Ellipsoidal Sets for Resilient and Robust Static Output-Feedback

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Abstract

The implementation of a controller, if not exact, may lead to the so-called fragility problem, *i.e.* the loss of expected closed-loop properties. In the present paper this difficult problem is dealt with considering robust static-output feedback (SOF) control for uncertain linear time-invariant systems. By analogy with robust analysis theory based on quadratic separation, a new formulation for the SOF design is shown to be a valuable way to tackle fragility issues. Indeed, the use of a quadratic separator for design purpose allows to define a whole resilient (non-fragile) set of SOF control laws. Results are formulated as matrix inequalities one of which is non-linear. A numerical algorithm based on non-convex optimisation is provided ant its running is illustrated on classical examples from literature.

Index Terms

Output-feedback, Quadratic Separation, Fragility, Robustness, LMI.

I. Introduction

One of the most challenging open problems in control theory is the synthesis of fixed-order or static output-feedback (SOF) controllers that meet desired performances and robustness specifications [3], [31]. Among all possible variations of this problem, this technical note considers the class of continuous-time, multi-variable, Linear Time-Invariant (LTI) systems. Lyapunov theory is employed to address stability and results are formulated in terms of matrix inequalities.

1) Methods: Within the chosen framework, all synthesis problems can be written as bilinear matrix inequalities (BMIs) and there exist two linearising changes of variables for the cases of state-feedback and full-order dynamic output-feedback. To date, no such linearising change of variables exists for SOF synthesis and one can conjecture that no convex formulation exists in the general case. Nevertheless, some authors have tackled the SOF problem within the linear matrix inequality (LMI) framework through non-optimal algorithmic approaches, [26], [4], [22], [11], [14], [9], [7], [17]. All of these complex algorithms have some specific convergence properties but lead to local results.

Our goal is to propose a new formulation of the SOF design problem based on the quadratic separation concept. The theoretical relationship between this concept and the new results are detailed in the next section. An algorithm based on [7] is proposed and tested in the last section of this technical note. Taken to this point the proposed design is comparable to existing methods but an additional contribution is that the applicability of this technique to both resilient and robust design simultaneously is demonstrated.

2) Robustness and resilience: Advanced industrial applications demand the synthesis of complex requirements. The specifications can have multiple aspects such as closed-loop performances and/or robustness. In this note we focus on robustness and point out the contributions of the quadratic separator-based SOF synthesis. Extensions to other specifications are given in [23].

Due to linearisation simplifications in modelling and to identification limitations, the LTI models are far from being determined with precision. This is the seminal consideration leading to the development of the so-called Robust Control theory. In particular, an important past result was the extension of LMI results to robust control. This was achieved for many uncertainty models at the expense of some degree of conservatism. The first robustness results on quadratic stability (see [1], [5]) have been extended recently to take parameter-dependent Lyapunov functions into account [8], [24]. Nevertheless, the robust SOF design suffers from being either highly non-linear or highly conservative.

The ellipsoidal set design [23] proves to be a contribution with respect to this problem. In addition, the proposed synthesis technique provides a new way to tackle fragility issues. Fragility concerns the closed-loop robustness with respect to uncertainty of the control law parameters. This issue formulated in [16] and to which many contributions have been made (e.g. [12], see also [20] for a balanced view), has significant repercussions for digital controller implementation. Different techniques have been proposed to deal with fragility issues. Some assume that the uncertainties are given while the control law itself has to be designed [34]. Others give a multiplicative structure to the uncertainty implying that the uncertainty depends on the parameters of the designed controller [33]. In all cases, the methodology is quite similar to robustness techniques.

The novel approach proposed in this paper is to keep the fragility in relation to the design. The synthesis is performed to design some quadratic separator that defines a whole set of control laws. The system is therefore resilient (non-fragile) to controller uncertainties as long as the parameters are kept within the designed ellipsoidal set.

3) Outline: First, some standard notations are introduced and ellipsoidal sets of matrices ("matrix ellipsoids") are defined. Second, the central result is given. Then SOF stabilisability is revisited from the perspective of quadratic separation. In section four the result is extended to robust stabilisability. The fifth section is devoted to fragility/resilience. In the last section all numerical aspects are drawn together and an algorithm is proposed and tested.

