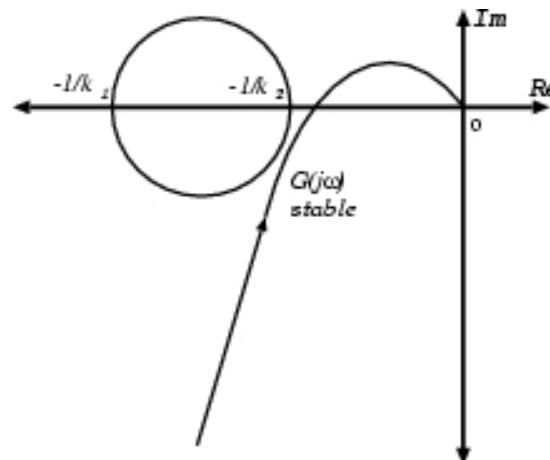


▲ $F = T(j\omega)$ is a transfer function

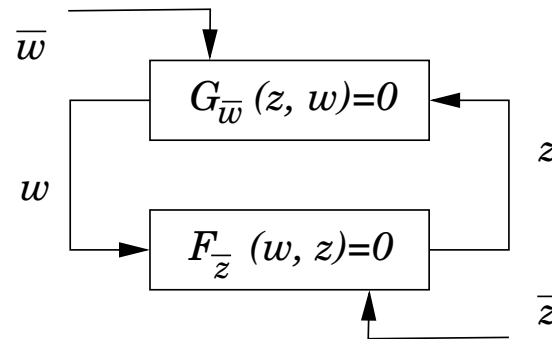
▲ $G(z)/z \in [-k_1, -k_2]$ is a sector-bounded gain

(i.e. the inverse graph of G is in $[-1/k_1, -1/k_2]$)

● Circle criterion : exists a quadratic separator (circle) for all ω



Another example : parameter-dependent Lyapunov function



▲ $F = A(\delta)$ parameter-dependent LTI state-space model

▲ $G = \mathcal{I}$ is an integrator

● Necessary and sufficient to have

$$\Theta(\delta) = \begin{bmatrix} 0 & -P(\delta) \\ -P(\delta) & 0 \end{bmatrix}$$

■ Direct relation with the IQC framework

▲ $F = T(j\omega)$ is a transfer matrix

▲ $G = \Delta$ is an operator known to satisfy an Integral Quadratic Constraint (IQC)

$$\int_{-\infty}^{+\infty} \begin{bmatrix} 1 & \Delta^*(j\omega) \end{bmatrix} \Pi(\omega) \begin{bmatrix} 1 \\ \Delta(j\omega) \end{bmatrix} d\omega \leq 0$$

● Stability of the closed-loop is guaranteed if for all ω

$$\begin{bmatrix} T^*(j\omega) & 1 \end{bmatrix} \Pi(\omega) \begin{bmatrix} T(j\omega) \\ 1 \end{bmatrix} > 0$$

▲ Knowing Δ the set of Δ how to choose $\Pi = \Theta$?

(i.e. the quadratic separator)

■ Some choices of quadratic separators \ominus

▲ Δ is full-bloc complex norm-bounded : $\Delta \in \{ \Delta^* \Delta \leq \bar{k}^2 \mathbf{1} \}$

\ominus should be such that

$$\begin{bmatrix} \mathbf{1} & \Delta^* \end{bmatrix} \begin{bmatrix} -\bar{k}^2 \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \Delta \end{bmatrix} \leq 0 \Rightarrow \begin{bmatrix} \mathbf{1} & \Delta^* \end{bmatrix} \ominus \begin{bmatrix} \mathbf{1} \\ \Delta \end{bmatrix} \leq 0$$

● \mathcal{S} -procedure ! [Yakubovitch 70's]

$$\exists \tau > 0 : \ominus \leq \tau \begin{bmatrix} -\bar{k}^2 \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{bmatrix}$$

■ Some choices of quadratic separators Θ

▲ $\Delta = \delta 1$ is scalar complex norm-bounded : $\Delta = \{ \delta 1 : |\delta| \leq \bar{k} \}$

\ominus should be such that

$$\begin{bmatrix} 1 & \delta^* \end{bmatrix} \begin{bmatrix} -\bar{k}^2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \delta \end{bmatrix} \leq 0 \Rightarrow \begin{bmatrix} 1 & \delta^* 1 \end{bmatrix} \ominus \begin{bmatrix} 1 \\ \delta 1 \end{bmatrix} \leq 0$$

● D -scaling

$$\exists D > 0 : \ominus \leq \begin{bmatrix} -\bar{k}^2 D & 0 \\ 0 & D \end{bmatrix}$$

■ Some choices of quadratic separators Θ

▲ $\Delta = \delta \mathbf{1}$ is scalar real norm-bounded : $\Delta = \{ \delta \mathbf{1} : |\delta| \leq \bar{k}, \delta = \delta^* \}$

● DG -scaling

$$\exists D > 0, G = -G^* : \Theta \leq \begin{bmatrix} -\bar{k}^2 D & G \\ G^* & D \end{bmatrix}$$

■ Some choices of quadratic separators Θ

▲ $\Delta = j\omega 1$ with $\omega \in \mathbb{R}$

● K-Y-P lemma

$$\exists P = P^* : \Theta \leq \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix}$$

■ Some choices of quadratic separators \ominus

$$\blacktriangle \Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \delta_2 \mathbf{1} \end{bmatrix}$$

Δ_1 if full-block complex norm-bounded in $\{ \Delta_1^* \Delta_1 \leq \bar{k}_1^2 \mathbf{1} \}$

δ_2 is scalar real norm-bounded in $\{ \delta_2 \mathbf{1} : |\delta_2| \leq \bar{k}_2, \delta_2 = \delta_2^* \}$

● One can take (full-block \mathcal{S} -procedure [Scherer], etc.)

$$\ominus = \left[\begin{array}{cc|cc} -\tau \bar{k}_1^2 \mathbf{1} & 0 & 0 & 0 \\ 0 & -\bar{k}_2^2 D & 0 & G \\ \hline 0 & 0 & \tau \mathbf{1} & 0 \\ 0 & G^* & 0 & D \end{array} \right]$$

■ μ -theory is a special case of IQC framework

▲ $F = T(j\omega)$ is a transfer matrix

▲ Δ is bloc-diagonal composed of

m_F full-bloc complex norm-bounded uncertainties

m_c scalar complex norm-bounded uncertainties

m_r scalar real norm-bounded uncertainties

▲ All uncertainties bounded by same \bar{k} (at the expense of modifying $T(j\omega)$)

▲ Goal : $k_m = \max \bar{k}$

● If $\mu = \frac{1}{k_m} < 1$ stability is proved.

