# Delay-Dependent Stability Analysis of Linear Time Delay Systems

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**Abstract**

Stability of time delay systems is investigated considering the delay-dependent case. The system without delays is assumed stable and conservative conditions are derived for finding the maximal delay that preserves stability. The problem is treated in the quadratic separation framework and the resulting criteria are formulated as feasibility problems of Linear Matrix Inequalities. The construction of the results relies on a fractioning of the delay. As the fractioning becomes thinner, the results prove to be less and less conservative. An example show the effectiveness of the proposed technique.

## 1 Introduction

During the last decade, the stability analysis of time delay systems with LMI based methods have attracted a large number of researchers (Moon *et al.*, 2001; Park, 1999; Xu and Lam, 2005; Fridman and Shaked, 2002). The criteria developed can be classified into two major categories: the delay independent case and the delay dependent case. The first category assumes the delay unknown and possibly unbounded. The criteria therefore does not depend on the size of the delay, whereas the second category makes use of information on the maximal length of the delay. The delay-dependent case starts form a system stable without delays and looks for the maximal delay that preserves stability. Conservatism of the developed methods is illustrated on examples by the maximal attainable value on the non destabilizing delay. The present paper improves significantly this attainable value as illustrated on an example.

Generally, all the methods involve a Lyapunov functional (or function), and more or less tight techniques to bound some cross terms (Kolmanovskii and Richard, 1999; Suplin *et al.*, 2004). These choices of specific Lyapunov functions and over-bounding techniques are the origin of conservatism. In the present paper we show that these conservative steps have an alternative interpretation in terms of quadratic separation.

The concept of Topological Separation (Safonov, 1980) is an alternative framework to Lyapunov Theory for proving stability of systems. In that framework, stability (or well-posedness) is defined with respect to some feedback connection and is proved at the expense of finding a separator function. Depending on the feedback connection modeling, this separator has direct relation with Lyapunov functionals. In case of linear systems, the separator may be chosen as a quadratic function as demonstrated in (Iwasaki and Hara, 1998). These last results where extended for descriptor systems in (Peaucelle *et al.*, 2005) and are applied to delay-dependent stability analysis in the present paper.

A closely related, alternative to the Lyapunov framework, approach can also be found in (Zhang *et al.*, 2001; Gu *et al.*, 2003). In these, delay dependent results are obtained applying small-gain methodology (which is a sub-case of the general quadratic separation) to an artificially modified "comparison" system that involves not only the Laplace operator $s$ and the delay operator $e^{-sh}$ but also a combination of these two: $(e^{-sh} - 1)/s$. Delay-dependent results are derived due to the fact that this last operator is norm bounded by $h$.

Taking advantage of the same combined operator, the present paper extends existing results by introducing artificially fractions of the delay. This methodology that can be compared to the artificial augmentation of systems used in (Ebihara *et al.*, 2005), leads to a sequence of LMI conditions of growing dimensions that prove to have decreasing conservatism.

The paper is organized as follows. In the next section, some preliminaries are given and the result of quadratic separation is recalled. The third section is then devoted to the main delay-dependent LMI result that depends on a fractioning of the delay. Section 4 is dedicated to the relationship between the results based on quadratic separation and the usual Lyapunov-Kasovskii framework. The Lyapunov like interpretation then allows in the following section to prove conservatism reduction as the fractioning grows. Section 6 illustrates the theoretical results on a classical academic example. Finally the papers closes on some conclusions are prospectives.

## 2 Preliminaries

### *2.1 Notations*

Throughout the paper, the following notations are used. For a two symmetric matrices, $A$ and $B$, $A > (\geq) B$ means that $A - B$ is (semi-) positive definite. $A^T$ denotes the transpose of $A$, $1_n$ and $0_{m,n}$ denote the respectively the identity matrix of size $n$ and null matrix of size $m \times n$. If the context allows it the dimensions of these matrices are often omitted. For a given matrix $B \in \mathbb{R}^{m\times n}$ such that $rank(B) = r$, we define $B^{\perp} \in \mathbb{R}^{n\times(n-r)}$ the right orthogonal complement of $B$ by $BB^{\perp} = 0$ and $B^{\perp}B^{\perp T} > 0$. The set of complex numbers of the right-half of the complex plane is denoted $\mathbb{C}^+ = \{s \in \mathbb{C} \ : \ s + s^* \geq 0\}$.