II. Preliminaries

Notations are standard. $\mathbb{R}^{m\times n}$ is the set of m-by-n real matrices and $\mathbb{S}^n$ is the subset of symmetric matrices in $\mathbb{R}^{n\times n}$. A' is the transpose of the matrix A. $\mathbb{1}$ and $\mathbb{0}$ are respectively the identity and the zero matrices of appropriate dimensions. For symmetric matrices, $> \ (\geq)$ is the Löwner partial order, i.e., $A > (\geq) B$ if and only if $A - B$ is positive (semi) definite. In matrix inequalities as well as in the problem formulations, the decision variables are in bold face (for instance $\mathbf{P}$ is a decision variable to be found to attest some property while P stands for a given solution). Assuming Σ_1 and Σ_2 are two systems with appropriate input/output vector dimensions, the interconnected system of Figure 1 is denoted $\Sigma_1 \star \Sigma_2$. To prevent confusion in the vectors defining the interconnection, we use the following notation $\Sigma_1 \overset{u,y}{\star} \Sigma_2$.

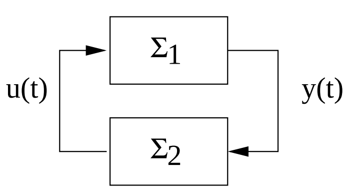

Fig. 1. Interconnected systems

Throughout this paper a particular set of matrices is used. Due to the notations and by extension of the notion of $\mathbb{R}^n$ ellipsoids, these sets are referred to as matrix ellipsoids of $\mathbb{R}^{m\times p}$.

Definition 1: Given three matrices $X \in \mathbb{S}^p$, $Y \in \mathbb{R}^{p\times m}$ and $Z \in \mathbb{S}^m$, the $\{X, Y, Z\}-$ellipsoid of $\mathbb{R}^{m\times p}$ is the set of matrices $\mathbf{K}$ satisfying the following matrix inequalities:

$$Z > \mathbb{0} \qquad \begin{bmatrix} \mathbb{1} & \mathbf{K}' \end{bmatrix} \begin{bmatrix} X & Y \\ Y' & Z \end{bmatrix} \begin{bmatrix} \mathbb{1} \\ \mathbf{K} \end{bmatrix} \leq \mathbb{0} \tag{1}$$

By definition, $K_o \triangleq -Z^{-1}Y'$ is the centre of the ellipsoid and $R \triangleq K_o' Z K_o - X$ is the radius. The inequalities (1) can also be written as: $Z > \mathbb{0} \quad (\mathbf{K} - K_o)' Z (\mathbf{K} - K_o) \leq R$

This definition shows that matrix ellipsoids are special cases of matrix sets defined by a quadratic matrix inequality. It may be possible to define some hyperbolic or parabolic sets in the same way. This paper addresses only ellipsoids. They satisfy the constraint $Z > \mathbb{0}$. Some properties of these sets are: i) A matrix ellipsoid is a compact convex set. ii) The $\{X, Y, Z\}-$ellipsoid is non-empty if and only if the radius ($R \geq \mathbb{0}$) is positive semi-definite. This property can also be expressed as $X \leq Y Z^{-1} Y'$.

III. Design revisited with quadratic separation

The key result of this paper is related to topological separation [27], [10]. The stability of interconnected systems (Figure 1) is equivalent to the existence of a topological separator between the graph of the first system (Σ_1) and the inverse graph of the other (Σ_2). Based on this general result, major contributions have been made for robust control (e.g. [5], [13], [21], [28], [29]).

The topological separation proves to be fertile ground for robustness analysis; however, to our knowledge, it has never been considered for synthesis purposes. The SOF synthesis problem can be written as an interconnected system as in Figure 1 where the first system Σ_1 is the given model and $\Sigma_2 = \mathbf{K}$ is the SOF matrix to compute. From a separation point of view, SOF design is equivalent to finding an operator that performs the topological separation between Σ_1 and the inverse graph of some linear transformation. When $\Sigma_2 = \mathbf{K}$ is a linear transformation, Iwasaki [13] proved that the separator can be chosen without conservatism among constant quadratic operators. SOF design can therefore be replaced by the synthesis of some quadratic separator.

Consider interconnected systems as in Figure 1, where the first system is LTI with the state-space representation (Σ) and the second system is an SOF gain (K):

$$\Sigma \quad : \quad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \qquad u(t) = Ky(t) \tag{2}$$

$x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input control vector, $y \in \mathbb{R}^p$ is the output measure vector and $\Sigma \star K$ is the closed-loop system defined by (2). The quadratic separation design is as follows.