● Convex problem with DG -scalings

(LMI for fixed \bar{k} , else Generalized Eigenvalue Problem)

- [Meinsma 97] DG -scalings are lossless if

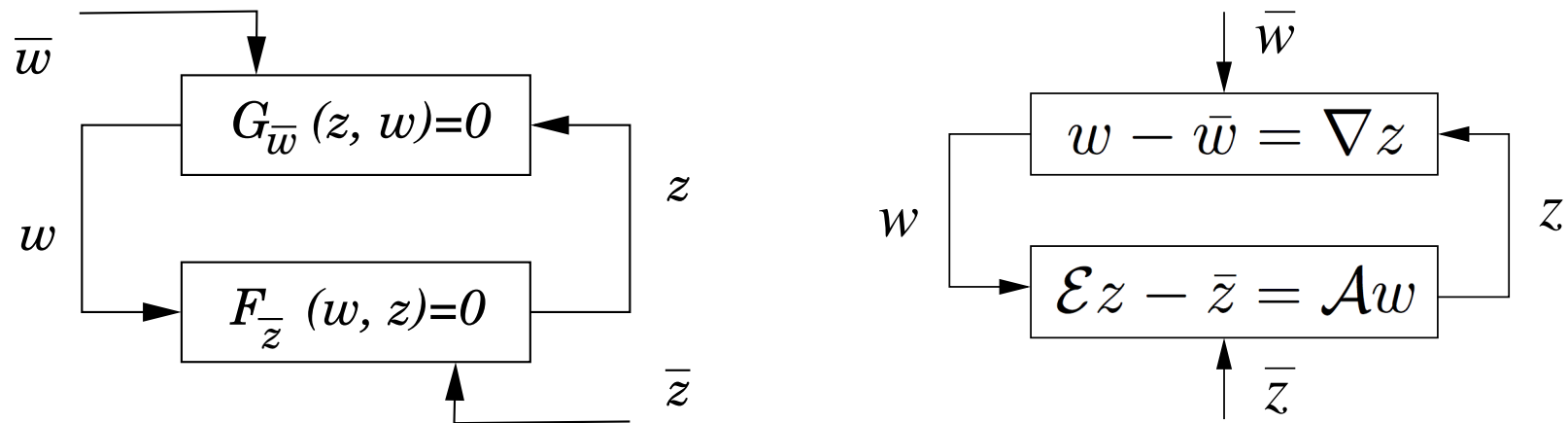
$$2(m_c + m_r) + m_F \leq 3$$

- ▲ In μ -theory needed to test for all $\omega \in \mathbb{R}$: griding techniques and ...

- Alternative is to consider s^{-1} as a scalar uncertainty
(treated as additional complex scalar bloc)

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② IQS for the descriptor case



■ Linear implicit application in feedback loop with an uncertain operator

$$\underbrace{\mathcal{E}z(t) = \mathcal{A}w(t)}_F, \quad \underbrace{w(t) = [\nabla z](t)}_G \quad \nabla \in \mathbb{W}$$

● ∇ is bloc-diagonal contains scalar, full-bloc, LTI and LTV uncertainties and other operators such as integrator etc.

■ Integral Quadratic Separator

- Well-posedness etc. is defined for bounded signals in L_2

$$z \in \mathbb{R}^p, \quad \|z\|^2 = \text{Trace} \int_0^\infty z^*(t)z(t)dt < \infty$$

- ▲ With scalar product

$$\langle z|w \rangle = \text{Trace} \int_0^\infty z^*(t)w(t)dt$$

- ▲ Notation

$$\|z\|_T^2 = \text{Trace} \int_0^T z^*(t)z(t)dt, \quad \langle z|w \rangle_T = \text{Trace} \int_0^T z^*(t)w(t)dt$$

2 IQS for the descriptor case

■ Integral Quadratic Separation [Automatica'08, CDC'08]

- For the case of linear application with uncertain operator

$$\mathcal{E}z(t) = \mathcal{A}w(t) \quad , \quad w(t) = [\nabla z](t) \quad \nabla \in \mathbb{W}$$

where $\mathcal{E} = \mathcal{E}_1 \mathcal{E}_2$ with \mathcal{E}_1 full column rank,

- Integral Quadratic Separator (IQS) : $\exists \Theta$, matrix, solution of LMI

$$\begin{bmatrix} \mathcal{E}_1 & -\mathcal{A} \end{bmatrix}^{\perp*} \Theta \begin{bmatrix} \mathcal{E}_1 & -\mathcal{A} \end{bmatrix}^{\perp} > 0$$

and Integral Quadratic Constraint (IQC) $\forall \nabla \in \mathbb{W}$

$$\int_0^{\infty} \begin{pmatrix} \mathcal{E}_2 z(t) \\ [\nabla z](t) \end{pmatrix}^* \Theta \begin{pmatrix} \mathcal{E}_2 z(t) \\ [\nabla z](t) \end{pmatrix} dt \leq 0$$

■ Integral Quadratic Separation

▲ Proof of sufficiency is starting from

$$\mathcal{E}z(t) = \mathcal{A}w(t) + \bar{z}, \quad w(t) = [\nabla z](t) + \bar{w}$$

to prove using the LMI and IQC constraints that

$$\exists \lambda : \begin{array}{l} \forall \bar{z} \in L_2, \bar{w} \in L_2 \\ \forall \nabla \in \mathbb{W} \end{array}, \quad \left\| \begin{array}{c} \mathcal{E}z \\ w \end{array} \right\| \leq \lambda \left\| \begin{array}{c} \bar{z} \\ \bar{w} \end{array} \right\|$$

▲ Proof of necessity based on Finsler lemma and builds the Θ matrix

▲ Note that z is not required to be unique and bounded, only $\mathcal{E}z$

● The Theorem is necessary and sufficient, but...

2 IQS for the descriptor case

- For some given ∇ , \exists LMI conditions for Θ solution to IQC

$$\int_0^{\infty} \begin{pmatrix} \mathcal{E}_2 z(t) \\ [\nabla z](t) \end{pmatrix}^* \Theta \begin{pmatrix} \mathcal{E}_2 z(t) \\ [\nabla z](t) \end{pmatrix} dt \leq 0$$

- ▲ Θ is build out of IQS for elementary blocs of ∇
- ▲ Improved DG -scalings, full-bloc S-procedure, vertex separators...
- ▲ Building Θ and related LMIs is tedious but can be automatized (RoMuloC)
- ▲ It is conservative except in few special cases [Meinsma et al., 1997].

② IQS for the descriptor case

■ Integral Quadratic Separation & Slack variables [ROCOND'09]

- For the case of a polytopically uncertain linear application

$$\mathcal{E}(\xi)z(t) = \mathcal{A}(\xi)w(t) \quad , \quad w(t) = [\nabla z](t) \quad \nabla \in \mathbb{W}$$

where $\mathcal{E}(\xi) = \mathcal{E}_1(\xi)\mathcal{E}_2$ with $\mathcal{E}_1(\xi)$ full column rank and

$$\begin{bmatrix} \mathcal{E}_1(\xi) & \mathcal{A}(\xi) \end{bmatrix} = \sum \xi_v \begin{bmatrix} \mathcal{E}_1^{[v]} & \mathcal{A}^{[v]} \end{bmatrix} \quad , \quad \dot{\xi}_v = 0 \quad , \quad \xi_v \geq 0 \quad , \quad \sum \xi_v = 1$$

- Integral Quadratic Separators (IQS) : $\exists \Theta^{[v]}$ and G solutions of LMIs

$$\Theta^{[v]} > G \begin{bmatrix} \mathcal{E}_1^{[v]} & -\mathcal{A}^{[v]} \end{bmatrix} + \begin{bmatrix} \mathcal{E}_1^{[v]} & -\mathcal{A}^{[v]} \end{bmatrix}^* G^* > 0$$

and the Integral Quadratic Constraints (IQC) $\forall \nabla \in \mathbb{W}$.