### 2.2 Problem statement

Let consider the following time delay system:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-h), \ \forall t \geq 0 \\ x(t) = \phi(t), \ \forall t \in [-h, 0] \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the instantaneous state, $\phi$ is the initial condition and $A, A_d \in \mathbb{R}^{n\times n}$ are known constant matrices. We denote the state of the system $x_t$ as :

$$x_t(.) : \begin{cases} [-h, 0] \rightarrow \mathbb{R}^n \\ \theta \mapsto x_t(\theta) = x(t+\theta) \end{cases}$$

It is assumed that $A + A_d$ is Hurwitz, *i.e.* the system with zero delay $h = 0$ is stable. The goal of the paper is to give conditions for finding the largest interval $[0\ \bar{h}]$ such that for all $h$ in this interval the delay system is stable.

### 2.3 Review of quadratic separation

Well-posedness of feedback systems provides a fertile framework for stability analysis of non-linear and uncertain systems. Major results for robust stability analysis has been given in (Iwasaki and Hara, 1998) and references therein. The purpose of this section is to briefly recall some new tools on quadratic separation developed for robustness issues of descriptor systems (Peaucelle *et al.*, 2005), which is needed for the main theorem of this paper.

Consider two possibly non-square matrices $\mathcal{E}$ and $\mathcal{A}$ and an uncertain constant, complex valued, matrix $\nabla$ with appropriate dimensions that belongs to some set $\mathbb{W}$. For simplicity of notations we assume in the present paper that $\mathcal{E}$ is full column rank. We make no assumption on the uncertainty set $\mathbb{W}$.

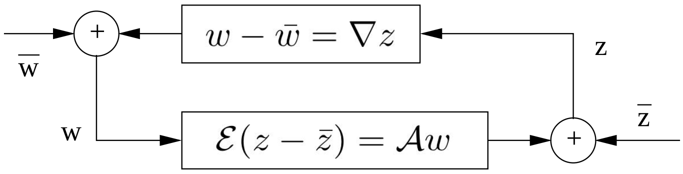


Figure 1. Feedback system

**Theorem 1** *(Peaucelle* et al.*, 2005) The uncertain feedback system of Figure 1 is well-posed if and only if there exists a Hermitian matrix $\Theta = \Theta^*$ satisfying both conditions*

$$\begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp *} \Theta \begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp} > 0 \tag{2}$$

$$\begin{bmatrix} 1 & \nabla^* \end{bmatrix} \Theta \begin{bmatrix} 1 \\ \nabla \end{bmatrix} \leq 0 \quad , \quad \forall \nabla \in \mathbb{W} \ . \tag{3}$$

*If $\mathcal{E}$ and $\mathcal{A}$ are real matrices, the equivalence still holds with $\Theta$ restricted to be a real matrix.*

## 3 Quadratic Separation for time delay system

Theorem 1 may be applied to the time-delay system by rewriting system (1) as an interconnected feedback of Figure 1. The basic idea is to embed the integrator and delay operators in an uncertain operator $\nabla$. To do so, the first step is to consider the integrator $s^{-1}$ as an uncertain operator constrained in the right-half of the complex plane (the system is stable if the feedback connected system is well-posed for all $s \in \mathbb{C}^+$, *i.e.* there is no poles in $\mathbb{C}^+$). Next, introduce the delay operator $d_h = e^{-hs}$ which operates on the system signals as:

$$d_h[x(t)] = x(t-h)$$

and which is bounded such that:

$$|d_h| \leq 1 \ , \ \forall s \in \mathbb{C}^+ \ , \ \forall h \geq 0 \ .$$

This last operator is norm bounded by 1 whatever the value of $h$. Based on this simple over bounding it is hence impossible to derive delay-dependent results. Therefore, as in (Zhang *et al.*, 2001) and (Gouaisbaut and Peaucelle, 2006), introduce the following bounded operator $\delta_h = s^{-1}(1-e^{-hs})$ that operates on the system signals as:

$$\delta_h[\dot{x}(t)] = x(t) - x(t-h)$$

and which is bounded such that:

$$|\delta_h| \leq h_m \ , \ \forall s \in \mathbb{C}^+ \ , \ 0 \leq h \leq h_m \ .$$

If the only information kept concerning these operators is their upper norm bound, one may easily expect to get quite conservative results. To reduce this predictable conservatism, consider fractions $h/r$ of the delay $h$. For such values one gets

$$|\delta_{h/r}| \leq \frac{h_m}{r} \ , \ \forall s \in \mathbb{C}^+ \ , \ 0 \leq h \leq h_m$$

and as the integer $r$ goes to infinity, this upper bound tends to be more and more precise.