Theorem 1: The LTI system Σ is stabilisable by static output-feedback if and only if there exist four matrices $\mathbf{P} \in \mathbb{S}^n$, $\mathbf{X} \in \mathbb{S}^p$, $\mathbf{Y} \in \mathbb{R}^{p\times m}$ and $\mathbf{Z} \in \mathbb{S}^m$ that simultaneously satisfy the following LMI constraints:

$$\mathbf{Z} > \mathbb{0} \qquad \mathbf{P} > \mathbb{0}$$
$$\begin{bmatrix} \mathbb{1} & \mathbb{0} \\ A & B \end{bmatrix}' \begin{bmatrix} \mathbb{0} & \mathbf{P} \\ \mathbf{P} & \mathbb{0} \end{bmatrix} \begin{bmatrix} \mathbb{1} & \mathbb{0} \\ A & B \end{bmatrix} < \begin{bmatrix} C & D \\ \mathbb{0} & \mathbb{1} \end{bmatrix}' \begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Y}' & \mathbf{Z} \end{bmatrix} \begin{bmatrix} C & D \\ \mathbb{0} & \mathbb{1} \end{bmatrix} \tag{3}$$

and the non-linear inequality constraint:

$$\mathbf{X} \leq \mathbf{Y}\mathbf{Z}^{-1}\mathbf{Y}' \tag{4}$$

Let (P, X, Y, Z) be a solution, then the non-empty $\{X,\, Y,\, Z\}$-ellipsoid is a set of stabilising gains.

Proof of sufficiency: Assume the constraints (3) and (4) are satisfied for some P, X, Y, Z matrices. Due to the properties of matrix ellipsoids the $\{X,\, Y,\, Z\}$-ellipsoid is non-empty. Take any element K. The last matrix inequality of (3) implies that for all $\begin{pmatrix} x' & u' \end{pmatrix} \neq 0$:

$$\begin{pmatrix} x \\ Ax+Bu \end{pmatrix}' \begin{bmatrix} \mathbb{0} & P \\ P & \mathbb{0} \end{bmatrix} \begin{pmatrix} x \\ Ax+Bu \end{pmatrix} < \begin{pmatrix} Cx+Du \\ u \end{pmatrix}' \begin{bmatrix} X & Y \\ Y' & Z \end{bmatrix} \begin{pmatrix} Cx+Du \\ u \end{pmatrix}$$

The definitions (2) imply that for all non-zero trajectories:

$$x'P\dot{x} + \dot{x}'Px < y' \begin{bmatrix} \mathbb{1} & K' \end{bmatrix} \begin{bmatrix} X & Y \\ Y' & Z \end{bmatrix} \begin{bmatrix} \mathbb{1} \\ K \end{bmatrix} y \leq 0$$

The closed-loop stability is assessed by the $V(x) = x'Px$ Lyapunov function. ■

Proof of necessity: Assume K is a stabilising SOF gain and $V(x) = x'Px$ is a Lyapunov certificate. By definition, $V(x)$ is positive definite ($P > \mathbb{0}$) and for all non-zero trajectories of the closed-loop system defined by equations (2), the derivative of $V(x)$ is negative:

$$\begin{bmatrix} KC & KD - \mathbb{1} \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} = 0 \quad \Longrightarrow \quad 0 > \begin{pmatrix} x \\ u \end{pmatrix}' \begin{bmatrix} \mathbb{1} & \mathbb{0} \\ A & B \end{bmatrix}' \begin{bmatrix} \mathbb{0} & P \\ P & \mathbb{0} \end{bmatrix} \begin{bmatrix} \mathbb{1} & \mathbb{0} \\ A & B \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix}$$

Applying the Finsler lemma [30], there exists a scalar τ such that:

$$\begin{bmatrix} \mathbb{1} & \mathbb{0} \\ A & B \end{bmatrix}' \begin{bmatrix} \mathbb{0} & P \\ P & \mathbb{0} \end{bmatrix} \begin{bmatrix} \mathbb{1} & \mathbb{0} \\ A & B \end{bmatrix} < \tau \begin{bmatrix} KC & KD - \mathbb{1} \end{bmatrix}' \begin{bmatrix} KC & KD - \mathbb{1} \end{bmatrix}$$

Inequality (3) is obtained with $X = \tau K'K$, $Y = -\tau K'$ and $Z = \tau\mathbb{1}$. The bottom-right block implies $\mathbb{0} < (KD - \mathbb{1})'Z(KD - \mathbb{1})$. Thus $Z > \mathbb{0}$ and the closed-loop system is necessarily well-posed. ∎

Remark 1: Let $\Sigma(s)$ be the transfer matrix of the LTI model. The symmetric matrix of theorem 1 composed of X, Y and Z is a topological separator between the graph of the system $\Sigma(s)$ and a stabilising set of controllers. One way to view this result to apply a variation of the Kalman-Yakubovich-Popov lemma on (3). See [23] for details.