▲ Inspired by [Geromel], [Trofino], [De Oliveira]

▲ [ROCOND'09] Methods for reducing size of the LMIs (without conservatism).

■ Robust analysis in IQS framework:

- 1- Write the robust analysis problem as a well-posedness problem

$$\mathcal{E}(\xi)z = \mathcal{A}(\xi)w, \quad w = \nabla z$$

- 2- Build Integral Quadratic Separators for each elementary bloc of ∇
- 3- Apply the IQS results to get (conservative) LMIs

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- Induced L_2 norm (H_∞ in the LTI case)

$$E\dot{x} = Ax + Bv, \quad g = Cx + Dv$$

- ▲ Prove that system is asymptotically stable
- ▲ and $\|g\| < \gamma\|v\|$ for zero initial conditions $x(0) = 0$
(strict upper bound on the L_2 gain attenuation)

- Equivalent to well-posedness with respect to

Integrator with zero initial conditions $x(t) = [\mathcal{I}_1 \dot{x}](t) = \int_0^t \dot{x}(\tau) d\tau$

and signals such that $\|v\| \leq \frac{1}{\gamma} \|g\|$

③ Performance analysis in quadratic separation framework

■ Induced L_2 norm

$$E\dot{x} = Ax + Bv, \quad g = Cx + Dv$$

▲ Define ∇_{n2n} the fictitious non-causal uncertain operator such that

$$v = \nabla_{n2n}g \quad \text{iff} \quad \|v\| \leq \frac{1}{\gamma} \|g\|$$

● Induced L_2 norm problem is equivalent to well-posedness of

$$\underbrace{\begin{bmatrix} E & 0 \\ 0 & 1 \end{bmatrix}}_{\mathcal{E}} \underbrace{\begin{pmatrix} \dot{x} \\ g \end{pmatrix}}_z = \underbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}_{\mathcal{A}} \underbrace{\begin{pmatrix} x \\ v \end{pmatrix}}_w, \quad \nabla = \begin{bmatrix} \mathcal{I}_1 & 0 \\ 0 & \nabla_{n2n} \end{bmatrix}$$

3 Performance analysis in quadratic separation framework

■ Induced L_2 norm

$$\underbrace{\begin{bmatrix} E & 0 \\ 0 & 1 \end{bmatrix}}_{\mathcal{E}} \underbrace{\begin{pmatrix} \dot{x} \\ g \end{pmatrix}}_z = \underbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}_A \underbrace{\begin{pmatrix} x \\ v \end{pmatrix}}_w, \quad \nabla = \begin{bmatrix} \mathcal{I}_1 & 0 \\ 0 & \nabla_{n2n} \end{bmatrix}$$

● Elementary IQS for bloc \mathcal{I}_1 is

$$\Theta_{\mathcal{I}_1} = \begin{bmatrix} 0 & -P \\ -P & 0 \end{bmatrix} : P > 0$$

Indeed (recall $x(t) = [\mathcal{I}_1 \dot{x}](t) = \int_0^t \dot{x}(\tau) d\tau$ and $x(0) = 0$)

$$\left\langle \begin{pmatrix} \dot{x} \\ \mathcal{I}_1 \dot{x} \end{pmatrix} \middle| \Theta_{\mathcal{I}_1} \begin{pmatrix} \dot{x} \\ \mathcal{I}_1 \dot{x} \end{pmatrix} \right\rangle_T = -x^*(T) P x(T) \leq 0$$

③ Performance analysis in quadratic separation framework

■ Induced L_2 norm

$$\underbrace{\begin{bmatrix} E & 0 \\ 0 & 1 \end{bmatrix}}_{\mathcal{E}} \underbrace{\begin{pmatrix} \dot{x} \\ g \end{pmatrix}}_z = \underbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}_A \underbrace{\begin{pmatrix} x \\ v \end{pmatrix}}_w, \quad \nabla = \begin{bmatrix} \mathcal{I}_1 & 0 \\ 0 & \nabla_{n2n} \end{bmatrix}$$

● Elementary IQS for bloc ∇_{n2n} is (small gain theorem)

$$\Theta_{\nabla_{n2n}} = \begin{bmatrix} -\tau \mathbf{1} & 0 \\ 0 & \tau \gamma^2 \mathbf{1} \end{bmatrix} : \tau > 0$$

Indeed (recall $v = \nabla_{n2n} g$ iff $\|v\| \leq \frac{1}{\gamma} \|g\|$)

$$\left\langle \begin{pmatrix} g \\ \nabla_{n2n} g \end{pmatrix} \middle| \Theta_{\nabla_{n2n}} \begin{pmatrix} g \\ \nabla_{n2n} g \end{pmatrix} \right\rangle = \tau (-\|g\|^2 + \gamma^2 \|v\|^2) \leq 0$$

③ Performance analysis in quadratic separation framework

- Apply IQS and get (for non-descriptor case $E = 1$)

$$P > 0, \quad \tau > 0$$
$$\begin{bmatrix} A^*P + PA + \tau C^*C & PB + \tau C^*D \\ B^*P + \tau D^*C & -\tau\gamma^2\mathbf{1} + \tau D^*D \end{bmatrix} < 0$$

which is the classical H_∞ result.

- No difficulty to generate LMIs for descriptor case
& if there are more blocs in ∇ such as uncertainties ...

- Impulse to norm performance (H_2 in the LTI case if $D = 0$)

$$E\dot{x} = Ax + Bv, \quad g = Cx + Dv$$

- ▲ Prove that system is asymptotically stable
- ▲ and $\|g\| < \gamma$ if $v = \alpha\delta(t)\mathbf{1}_m$, $|\alpha| \leq 1$ and zero initial conditions $x(0) = 0$
- ! The Dirac delta function $\delta(t)$ is not in L_2
- Impulse inputs define jumps of the state

③ Performance analysis in quadratic separation framework

- Impulse to norm performance (H_2 in the LTI case if $D = 0$)

$$E\dot{x} = Ax + Bv, \quad g = Cx + Dv$$

- ▲ Prove that system is asymptotically stable
- ▲ and $\|g\| < \gamma$ if $v = \alpha\delta(t)\mathbf{1}_m$, $|\alpha| \leq 1$ and zero initial conditions $x(0) = 0$
- Redefinition of the problem :

$$Ex(0) = \alpha B, \quad g(0) = \alpha D$$

$$E\dot{x}(t > 0) = Ax(t > 0), \quad g(t > 0) = Cx(t > 0)$$

- ▲ Prove that system is asymptotically stable
- ▲ and $\|g\| < \gamma$ for all $\alpha \leq 1$
- Need to describe initial conditions as signals in L_2

■ Square-root of the shifted delta function φ_θ :
$$\begin{cases} L_2 \longrightarrow L_2 \\ x \longmapsto \varphi_\theta x \end{cases}$$

with properties that φ_θ is linear, and whatever x, y in L_2 and whatever P :

$$[\varphi_\theta y]^*(t)P[\varphi_\theta x](t) = \delta(t - \theta)y^*(t)Px(t)$$

$$[\varphi_{\theta_1} y]^*(t)P[\varphi_{\theta_2} x](t) = 0 \quad \text{if } \theta_1 \neq \theta_2$$

● A formal definition:

$[\varphi_\theta x](t) = \varphi(t - \theta)x(t)$ where φ is the limit of complex valued functions

$$\varphi(t) = \lim_{\epsilon \rightarrow 0} \frac{\sqrt{\epsilon/\pi}}{t + j\epsilon} \quad \left(\lim_{\epsilon \rightarrow 0} \frac{\epsilon/\pi}{(t - j\epsilon)(t + j\epsilon)} = \delta(t) \right)$$

● $\varphi_0 x$ is an L_2 signal that contains the information $x(0)$.