Choose any integer $r \geq 1$. Based on the upper considerations and introducing redundant equations involving the fractions of the delay, system (1) matches the framework of Figure 1 with:

$$\overbrace{\left[\begin{array}{c|c|c} 1_n & 0 & 0 \\ \hline 0 & 1_{rn} & 0 \\ \hline 0 & 0 & 0 \\ \hline 1_n & 0 & -1_n \end{array}\right]}^{\mathcal{E}} z(t) = \overbrace{\left[\begin{array}{c|cc|c} A & 0 & A_d & 0 \\ \hline 1_n & 0 & 0 & 0 \\ 0 & 1_{(r-1)n} & 0 & \\ \hline -1_n & 1_n & 0 & 1_n \\ \hline 0 & \multicolumn{2}{c|}{0} & 0 \end{array}\right]}^{\mathcal{A}} w(t) \tag{4}$$

where

$$z(t) = \left(\begin{array}{c} \dot{x}(t) \\ \hline x(t) \\ x(t - \frac{1}{r}h) \\ \vdots \\ x(t - \frac{r-1}{r}h) \\ \hline \dot{x}(t) \end{array}\right) \ , \ w(t) = \left(\begin{array}{c} x(t) \\ \hline x(t - \frac{1}{r}h) \\ x(t - \frac{2}{r}h) \\ \vdots \\ x(t-h) \\ \hline x(t) - x(t - \frac{h}{r}) \end{array}\right) \tag{5}$$

along with the set defined by :

$$\mathbb{W} = \left\{ \left[\begin{array}{c|c|c} s^{-1}1_n & 0 & 0 \\ \hline 0 & d_{h/r}1_{rn} & 0 \\ \hline 0 & 0 & \delta_{h/r}1_n \end{array}\right] \ , \ s^{-1} \in \mathbb{C}^+ \ , \ 0 \leq h \leq h_m \right\} . \tag{6}$$

We aim at proving the robust stability of system (4) for all values of the uncertainty defined by the set (6). The original problem recast in this quadratic separation framework gives the following result when applying Theorem 1.

**Theorem 2** *Let an integer $r \geq 1$. If there exist a positive definite matrices $P \in \mathbb{R}^{n\times n}$ and two positive semi-definite matrices $Q \in \mathbb{R}^{rn\times rn}$, $R \in \mathbb{R}^{n\times n}$ such that:*

$$M_P^T \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} M_P + M_R^T \begin{bmatrix} \frac{h_m}{r}R & 0 \\ 0 & -\frac{r}{h_m}R \end{bmatrix} M_R + M_Q^T \begin{bmatrix} Q & 0 \\ 0 & -Q \end{bmatrix} M_Q < 0 \qquad (7)$$

*with*

$$M_P = \left[\begin{array}{c} A \ 0_{n,(r-1)n} \ A_d \\ \hline 1_n \ 0_{n,rn} \end{array}\right], \quad M_R = \left[\begin{array}{c} A \ 0_{n,(r-1)n} \ A_d \\ \hline -1_n \ 1_n \ 0_{n,(r-1)n} \end{array}\right], \quad M_Q = \left[\begin{array}{c} 1_{rn} \ 0_{rn,n} \\ \hline 0_{rn,n} \ 1_{rn} \end{array}\right]$$

*then system (1) is asymptotically stable for any delay $h$, such that $h \leq h_m$.*

*Proof :* Assume the system (1) to be modeled as an interconnected system of the form (4) with the uncertainty set (6). A quadratic separator for this set of uncertainties can be chosen as

$$\Theta = \begin{bmatrix} 0 & 0 & 0 & -P & 0 & 0 \\ 0 & -Q & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{h_m}{r}R & 0 & 0 & 0 \\ -P & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & Q & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{r}{h_m}R \end{bmatrix} \qquad (8)$$

with $P > 0$, $Q \geq 0$ and $R \geq 0$. Indeed simple calculations show that (3) is satisfied. Consider now condition (2). An admissible right orthogonal for $\begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}$ is given by:

$$\left[\begin{array}{c|c|c|c|c|c} 1_n & 0 & 0 & -A & 0 \ -A_d & 0 \\ \hline 0 & 1_{rn} & 0 & \begin{array}{c} -1_n \\ 0 \end{array} & \begin{array}{cc} 0 & 0 \\ -1_{(r-1)n} & 0 \end{array} & 0 \\ \hline 0 & 0 & 0 & 1_n & -1_n \ 0 & -1_n \\ \hline 1_n & 0 & -1_n & 0 & 0 & 0 \end{array}\right]^{\perp} = \left[\begin{array}{c|c} A & 0 \ A_d \\ \hline 1_n & 0 \quad 0 \\ 0 & 1_{(r-1)n} \ 0 \\ \hline A & 0 \ A_d \\ \hline 1_n & 0 \\ \hline 0 & 1_{rn} \\ \hline 1_n & -1_n \ 0 \end{array}\right]$$