Remark 2: The topological separation applies for any linear or non-linear interconnected systems. Therefore the result of theorem 1 is also a sufficient condition for non-linear control. It implies that any control law constrained by the quadratic constraint:

$$\begin{pmatrix} y(t) \\ u(t) \end{pmatrix}' \begin{bmatrix} X & Y \\ Y' & Z \end{bmatrix} \begin{pmatrix} y(t) \\ u(t) \end{pmatrix} \leq 0$$

stabilises the system Σ. The set of stabilising control laws is therefore not limited to the static output-feedback class of controllers.

IV. Robust stabilisation

As previously stated, quadratic separation has been used extensively for robust analysis purposes. It is now applied to simultaneously handle robustness and design.

Consider the LTI system Σ_{lft} where the input $w \in \mathbb{R}^{m_w}$ and output $z \in \mathbb{R}^{p_z}$ define an exogenous feedback of an uncertainty matrix Δ:

$$\Sigma_{\text{lft}} \quad : \quad \begin{cases} \dot{x}(t) = & Ax(t) + & B_w w(t) + & Bu(t) \\ z(t) = & C_z x(t) + & D_{zw} w(t) + & D_{zu} u(t) \\ y(t) = & Cx(t) + & D_{yw} w(t) + & Du(t) \end{cases} \qquad w(t) = \Delta z(t) \tag{5}$$

For any admissible uncertainty Δ, the uncertain model is an LTI system obtained through the interconnection $\Sigma_{\text{lft}}(\Delta) = \Sigma_{\text{lft}} \overset{w,z}{\star} \Delta$. The resulting state-space matrices are rational in the uncertain parameters. The interconnection defines a Linear Fractional Transformation (LFT). The uncertain parameters are all gathered in a unique matrix Δ. They are assumed to be constant parametric uncertainties and the uncertainty set is a matrix ellipsoid of $\mathbb{R}^{m_w \times p_z}$ defined by:

$$\mathbb{\Delta}_{\text{lft}} = \{X_{\text{lft}},\, Y_{\text{lft}},\, Z_{\text{lft}}\}\text{-ellipsoid}$$

In order to guarantee that the nominal system $\Sigma_{\text{lft}}(\mathbb{0})$ is included in the set of realisations $\Sigma_{\text{lft}}(\Delta)$, the matrix X_{lft} is assumed to be negative semi-definite ($X_{\text{lft}} \leq \mathbb{0}$). The two matrices X_{lft} and Y_{lft} are assumed not to be zero simultaneously so that the set does not reduce to the singleton $\mathbb{0}$.

Such uncertainty sets are also known as $\{X_{\text{lft}},\, Y_{\text{lft}},\, Z_{\text{lft}}\}$-dissipative uncertainties. As reported for instance in [18], [25], [29] and [32], this modelling of uncertainties contains the well known norm-bounded uncertainties ($\{-\mathbb{1},\, \mathbb{0},\, \mathbb{1}\}$-dissipative) and positive real uncertainties ($\{\mathbb{0},\, -\mathbb{1},\, \mathbb{0}\}$-dissipative) that lead respectively to the small gain and passivity frameworks.

Define the three matrices:

$$N_1 = \begin{bmatrix} \mathbb{1} & \mathbb{0} & \mathbb{0} \\ A & B_w & B \end{bmatrix} \qquad N_2 = \begin{bmatrix} C_z & D_{zw} & D_{zu} \\ \mathbb{0} & \mathbb{1} & \mathbb{0} \end{bmatrix} \qquad N_3 = \begin{bmatrix} C & D_{yw} & D \\ \mathbb{0} & \mathbb{0} & \mathbb{1} \end{bmatrix}$$

Theorem 2: The uncertain LTI system $\Sigma_{\text{lft}}(\Delta)$ with $\Delta \in \mathbb{\Delta}_{\text{lft}}$ is robustly stabilisable by static output-feedback if and only if there exist four matrices $\mathbf{P} \in \mathbb{S}^n$, $\mathbf{X} \in \mathbb{S}^p$, $\mathbf{Y} \in \mathbb{R}^{p \times m}$, $\mathbf{Z} \in \mathbb{S}^m$ and a scalar $\boldsymbol{\tau}_{\textbf{lft}}$ that simultaneously satisfy the non-linear constraint (4) and the following LMI constraints:

$$\begin{gathered} \boldsymbol{\tau}_{\textbf{lft}} > 0 \qquad\qquad \mathbf{Z} > \mathbb{0} \qquad\qquad \mathbf{P} > \mathbb{0} \\ N_1' \begin{bmatrix} \mathbb{0} & \mathbf{P} \\ \mathbf{P} & \mathbb{0} \end{bmatrix} N_1 < \boldsymbol{\tau}_{\textbf{lft}}\, N_2' \begin{bmatrix} X_{\text{lft}} & Y_{\text{lft}} \\ Y_{\text{lft}}' & Z_{\text{lft}} \end{bmatrix} N_2 + N_3' \begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Y}' & \mathbf{Z} \end{bmatrix} N_3 \end{gathered} \tag{6}$$