3 Performance analysis in quadratic separation framework

■ Impulse to norm performance equivalent to well-posedness of

$$\underbrace{\begin{bmatrix} E & 0 & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathcal{E}} \underbrace{\begin{pmatrix} \varphi_0 x \\ \dot{x} \\ \varphi_0 g \\ g \end{pmatrix}}_z = \underbrace{\begin{bmatrix} 0 & B \\ A & 0 \\ 0 & D \\ C & 0 \end{bmatrix}}_A \underbrace{\begin{pmatrix} x \\ v \end{pmatrix}}_w, \quad \nabla = \begin{bmatrix} \mathcal{I}_2 & 0 \\ 0 & \nabla_{i2n} \end{bmatrix}$$

▲ \mathcal{I}_2 is the integrator with non-zero initial conditions

$$x(t) = \begin{bmatrix} \mathcal{I}_2 \\ \end{bmatrix} \begin{pmatrix} \varphi_0 x \\ \dot{x} \end{pmatrix} (t) = x(0) + \int_0^t \dot{x}(\tau) d\tau$$

$$v = \nabla_{i2n} \begin{pmatrix} \varphi_0 g \\ g \end{pmatrix} : v = \alpha \varphi_0 \mathbf{1}_m, \quad |\alpha| \leq \frac{1}{\gamma} \left\| \begin{pmatrix} \varphi_0 g \\ g \end{pmatrix} \right\|$$

3 Performance analysis in quadratic separation framework

- Elementary IQS for bloc \mathcal{I}_2 is

$$\Theta_{\mathcal{I}_2} = \left[\begin{array}{cc|c} -P & 0 & 0 \\ 0 & 0 & -P \\ \hline 0 & -P & 0 \end{array} \right] : P > 0$$

Indeed (recall $x(t) = [\mathcal{I}_2 \begin{pmatrix} \varphi_0 x \\ \dot{x} \end{pmatrix}](t) = x(0) + \int_0^t \dot{x}(\tau) d\tau$)

$$\left\langle \begin{pmatrix} \varphi_0 x \\ \dot{x} \\ \hline x \end{pmatrix} \middle| \Theta_{\mathcal{I}_2} \begin{pmatrix} \varphi_0 x \\ \dot{x} \\ \hline x \end{pmatrix} \right\rangle_T = -\text{Trace}(x^*(T) P x(T)) \leq 0$$

3 Performance analysis in quadratic separation framework

- Elementary IQS for bloc ∇_{i2n} is

$$\Theta_{\nabla_{i2n}} = \left[\begin{array}{cc|c} -\tau \mathbf{1} & 0 & 0 \\ 0 & -\tau \mathbf{1} & 0 \\ \hline 0 & 0 & Q \end{array} \right] : \text{Trace}(Q) < \tau \gamma^2$$

Indeed (recall $v = \nabla_{i2n} \begin{pmatrix} \varphi_0 g \\ g \end{pmatrix} : v = \alpha \varphi_0 \mathbf{1}_m$, $|\alpha| \leq \frac{1}{\gamma} \left\| \begin{pmatrix} \varphi_0 g \\ g \end{pmatrix} \right\|$)

$$\left\langle \begin{pmatrix} \varphi_0 g \\ g \\ v \end{pmatrix} \middle| \Theta_{\nabla_{i2n}} \begin{pmatrix} \varphi_0 g \\ g \\ v \end{pmatrix} \right\rangle = -\tau \left\| \begin{pmatrix} \varphi_0 g \\ g \end{pmatrix} \right\|^2 + \alpha^2 \text{Trace}(Q) \leq 0$$

- Apply IQS and get (for non-descriptor case $E = 1$)

$$P > 0 \quad , \quad \tau > 0 \quad , \quad \text{Trace}(Q) \leq \tau\gamma^2$$
$$A^*P + PA + \tau C^*C < 0 \quad , \quad Q > B^*PB + \tau D^*D$$

which is the classical H_2 result (when $D = 0$) as expected.

- No difficulty to generate LMIs for descriptor case
& if there are more blocs in ∇ such as uncertainties ...

■ Impulse to peak performance

$$E\dot{x} = Ax + Bv, \quad g = Cx + Dv$$

▲ Prove that system is asymptotically stable

▲ and $\max_{t \geq 0} \|g(t)\| < \gamma$ if $v = \delta(t)\alpha$, $\|\alpha\| \leq 1$ and $x(0) = 0$

● Redefinition of the problem :

▲ Let $\theta = \arg \max_{t \geq 0} \|g(t)\|$ (unknown positive or zero)

$$Ex(0) = B\alpha, \quad g(0) = D\alpha$$

$$E\dot{x}(\theta > t > 0) = Ax(\theta > t > 0), \quad g(\theta) = Cx(\theta)$$

▲ Prove that system is asymptotically stable

▲ and $\|g(0)\| < \gamma$, $\|g(\theta)\| < \gamma$ for all $\|\alpha\| \leq 1$

■ Need to describe final conditions.

3 Performance analysis in quadratic separation framework

■ Truncation operator \mathbb{T}_θ :
$$\begin{cases} L_2 \longrightarrow L_2 \\ x \longmapsto \mathbb{T}_\theta x \end{cases}$$

with properties
$$\begin{cases} [\mathbb{T}_\theta x](t) = x(t) & \forall t \in [0, \theta] \\ [\mathbb{T}_\theta x](t) = 0 & \forall t > \theta \end{cases}$$

▲ Integration \mathcal{I}_3 maps
$$\begin{pmatrix} \varphi_0 x \\ \mathbb{T}_\theta \dot{x} \end{pmatrix} \text{ to } \begin{pmatrix} \mathbb{T}_\theta x \\ \varphi_\theta x \end{pmatrix}$$

$$\left[\begin{array}{l} \mathcal{I}_3 \begin{pmatrix} \varphi_0 x \\ \mathbb{T}_\theta \dot{x} \end{pmatrix} \\ \mathcal{I}_3 \begin{pmatrix} \varphi_0 x \\ \mathbb{T}_\theta \dot{x} \end{pmatrix} \end{array} \right] (t) = x(0) + \int_0^t \dot{x} d\tau = x(t) = \mathbb{T}_\theta x(t) , \quad \forall t \in [0, \theta]$$
$$\left[\begin{array}{l} \mathcal{I}_3 \begin{pmatrix} \varphi_0 x \\ \mathbb{T}_\theta \dot{x} \end{pmatrix} \end{array} \right] (t) = x(0) + \int_0^\theta \dot{x} d\tau = x(\theta) , \quad \forall t > \theta .$$