Applying this choice of $\begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp}$ to (2) gives exaclty the LMI (7). ■

**Remark 3**

- *It is shown in (Gouaisbaut and Peaucelle, 2006) that for $r = 1$, the proposed result is equivalent to the main classical results of the literature (Han, 2002; Fridman, 2002; Xu and Lam, 2005; Suplin* et al.*, 2004). Moreover, in the fifth section of the paper it is proved that for $r > 1$ the present results are less conservative.*
- *The number of variables involved in the LMI (7) is $\frac{rn(rn+1)}{2} + n(n+1)$ and the number of rows of the matrix constraint is $(r+1)n$. As proved in (Gouaisbaut and Peaucelle, 2006) the size of the semi-definite programing problem for $r = 1$ is lower than those of previous results.*
- *If $(P, Q, R)$ are solution to (7) for some value $h_m$, then the condition also holds with the same $(P, Q, R)$ and any $h \leq h_m$.*
- *As for all existing results, finding the maximal $h_m$ that satisfies (7) is not a linear optimization problem over LMI constraints. Nevertheless, maximizing $h_m$ may be done quite efficiently with a line search or a dichotomy method by iterating on $h_m$ and solving the LMIs at each step. This is the procedure utilized in the numerical example of section 6.*

At this point we have demonstrated a new approach for solving the delay-dependent stability analysis problem based on quadratic separation framework. The next section is devoted to showing that this result may as well be interpreted in the classical Lyapunov-Krasovskii framework. Based on this interpretation, we then prove that the conservatism of Theorem 2 is reduced as $r$ grows.

## 4 Lyapunov Counterpart

**Proposition 1** *If $(P, Q, R)$ is a feasible solution to the LMI (7) then for all delays $h \leq h_m$ the functional*

$$\begin{aligned} V(x_t) = x(t)^T P x(t) + \int_{t-\frac{h}{r}}^{t} \int_{\nu}^{t} \dot{x}(\omega)^T R \dot{x}(\omega) d\omega d\nu \\ + \int_{t-\frac{h}{r}}^{t} \begin{pmatrix} x(\omega) \\ x(\omega - \frac{1}{r}h) \\ \vdots \\ x(\omega - \frac{r-1}{r}h) \end{pmatrix}^T Q \begin{pmatrix} x(\omega) \\ x(\omega - \frac{1}{r}h) \\ \vdots \\ x(\omega - \frac{r-1}{r}h) \end{pmatrix} d\omega \end{aligned} \tag{9}$$

*is a Lyapunov-Krasovskii functional that proves the stability of system (1).*

*Proof :* First note that since $P$ is positive definite ($\exists \epsilon_1 > 0 \ : \ P \geq \epsilon_1 1$) and $Q$ and $R$ are positive semi-definite, the first Lyapunov-Krasovskii condition holds:

$$V(x_t) \geq x_t^T(0) P x_t(0) \geq \epsilon_1 \|x_t(0)\|^2 \ .$$

Secondly, let us prove that there exists a positive $\epsilon_2$ such that the derivative of the functional $\dot{V}(x_t) \leq -\epsilon_2 \|x_t(0)\|^2$ along the system trajectories. The derivative of (9) writes as:

$$\begin{aligned}\dot{V}(x_t) &= 2\dot{x}(t)Px(t) + \tfrac{h}{r}\dot{x}^T(t)R\dot{x}(t) - \int_{t-\frac{h}{r}}^{t} \dot{x}(\omega)^T R\dot{x}(\omega)d\omega \\ &+ \begin{pmatrix} x(t) \\ x(t-\frac{1}{r}h) \\ \vdots \\ x(t-\frac{r-1}{r}h) \end{pmatrix}^T Q \begin{pmatrix} x(t) \\ x(t-\frac{1}{r}h) \\ \vdots \\ x(t-\frac{r-1}{r}h) \end{pmatrix} - \begin{pmatrix} x(t-\frac{1}{r}h) \\ x(t-\frac{2}{r}h) \\ \vdots \\ x(t-\frac{r}{r}h) \end{pmatrix}^T Q \begin{pmatrix} x(t-\frac{1}{r}h) \\ x(t-\frac{2}{r}h) \\ \vdots \\ x(t-\frac{r}{r}h) \end{pmatrix}\end{aligned}$$