Let $(P, X, Y, Z, \tau_{\text{lft}})$ be a feasible solution, then the non-empty $\{X,\, Y,\, Z\}$-ellipsoid is a set of robustly stabilising gains.

The proof, omitted for conciseness, essentially follows the same lines as the proof of theorem 1. The starting point are the LMI analysis conditions for closed-loop robust stability. Details can be found in [23].

Remark 3: Both interconnection operators Δ and K are taken into account using the same theory of quadratic separation. The first interconnected system K exists if a quadratic separator built out of the matrices X, Y and Z exists (see remark 1). Robustness is achieved if a quadratic separator exists "between" the uncertainty set and the nominal system respectively. For the considered dissipative uncertainties the separator is losslessly parameterised by the scalar $\boldsymbol{\tau}_{\mathbf{lft}}$. Precise definition of the separation is given in [23].

Remark 4: Recall that μ-theory is closely related to quadratic separation. Indeed, the interconnected systems of Figure 1 can be seen respectively as a constant matrix $\Sigma_1 = M$ and a structured operator $\Sigma_2 = \Omega$ such that:

$$\Omega = \text{diag}(\tilde{\omega}_1 \mathbb{1}, \ldots, \tilde{\omega}_{m_r} \mathbb{1}, \omega_1 \mathbb{1}, \ldots, \omega_{m_c} \mathbb{1}, \Omega_1, \ldots, \Omega_{m_F}) \tag{7}$$

It is composed of m_r repeated real scalar blocks, m_c repeated complex scalar blocks and m_F full-blocks. Results about conservatism of μ-analysis are given in [19]. Losslessness of finite dimensional convex scalings is proved if $2(m_r + m_c) + m_F \leq 3$. By tedious but trivial manipulations, the result extends to ellipsoidal matrix constraints on the elements in Ω.

The considered robust SOF design problem fits in that framework with M defined using the matrices A, $B_w \ldots D$ and with operator Ω composed of $s^{-1}\mathbb{1}_n$, Δ (full block real uncertainty) and K (full block real SOF gain): $2(m_r + m_c) + m_F = 4 > 3$. Nevertheless, μ-theory results are not in contradiction with the fact that theorem 2 is lossless. Losslessness is achieved at the expense of entire freedom on the set of controllers (it can degenerate to a single point). In other words, theorem 2 parameterises all robustly stabilising SOF gains but not all robust stabilising $\{X,\, Y,\, Z\}$-ellipsoids.

Meanwhile, for the SOF stabilisation problem of theorem 1, the formula can be written as $2(m_r+m_c)+m_F = 3$. The theorem is strongly lossless in the sense that not only are all stabilising SOF gains parameterised by inequalities (3) and (4), but all stabilising $\{X,\, Y,\, Z\}$-ellipsoids are as well.

V. Resilience and robustness

1) Resilient: non-fragile: Another effect of the design of stabilising $\{X,\, Y,\, Z\}$-ellipsoids is to handle fragility. The method allows one to ensure that an SOF control be non-fragile (resilient). While robustness applies to properties of the closed-loop system with respect to modelling uncertainties, resilience implies invariance of the closed-loop properties with respect to control implementation errors. These errors cannot be known *a priori* because they depend on the synthesis results. This aspect makes resilience and robustness slightly different questions.

Corollary 1: Assume the matrices P, X, Y and Z satisfy the constraints (3) and (4). Take the central controller $K_o = -Z^{-1}Y'$ and the radius R of the $\{X, Y, Z\}$-ellipsoid. The closed-loop system $\Sigma \star K$ is resilient to any additive uncertainty $\Delta_K(t)$ such that:

$$K = K_o + \Delta_K \qquad\qquad \Delta_K' Z \Delta_K \leq R \tag{8}$$

Proof : Write that $K = K_0 + \Delta_K$ belongs to the $\{X, Y, Z\}$-ellipsoid and apply remark 2.