3 Performance analysis in quadratic separation framework

- Impulse to peak performance equivalent to well-poedness of

$$\underbrace{\left[\begin{array}{cc|cc} E & 0 & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]}_{\mathcal{E}} \underbrace{\begin{pmatrix} \varphi_0 x \\ \frac{\mathbb{T}_\theta \dot{x}}{\varphi_0 g} \\ \varphi_\theta g \end{pmatrix}}_z = \underbrace{\left[\begin{array}{cc|cc} 0 & 0 & 0 & B \\ A & 0 & 0 & 0 \\ 0 & 0 & D & 0 \\ 0 & C & 0 & 0 \end{array} \right]}_A \underbrace{\begin{pmatrix} \mathbb{T}_\theta x \\ \frac{\varphi_\theta x}{v_0} \\ v_\theta \end{pmatrix}}_w$$

$$\nabla = \begin{bmatrix} \mathcal{I}_3 & 0 & 0 \\ 0 & \nabla_{i2p,0} & 0 \\ 0 & 0 & \nabla_{i2p,\theta} \end{bmatrix}$$

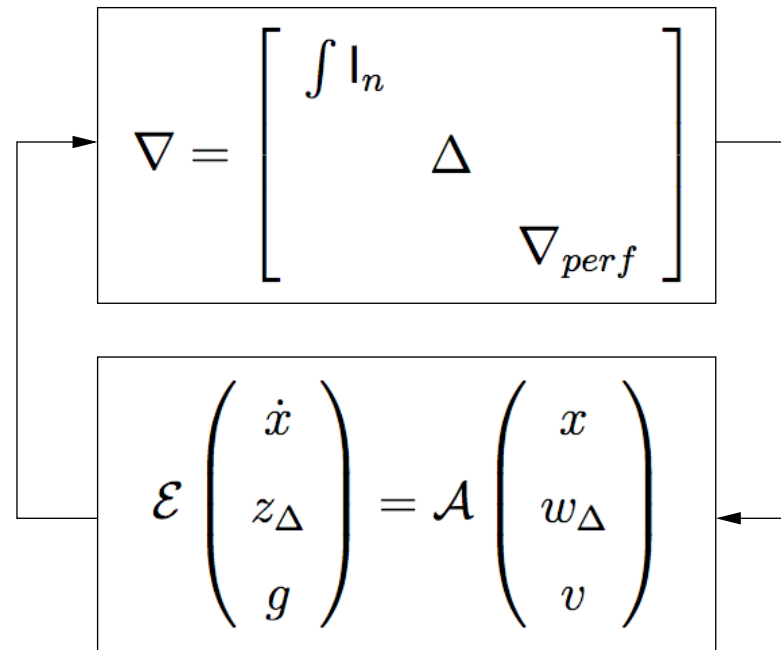
where $v_\theta = \nabla_{i2p,\theta} \varphi_\theta g : v = \varphi_0 \bar{v}$, $\bar{v}^* \bar{v} \leq \frac{1}{\gamma^2} \langle \varphi_\theta g | \varphi_\theta g \rangle$

... LMIs can be produced in the same way as for other performances...

③ Performance analysis in quadratic separation framework

■ Generic robust performance analysis problem:

● Well-posedness of



▲ $\int 1_n$ integrator

▲ Δ matrix of uncertainties

▲ ∇_{perf} operator related to performances

(induced L_2 , H_∞ / H_2 , impulse-to-norm, norm-to-peak / impulse-to-peak)

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- ⑤ The Romuald toolbox

4 System augmentation and conservatism reduction

■ General formulation of robust performance analysis

● Well-posedness of

$$\underbrace{\mathcal{E}z(t) = \mathcal{A}w(t)}_F, \quad \underbrace{w(t) = [\nabla z](t)}_G \quad \nabla \in \mathbb{W}$$

where ∇ contains

▲ integrator $\mathcal{I}_1, \mathcal{I}_2$ or \mathcal{I}_3 (or delay operators for discrete-time systems)

▲ performance operator $\nabla_{n2n}, \nabla_{i2n}$ or ∇_{i2p}

▲ delay operators $\left\{ \begin{array}{l} x(t-d) = [\mathcal{D}_0 x](t) \\ x(t) - x(t-d) = [\mathcal{D}_1 \dot{x}](t) \\ \dots \end{array} \right.$ see [Gouaisbaut]

▲ uncertainties Δ of norm-bounded type (and others : polytopes...)

● LMI results based on DG -scaling type separators

■ May be conservative as soon as more than 2 blocs !

4 System augmentation and conservatism reduction

■ Towards less-conservative conditions: System augmentation

▲ Example of stability of uncertain system with parametric uncertainty ($\dot{\delta} = 0$)

$$\dot{x} = (A + \delta B_{\Delta}(1 - \delta D_{\Delta})^{-1}C_{\Delta})x$$

▲ Corresponds to well-posedness of

$$\begin{pmatrix} \dot{x} \\ z_{\Delta} \end{pmatrix} = \begin{bmatrix} A & B_{\Delta} \\ C_{\Delta} & D_{\Delta} \end{bmatrix} \begin{pmatrix} x \\ w_{\Delta} \end{pmatrix}, \quad \nabla = \begin{bmatrix} \mathcal{I}_1 \mathbf{1}_n & 0 \\ 0 & \delta \mathbf{1}_m \end{bmatrix}$$

▲ [Meinsma] rule indicates results may be conservative

4 System augmentation and conservatism reduction

▲ Well-posedness of

$$\begin{pmatrix} \dot{x} \\ z_\Delta \end{pmatrix} = \begin{bmatrix} A & B_\Delta \\ C_\Delta & D_\Delta \end{bmatrix} \begin{pmatrix} x \\ w_\Delta \end{pmatrix}, \quad \nabla = \begin{bmatrix} \mathcal{I}_1 \mathbf{1}_n & 0 \\ 0 & \delta \mathbf{1}_m \end{bmatrix}$$

● adding the fact that $\dot{w}_\Delta = \delta \dot{z}_\Delta$, is also equivalent to well-posedness of

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -C_\Delta & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array} \right] \begin{pmatrix} \dot{z}_\Delta \\ \dot{x} \\ z_\Delta \\ \dot{z}_\Delta \end{pmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & A & B_\Delta & 0 \\ 0 & C_\Delta & D_\Delta & 0 \\ 0 & 0 & 0 & D_\Delta \\ 1 & 0 & 0 & 0 \end{array} \right] \begin{pmatrix} z_\Delta \\ x \\ w_\Delta \\ \dot{w}_\Delta \end{pmatrix}$$

$$\nabla = \begin{bmatrix} \mathcal{I}_1 \mathbf{1}_{m+n} & 0 \\ 0 & \delta \mathbf{1}_{2m} \end{bmatrix}$$

4 System augmentation and conservatism reduction

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -C_{\Delta} & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array} \right] \begin{pmatrix} \dot{z}_{\Delta} \\ \dot{x} \\ z_{\Delta} \\ \dot{z}_{\Delta} \end{pmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & A & B_{\Delta} & 0 \\ 0 & C_{\Delta} & D_{\Delta} & 0 \\ 0 & 0 & 0 & D_{\Delta} \\ 1 & 0 & 0 & 0 \end{array} \right] \begin{pmatrix} z_{\Delta} \\ x \\ w_{\Delta} \\ \dot{w}_{\Delta} \end{pmatrix}$$

$$\nabla = \begin{bmatrix} \mathcal{I}_1 \mathbf{1}_{m+n} & 0 \\ 0 & \delta \mathbf{1}_{2m} \end{bmatrix}$$