Condition (7) implies for any $h \leq h_m$ that there exists a positive $\epsilon_2$ such that:

$$M_P^T \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} M_P + M_R^T \begin{bmatrix} \frac{h}{r}R & 0 \\ 0 & -\frac{r}{h}R \end{bmatrix} M_R + M_Q^T \begin{bmatrix} Q & 0 \\ 0 & -Q \end{bmatrix} M_Q \leq -\epsilon_2 1 \ .$$

Pre and post-multiply this inequality by respectively $w^T(t)$ and $w(t)$ defined in (5). Due to the system equations (4) one gets

$$\begin{aligned}&2\dot{x}(t)Px(t) + \tfrac{h}{r}\dot{x}^T(t)R\dot{x}(t) - \tfrac{r}{h}(x(t) - x(t-\tfrac{h}{r}))^T R(x(t) - x(t-\tfrac{h}{r})) \\ &+ \begin{pmatrix} x(t) \\ x(t-\frac{1}{r}h) \\ \vdots \\ x(t-\frac{r-1}{r}h) \end{pmatrix}^T Q \begin{pmatrix} x(t) \\ x(t-\frac{1}{r}h) \\ \vdots \\ x(t-\frac{r-1}{r}h) \end{pmatrix} - \begin{pmatrix} x(t-\frac{1}{r}h) \\ x(t-\frac{2}{r}h) \\ \vdots \\ x(t-\frac{r}{r}h) \end{pmatrix}^T Q \begin{pmatrix} x(t-\frac{1}{r}h) \\ x(t-\frac{2}{r}h) \\ \vdots \\ x(t-\frac{r}{r}h) \end{pmatrix} < -\epsilon_2 \|x_t(0)\|^2\end{aligned}$$

Jensen's inequality (Gu *et al.*, 2003) that states:

$$-\int_{t-\frac{h}{r}}^{t} \dot{x}^T(\theta)R\dot{x}(\theta)d\theta \leq -\frac{r}{h}\left(x(t) - x(t-\frac{h}{r})\right)^T R\left(x(t) - x(t-\frac{h}{r})\right)$$

concludes the proof. ∎

## 5 Reduced conservatism

**Proposition 2** *Let $\bar{h}_r$ be the maximal delay obtained by Theorem 1 for a chosen fractioning $r$. One has:*

$$\bar{h}_r \leq \bar{h}_{r+1} \quad .$$

*Proof :* Take $h = \frac{r}{r+1}\bar{h}_r$. Due to Theorem 1 there exist $P$, $Q$ and $R$ that satisfy condition (10) for the fractioning $r$ and this value of $h$. Following the proof of Proposition 1 it implies that for all non zero trajectories of the system one has

$$2\dot{x}(t)Px(t) + \tfrac{h}{r}\dot{x}^T(t)R\dot{x}(t) - \tfrac{r}{h}(x(t) - x(t-\tfrac{h}{r}))^T R(x(t) - x(t-\tfrac{h}{r}))$$

$$+ \begin{pmatrix} x(t) \\ x(t-\frac{1}{r}h) \\ \vdots \\ x(t-\frac{r-1}{r}h) \end{pmatrix}^T Q \begin{pmatrix} x(t) \\ x(t-\frac{1}{r}h) \\ \vdots \\ x(t-\frac{r-1}{r}h) \end{pmatrix} - \begin{pmatrix} x(t-\frac{1}{r}h) \\ x(t-\frac{2}{r}h) \\ \vdots \\ x(t-\frac{r}{r}h) \end{pmatrix}^T Q \begin{pmatrix} x(t-\frac{1}{r}h) \\ x(t-\frac{2}{r}h) \\ \vdots \\ x(t-\frac{r}{r}h) \end{pmatrix} < 0 \quad .$$