■

Remark 5: The strong losslessness of theorem 1 (as defined in remark 4), implies that the constraints (3) and (4) describe exactly all resilient stabilising SOF ellipsoids. In other words: for a given $\{X, Y, Z\}$-ellipsoid, if the LMI (3) in the unknown $\mathbf{P}$ is infeasible, then there is at least one gain K inside the ellipsoid that destabilises the closed-loop.

Remark 6: The same type of corollary as 1 applies to the robust stability of theorem 2. In that case, both robustness with respect to uncertainties Δ and resilience with respect to Δ_K are guaranteed. Unfortunately, as shown in remark 4, strong losslessness does not hold for theorem 2: some resilient $\{X, Y, Z\}$-ellipsoids may not be described by the LMIs (6).

Corollary 1 illustrates general resilience properties of ellipsoidal SOF design. It gives, *a posteriori*, an admissible set of uncertainties. The two following corollaries give *a priori* requirements on the ellipsoidal set.

Corollary 2: Assume the matrices P, X, Y and Z satisfy (3) with the constraints:

$$\mathbf{Z} = \mathbb{1} \qquad\qquad \mathbb{0} < \rho\mathbb{1} \leq \mathbf{Y}\mathbf{Y}' - \mathbf{X} \tag{9}$$

then the closed loop system $\Sigma \star K$ is resilient to any additive norm-bounded uncertainty $\Delta_K(t)$ such that: $K = K_o + \Delta_K \qquad \|\Delta_K\|^2 = \Delta_K' \Delta_K \leq \rho\mathbb{1}$

Proof : The proof is due to corollary 1 with the restriction $\mathbf{Z} = \mathbb{1}$. ■

Corollary 3: Assume the matrices P, X, Y and Z satisfy (3) along with the constraint:

$$\mathbf{X} \leq (1 - \bar{\delta}^2)\mathbf{Y}\mathbf{Z}^{-1}\mathbf{Y}' \tag{10}$$

then the closed-loop system $\Sigma \star K$ is resilient to any multiplicative uncertainty (see [6], [33]) $\Delta_K(t)$ such that: $K = K_o + \delta K_o = (1+\delta)K_o \qquad |\delta| \leq \bar{\delta}$

Proof: Write that condition (1) holds for all $K = K_o + \delta K_o$. ■

2) Robust, resilient and simultaneous stabilisation:

Define the general multi-performance problem:

For a family of (possibly uncertain) models $\left\{\Sigma^{[i]}(\Delta^{[i]})\right\}$*,*
find a common (possibly resilient) controller $\mathbf{K}$*,*
that performs for each closed-loop $\Sigma^{[i]}(\Delta^{[i]}) \star \mathbf{K}$ *a given (robust) performance,* $\Pi^{[i]}$*.*

$\Pi^{[i]}$ may be stabilisation requirements ($\mathbf{K}$ stabilises (robustly) all of the models simultaneously), but can also be other performances [23]. To formulate such design with existing methods would imply defining some non-linear constrained problem (BMIs for example) for each performance and then solving all the non-linear constraints simultaneously. The numerical complexity would explode as the number of specifications grew.

This is not the case when design is considered from a quadratic separation point of view. A contribution of this note is the formulation of general multi-performance design, without any additional assumption or conservatism, in such a manner that the same algorithm may solve all such SOF design problems.

For each model $\Sigma^{[i]}(\Delta^{[i]})$ *associated to performance* $\Pi^{[i]}$*, find the matrix unknowns (Lyapunov matrices,* $\mathbf{P}^{[i]}$*, scalings* $\boldsymbol{\tau}_{\mathbf{lft}}^{[i]}$*...), plus the three common matrices* $\mathbf{X}$*,* $\mathbf{Y}$ *and* $\mathbf{Z}$*, constrained by all the concatenated LMI constraints and the unique non-linear constraint (4).*

In other words, to solve a multi-objective problem, we need only to increase the number of unknowns and the number of LMI constraints accordingly for each objective and each model. From a numerical point of view, it increases the computational burden and reduces the domain where the $\{X,\ Y,\ Z\}$-ellipsoid can be found. From a theoretical point of view, there is no additional conservatism to solving a multi-objective control problem compared to single-objective control problems. Losslessness is obtained by the use of model-dependent Lyapunov functions.

All design problems are equivalent to finding a feasible solution (Q, X, Y, Z) to the constraints summarised as:

$$\mathcal{L}(\mathbf{Q}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) < \mathbb{0} \qquad \text{and} \qquad \mathbf{X} \leq \mathbf{Y}\mathbf{Z}^{-1}\mathbf{Y}' \tag{11}$$

where $\mathbf{Q}$ represents all the stacked variables such as the Lyapunov matrices and other separators, and where $\mathcal{L}(\cdot)$ is a linear matrix operator. The first constraint $\mathcal{L}(\mathbf{Q}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) < \mathbb{0}$ is convex and there exist efficient numerical tools to solve such LMI constraints. The main difficulty comes from the non-linear constraint.