- ▲ It is descriptor model.
- More decisions variables in the separator (increased dimensions of ∇)

4 System augmentation and conservatism reduction

- Lyapunov function is with respect to the augmented state
(vector involved in the integrator operator)

$$\begin{pmatrix} z_{\Delta}^* & x^* \end{pmatrix} P \begin{pmatrix} z_{\Delta} \\ x \end{pmatrix}$$

- ▲ Recalling that



$$z_{\Delta} = \delta(1 - \delta D_{\Delta})^{-1} C_{\Delta} x$$

the result corresponds to looking for a parameter dependent Lyapunov function

$$x^* \begin{bmatrix} \delta(1 - \delta D_{\Delta})^{-1} C_{\Delta} \\ 1 \end{bmatrix}^* P \begin{bmatrix} \delta(1 - \delta D_{\Delta})^{-1} C_{\Delta} \\ 1 \end{bmatrix} x$$

- Proves to be less conservative than for LMIs obtained on original system.

4 System augmentation and conservatism reduction

- Towards less-conservative conditions: System augmentation
- Adding more equations for higher derivatives of the state:
 - less conservative LMI conditions
- Same technique works for time varying uncertainties
 - (if known bounds on derivatives)
- Has been applied successfully to time-delay systems [Gouaisbaut]:
 - gives sequences of LMI conditions with decreasing conservatism
- ▲ Related to SOS representations of positive polynomials [Sato 2009]:
 - conservatism decreases as the order of the representation is augmented
- No need to manipulate by hand LMIs (Schur complements etc.), polynomials...
- ▲ Does conservatism vanishes? Exactly? Asymptotically? 
- ▲ Is it possible to cope with non-linearities in the same way? 

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■ Freely distributed software to test the theoretical results

● Existing software : RoMulOC

`www.laas.fr/OLOCEP/romuloc`

▲ Contains some of the analysis results plus some state-feedback features

● Currently developed software : Romuald

▲ Dedicated to analysis of descriptor systems

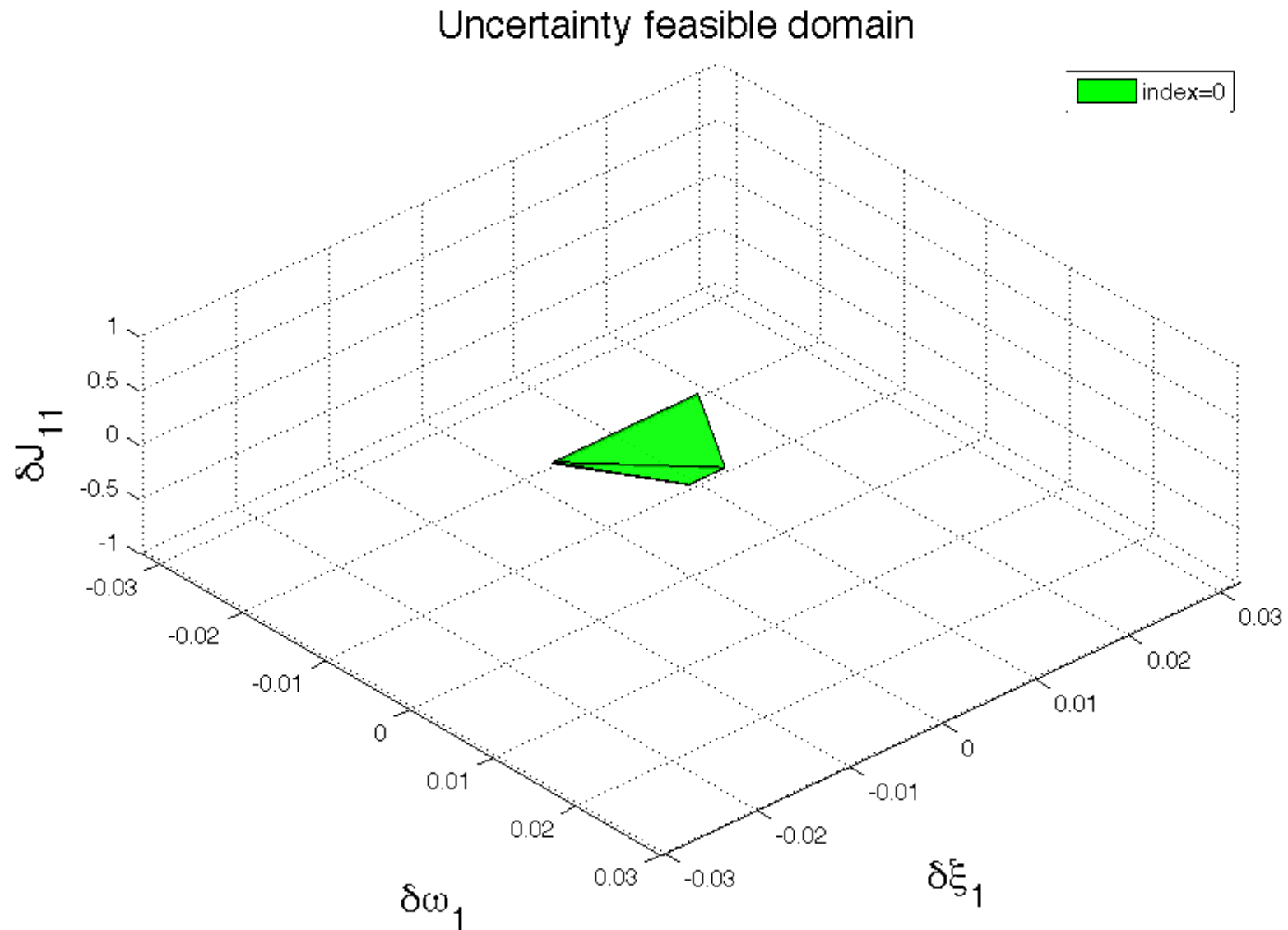
▲ Fully coded using the quadratic separation theory