Replace $h$ by its value $\frac{r}{r+1}\bar{h}_r$ and take $\tilde{Q} = \begin{bmatrix} Q & 0_{rn\times n} \\ 0_{n\times rn} & 0_n \end{bmatrix}$ to get:

$$2\dot{x}(t)Px(t) + \tfrac{\bar{h}_r}{r+1}\dot{x}^T(t)R\dot{x}(t) - \tfrac{r+1}{\bar{h}_r}(x(t) - x(t-\tfrac{\bar{h}_r}{r+1}))^T R(x(t) - x(t-\tfrac{\bar{h}_r}{r+1}))$$

$$+ \begin{pmatrix} x(t) \\ x(t-\frac{1}{r+1}\bar{h}_r) \\ \vdots \\ x(t-\frac{r}{r+1}\bar{h}_r) \end{pmatrix}^T \tilde{Q} \begin{pmatrix} x(t) \\ x(t-\frac{1}{r+1}\bar{h}_r) \\ \vdots \\ x(t-\frac{r}{r+1}\bar{h}_r) \end{pmatrix} - \begin{pmatrix} x(t-\frac{1}{r+1}\bar{h}_r) \\ x(t-\frac{2}{r+1}\bar{h}_r) \\ \vdots \\ x(t-\frac{r+1}{r+1}\bar{h}_r) \end{pmatrix}^T \tilde{Q} \begin{pmatrix} x(t-\frac{1}{r+1}\bar{h}_r) \\ x(t-\frac{2}{r+1}\bar{h}_r) \\ \vdots \\ x(t-\frac{r+1}{r+1}\bar{h}_r) \end{pmatrix} < 0 \quad .$$

Conversely, this inequality implies that condition (10) holds for the fractioning $r+1$ and $h_m = \bar{h}_r$. This concludes the proof. ■

## 6 Example

The computations are performed under MATLAB© using the SDP solver SeDuMi (Sturm, 1999) and the parser YALMIP (Löfberg, 2004).

Consider the following time delay system (1) with

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} \quad , \quad A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \tag{10}$$

This example has for long been used to illustrate delay-dependent results. The left-hand side of Table 1 summarizes these previously published results by giving the reference, the obtained bound and the number of variables involved in the corresponding LMIs. The right-hand gives the results of Theorem 2 for various fractionings.

Table 1
Maximum allowable delay

| Methods | $h_{max}$ | nb vars | Theorem 2 | $\bar{h}_r$ | nb vars |
|---|---|---|---|---|---|
| (Li and De Souza, 1997) | 0.8571 | 9 | r=1 | 4.4721 | 9 |
| (Niculescu *et al.*, 1996) | 0.99 | 11 | r=2 | 5.71 | 42 |
| (Moon *et al.*, 2001) | 4.3588 | 16 | r=3 | 5.96 | 75 |
| (Han, 2002) | 4.4721 | 9/18 | r=4 | 6.05 | 120 |
| (Fridman, 2002) | 4.47 | 27 | r=5 | 6.09 | 177 |
| (Xu and Lam, 2005) | 4.4721 | 17 | r=9 | 6.149 | 525 |
| (Suplin *et al.*, 2004) | 4.4721 | 38 | r=15 | 6.164 | 1407 |
| | | | r=25 | 6.169 | 3837 |
| Theoretical bound | 6.1725 | $\infty$ | r=30 | 6.171 | 5502 |

For no fractioning ($r = 1$) the new formulation is equivalent to those of previous results as proved in (Gouaisbaut and Peaucelle, 2006). Moreover it is the one with the less decision variables. For $r > 1$ the example illustrates conservatism reduction. Although the first gap between $r = 1$ and $r = 2$ is the most impressive, the tests going up to $r = 30$ show that the conservatism is reduced at each step. Moreover, this particular example of order 2 for which one may test high values or $r$ without being limited by numerical complexity, seems to indicate that the sequence $\{\bar{h}_r\}$ tends to the exact maximal non destabilizing delay.

A prospective question is to investigate whether the conservatism gap tends to zero as the fractioning goes to infinity. The answer to that question is purely theoretical since for medium or high order systems numerical complexity makes the conditions intractable before the limit is attained.

## 7 Conclusion

In this paper, we proposed a new criterion for testing the stability of linear time delay systems. This approach is based on the quadratic separation framework and a fractioning scheme for the delay operator which allows to reduce the conservatism of the technique.

Prospective work will include extensions to the case of multiple delays and uncertain systems. These two extensions are expected to be relatively straightforward assuming that the delays have common fractioning and that uncertainties enter the system in the same feedback connection form as in Figure 1. A more involved future work will be devoted to studying the relations between the proposed fractioned Lyapunov-Krasovskii functional and the discretized functional used in (Gu *et al.*, 2003).

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