When resilience with respect to multiplicative uncertainty with level $\bar{\delta}$ is specified, the second inequality is replaced by (10). This is the situation considered in the following section.

VI. Algorithm and examples

1) Cone complementarity algorithm: The numerical examples are solved using a first order iterative algorithm based on a cone complementarity technique [7].

Lemma 1: The considered problem is feasible if and only if zero is the global optimum of the optimisation problem:

$$\min \quad \text{trace}(\mathbf{TS}) \quad \text{s.t.} \left\{ \begin{array}{ll} \mathcal{L}(\mathbf{Q},\mathbf{X},\mathbf{Y},\mathbf{Z}) < \mathbb{0} & \mathbf{S} = \begin{bmatrix} \hat{\mathbf{X}} & \mathbf{Y} \\ \mathbf{Y}' & \mathbf{Z} \end{bmatrix} \geq \mathbb{0} \\ \mathbf{X} \leq (1-\bar{\delta}^2)\hat{\mathbf{X}} & \\ \mathbf{T_1} \geq \mathbb{1} & \mathbf{T} = \begin{bmatrix} \mathbf{T_1} & \mathbf{T_2} \\ \mathbf{T_2}' & \mathbf{T_3} \end{bmatrix} \geq \mathbb{0} \end{array} \right\} \tag{12}$$

Proof : With the constraints $T \geq \mathbb{0}$ and $S \geq \mathbb{0}$, trace$(TS) = 0$ implies that $TS = \mathbb{0}$. Therefore, after some manipulations, one gets: $\hat{X} = -T_1^{-1}T_2Y' = -T_1^{-1}(-T_1YZ^{-1})Y' = YZ^{-1}Y'$. Thus the non-linear constraint is satisfied: $X \leq (1-\bar{\delta}^2)\hat{X} = YZ^{-1}Y'$.
The converse implication is proved taking $\hat{X} = (1-\bar{\delta}^2)YZ^{-1}Y'$ and T such that $TS = \mathbb{0}$. ■

As in [7] and [17], the optimisation problem (12) can then be solved with a first order conditional gradient algorithm, also known as the Frank and Wolfe feasible direction method [2]. For conciseness its properties are not restated here. Note only that the linear objective trace($T_k\mathbf{S} + \mathbf{T}S_k$) is the relaxed objective of the non-linear function trace($\mathbf{TS}$). The obtained LMI optimisation is repeated iteratively with matrices T_k and S_k computed from each previous optimisation step. The obtained sequence, trace(T_kS_k), is strictly decreasing.

Remark 7: The stopping criteria of the usual gradient algorithm is either related to slow progress of the optimisation objective or to the achievement of trace$(TS) = 0$. In the first case, the algorithm fails due to flat behaviour or because it found a non-satisfactory local optimum. The second case is the expected success. Unfortunately, it cannot be achieved numerically and the stopping criteria is trace$(TS) \leq \epsilon$ where ϵ is a chosen accuracy level. The exact non-linear constraint may not be exactly satisfied.

As a matter of fact, since the equality constraint involving $\hat{X}$ is not the goal of the original problem (10), in the numerical examples below we adopted the following stopping criteria for the conditional gradient algorithm:

- If trace$(T_{k-1}S_{k-1} - T_k S_k)$ is below a chosen level, then STOP, the algorithm failed.
- As soon as $X \leq YZ^{-1}Y'$, STOP, a stabilising ellipsoid is found.

2) Examples: The algorithm has been tested on the examples of [17] with comparable results. For instance take exactly the same H_2/H_∞ problem given in example 1. Our algorithm converged in 16 iterations; the central gain of the obtained ellipsoid is $K_o = [\ -0.1812 \quad 0.5546\ ]'$ and the closed-loop analysis gives $\|\Sigma \star K_o\|_\infty = 0.4005$ and $\|\Sigma \star K_o\|_2 = 0.1325$. For conciseness, the comparisons are not detailed further here. In the next example we focus on the robust multi-performance design possibilities that have no equivalent in literature.