▲ Allows systematic system augmentation

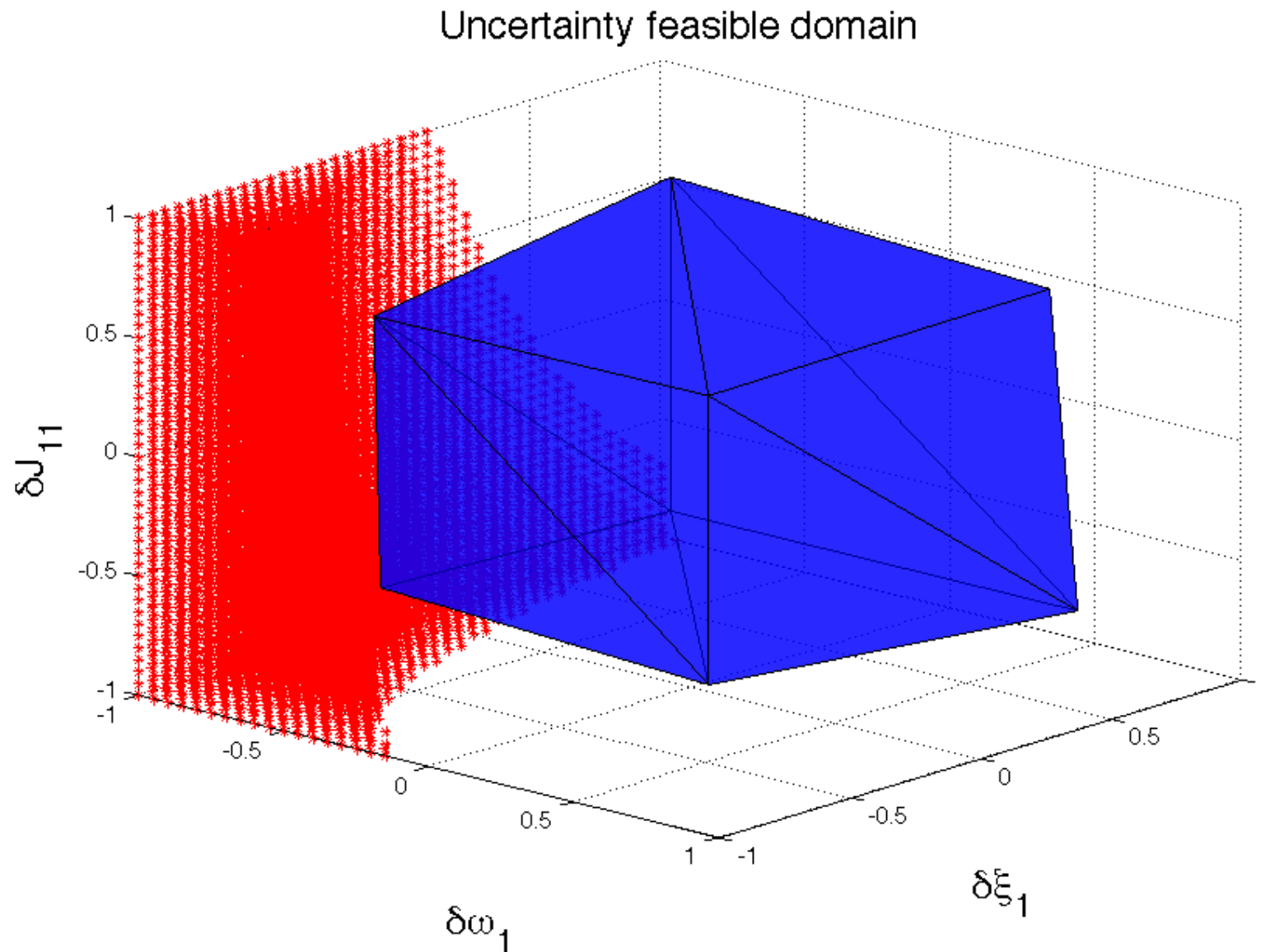
▲ First preliminary tests currently done for satellite and plane applications

```
>> quiz = ctrpb( OrderOfAugmentation ) + h2 (usys);  
>> result = solvesdp( quiz )
```