Consider the VTOL example with uncertain parameters. The nominal system is given by:

$$
\begin{array}{l}
A = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & p_{1o} & -0.7070 & p_{2o} \\ 0 & 0 & 1 & 0 \end{bmatrix} B_w = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} D_{zu} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \\
B = \begin{bmatrix} 0.4422 & 0.1761 \\ p_{3o} & -7.5922 \\ -5.5200 & 4.4900 \\ 0 & 0 \end{bmatrix} C_z = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} C = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{array}{l} D_{yw} = \mathbb{0} \\ D_{zw} = \mathbb{0} \\ D = \mathbb{0} \end{array}
\end{array}
\quad (13)
$$

and the three uncertainties are gathered in a diagonal matrix $\Delta = \mathrm{diag}(\Delta_{p_1}, \Delta_{p_2}, \Delta_{p_3})$ with:

$$
|\Delta_{p_1}| \leq 0.05 \quad |\Delta_{p_2}| \leq 0.01 \quad |\Delta_{p_3}| \leq 0.04
$$

The chosen modelling of uncertainties does not allow taking into account the structured nature of Δ. It will therefore be incorporated into a larger uncertainty domain $\mathbb{\Delta}_{\text{lft}}$ defined as the $\{X_{\text{lft}}$, $\mathbb{0}$, $\mathbb{1}\}$-ellipsoid where $X_{\text{lft}} = \mathrm{diag}(-0.05^2, -0.01^2, -0.04^2)$.

In [15] the nominal values are $p_{1o} = 0.3681$, $p_{2o} = 1.42$ and $p_{3o} = 3.5446$. The uncertain system with these values of the nominal parameters is denoted $\Sigma^{[1]}(\Delta)$. Take $\Sigma^{[2]}(\Delta)$, the same system, but defined for another operating point such that $p_{1o} = 0.3681 + 0.05$, $p_{2o} = 1.42 - 0.01$ and $p_{3o} = 3.5446 + 0.05$. By recursion define $\Sigma^{[i]}(\Delta)$, i=1,...,N.

The design problem is to stabilise quadratically all uncertain systems with overlapping values of the parameters. All systems $\Sigma^{[i]}(\Delta)$ have to be stabilised by the same SOF controller. The algorithm is tested, for $\bar{\delta} = 0.5$ (i.e. accepts variation of 50% of the controller parameters) and for several choices of N. The results are given in Table I. For each value of N the table gives the number of iterations before convergence, the computation time on a SunBlade100 work station, the value of trace(TS) when the algorithm stopped, a controller inside the obtained $\{X, Y, Z\}$-ellipsoid (see Figure 2) and, for this value of K, the maximal real part of all closed-loop poles computed on nominal systems.

N	iter	sec.	Trace(TS)	K	max poles
3	20	38	0.02	[-0.3 0.8]'	-0.0814
10	17	64	0.1	[-0.4 0.9]'	-0.0740
20	23	134	0.04	[-0.5 0.9]'	-0.0675
30	68	636	0.05	[-0.6 1.2]'	-0.0666

TABLE I

NUMERICAL RESULTS

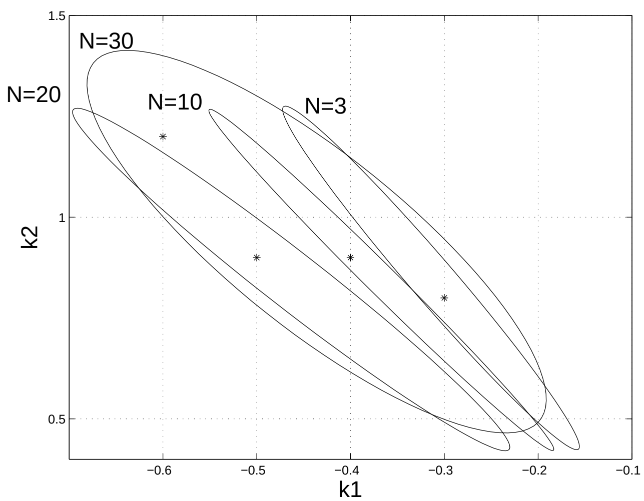

Fig. 2. Ellipsoids of controllers

VII. Conclusions

A new quadratic separation framework has been defined for robust static output-feedback synthesis. Even if this approach does not allow to escape from the non-convex nature of this problem, it provides a novel point of view on SOF design. In particular, fragility issues related to the physical implementation of robust and/or optimal controllers is shown to be naturally dealt with in this context. An algorithm based on non-convex optimisation techniques is proposed. It is used to compute ellipsoidal sets of resilient SOF control laws. Its efficiency is illustrated by examples. The versatility of the proposed set-up offers different potentialities such as multi-objective control and will be extended to the case of state-feedback and dynamic output-feedback.

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