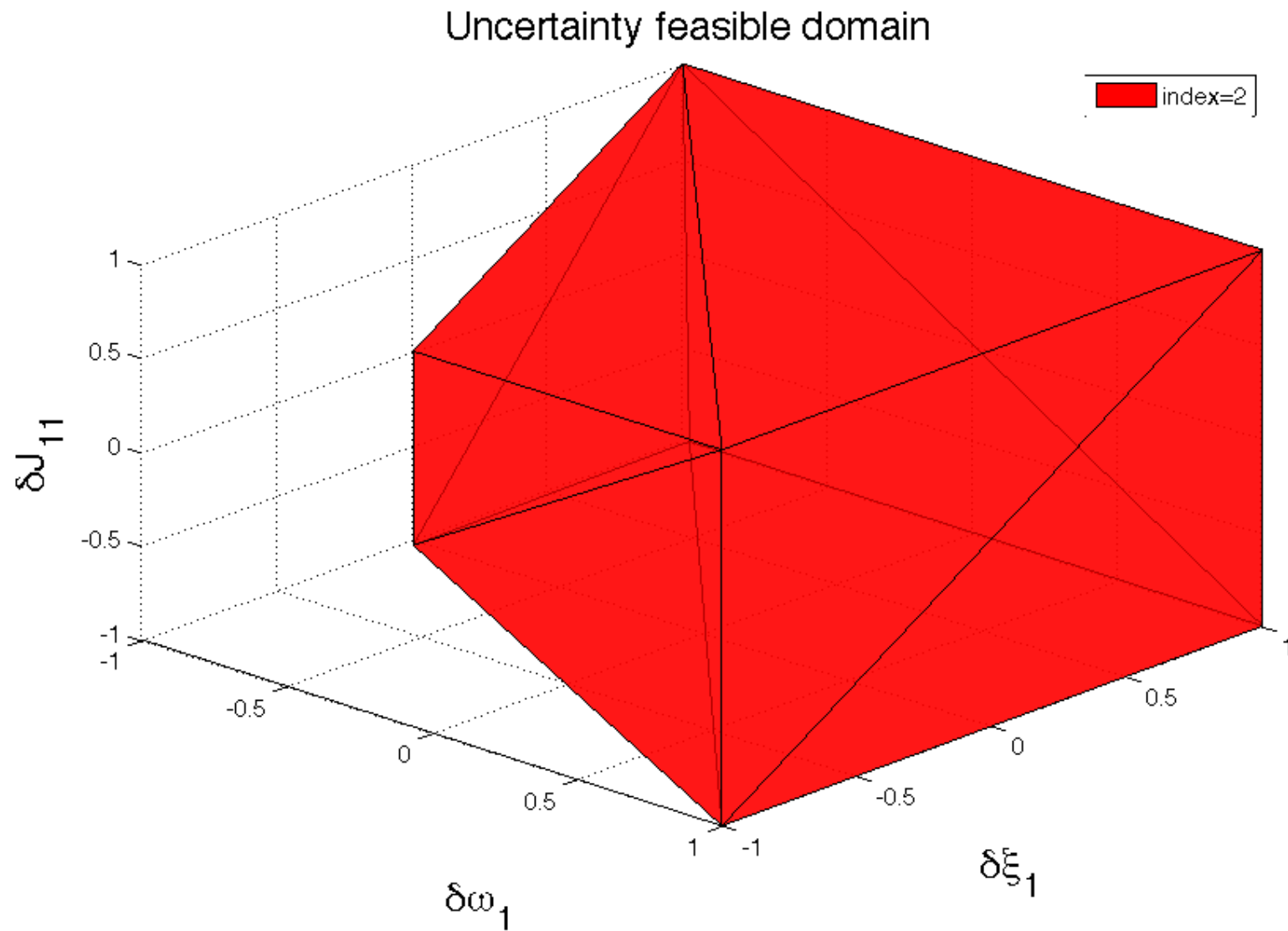
5 The Romuald toolbox



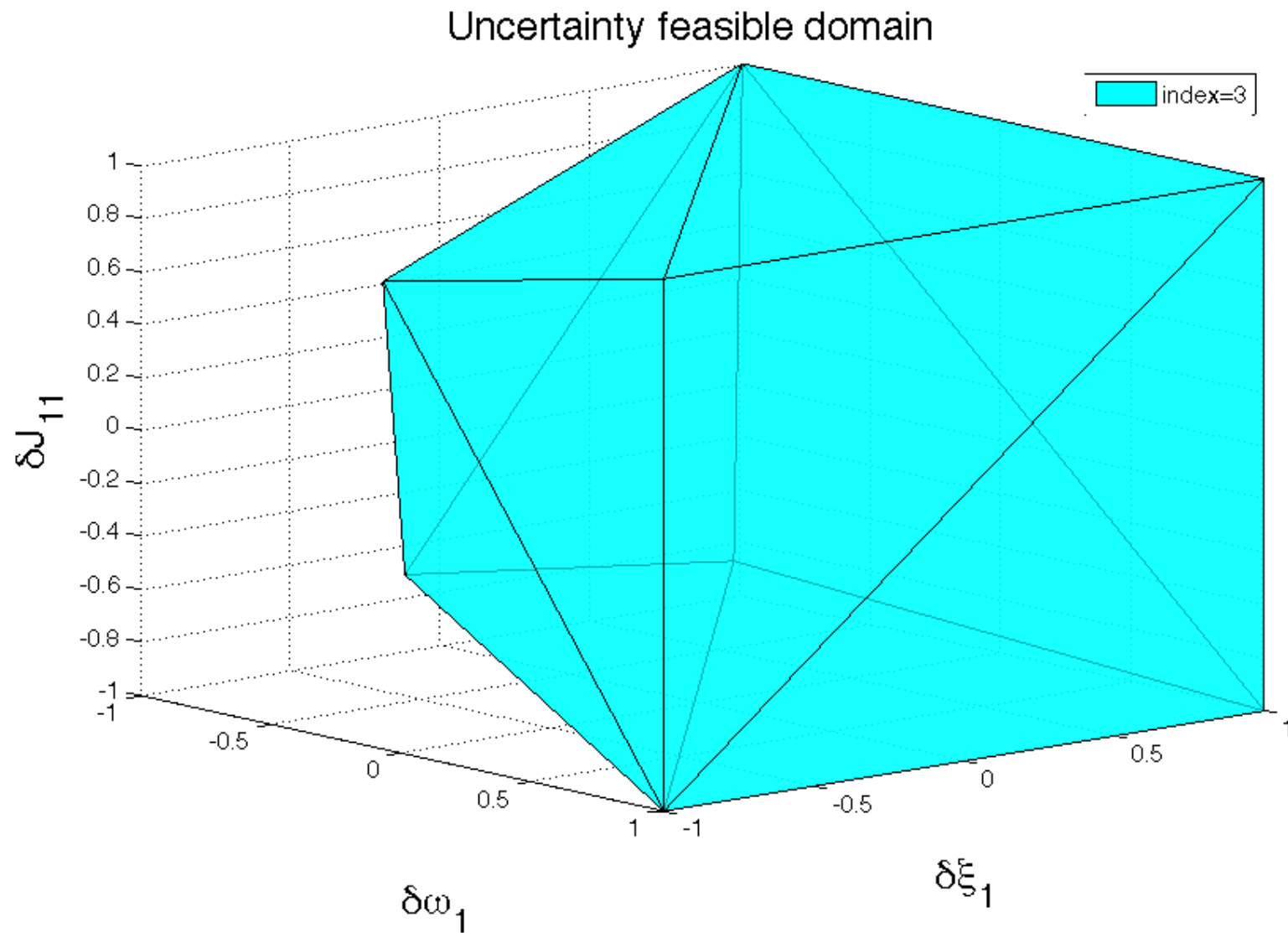
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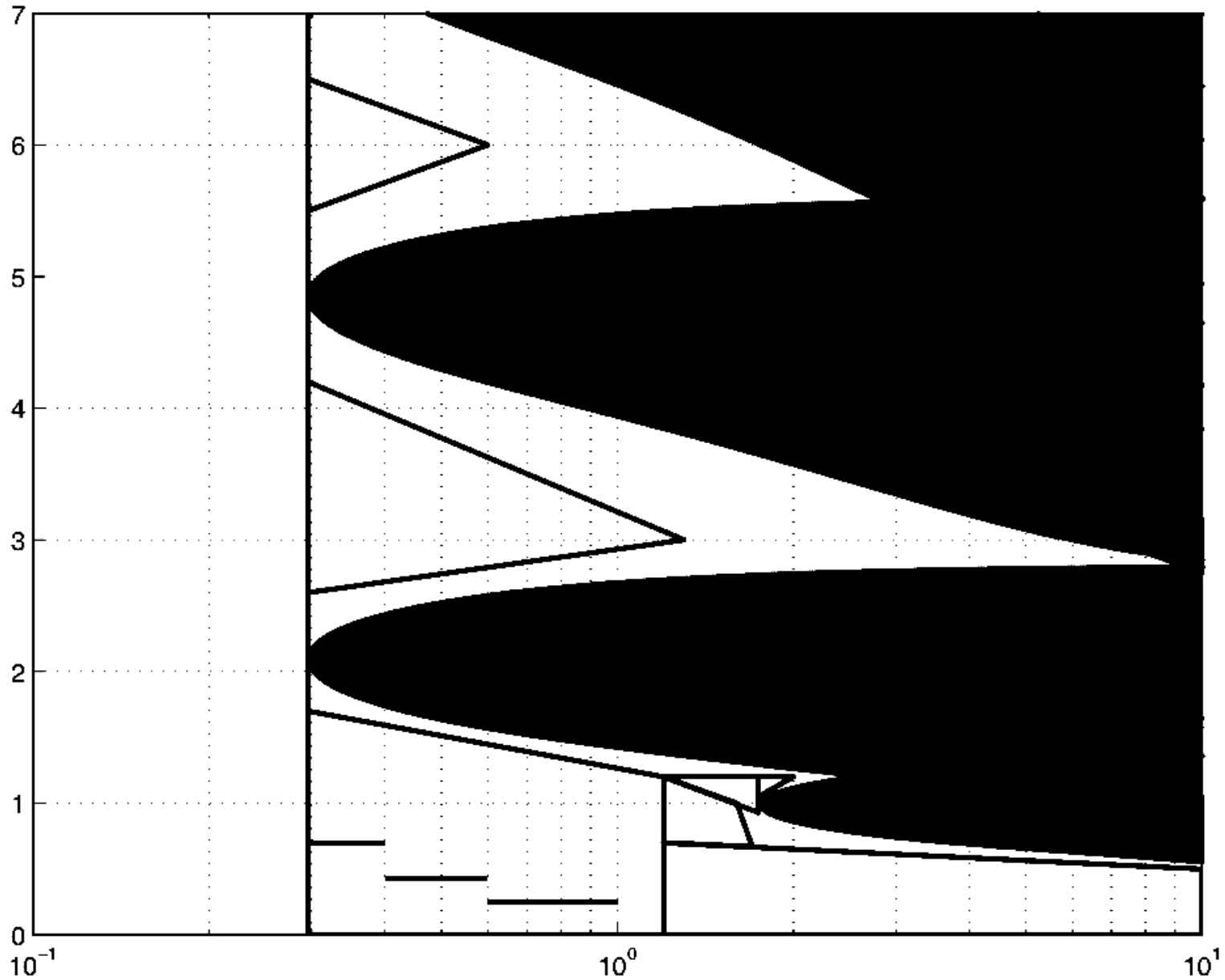
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Conclusions

