Robust adaptive L_2-gain control of polytopic MIMO LTI systems - LMI results *†

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Abstract

Passification-based direct adaptive control is considered for polytopic uncertain linear time-invariant multi-input multi-output systems. Linear Matrix Inequality based results are provided to guarantee that the adaptive algorithm passifies the system whatever the uncertain parameters in some given set. The contributions are based on the introduction of a parallel feed-forward shunt that liberates from the strong equality constraint $PB = C^T G^T$ often used for strict passification. The shunt, combined with the introduction of slack variables, allows to conclude with easy to test conditions without restrictions on which data is uncertain in the process model. The resulting adaptive control is such that the control gain is bounded in a chosen set whatever bounded disturbances. Moreover, it proves to be not worse than computable parameter-dependent static output-feedback controls with respect to L_2 gain attenuation. A simple academic example illustrates the results. **Keywords:** Passivity, Robustness, Adaptive Control, LMI, output-feedback, polytopic systems, L_2 gain.

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1 Introduction

Adaptive control techniques are conceived with the idea that on line modification of the control algorithm is needed in order to reject the inevitable uncertainties and disturbances. One such adaptive scheme [7, 1] relies on adaptive parameter estimation which gives, either directly or after some computation, the data for tuning the controller. Yet another adaptation scheme [5, 8] takes advantage of passivity properties to perform directly the tuning of the controller gains parameters with the sole output measurements. The paper is dedicated to robustness issues in this passification-based strategy (also called simple adaptive control in [8]).

The case of state-space linear time-invariant (LTI) systems is studied ($\dot{x} = Ax+Bu$, $y = Cx$). The considered adaptive control of such systems is proved to be possible if there exists a combination of the outputs $z_P = Gy$ such that the system from its inputs u to the artificial outputs z_P is hyper minimum phase [6]. Equivalently it corresponds to the existence of a static output-feedback $u = Fy + w_P$ such that the closed-loop from w_P to z_P is strictly passive (the open-loop system is sometimes called *almost strictly passive* [8] in that case). Last equivalence, the system is then also closed-loop passive with the adaptive control. Existence of a static feedback is therefore a tool for proving properties of adaptive control and, for a given matrix G, static feedback passifiability can be formulated in terms of Linear Matrix Inequalities (LMIs). In the paper, the G matrix is assumed to be given, obtained based on known properties of the system or obtained via methodologies such as those provided in [2, 11].

At this stage a question arises: why performing adaptive control if static feedback is available? The answer is that adaptive control is expected to be robust. Yet robustness needs to be proved and this is the purpose of the exposed research. Following the equivalence with static feedback control, the proof is obatined by the existence of a parameter-dependent passifying static feedback. In a sense this indicates, for particular situations, that the existence of an estimation/gain-tuning adaptive control implies the validity of a simple passivity-based adaptive law.

In [11], LMI-based results are derived for the existence of a parameter-dependent static feedback that passifies the system under the assumption that only the A matrix of the LTI system is affected by uncertainties. This limitation may seem inevitable if considering strict passivity of systems without feed-through gains. Indeed it relies on an inequality constrain $PB = C^T G^T$ which is hard to satisfy if the B and C matrices are imperfectly known. Inspired by the results of [8], a parallel feedforward gain (also known as a *shunt*) is introduced in order to extend the results to systems where all matrices are uncertain. LMI results are produced that allow the simultaneous design of the feedback and shunt gains that both have the same polytopic structure as the uncertain system. Similarly to the results of [13] (that considers state-feedback and not output-feedback), robustness is proved using parameter-dependent storage functions thanks to the slack-variable methodology of [10].

Another characteristic of the results produced in this paper is that not only passivity and stability are expected for the closed-loop system, but L_2-gain attenuation is evaluated as well. This L_2 gain attenuation performance is measured with respect to input/output signals w_L/z_L independent of those (w_P/z_P) used to define passivity properties. With respect to L_2-gain performance, adaptive control is proved to be always better (not worse) than parameter-dependent static output-feedback and that, whatever the input/output signals used to define the performance. This result is obtained under the practical restriction that control gains are bounded and this restriction is achieved via a dead-zone corrective term in the adaptation algorithm to which is added a logarithmic-type barrier behavior. The corrective term proves also to bring good convergence properties even for the case with noisy measurements which is we believe an improvement compared to closely related results of [3]. Moreover, the conditions to be tested are LMI rather than BMI, thus are much more simple to solve with appropriate solvers such as those available using YALMIP [9].

The paper is organized as follows. First, a brief section is devoted to the description of the considered static parameter-dependent and adaptive control schemes. In particular, the adaptive control is composed of a dead-zone type function which is needed for proving closed-loop stability and which, as suggested in [1], has disturbance rejection properties. In the third section, the passification problem with L_2-gain attenuation is formulated and LMI conditions are provided for the case of LTI systems without uncertainties. The fourth section exposes the robust LMI results and finally a numerical example is treated in the last section.

Notations: $\mathbb{R}^{m\times n}$ is the set of m-by-n real matrices. A^T is the transpose of the matrix A. $\mathbf{1}$ and $\mathbf{0}$ are respectively the identity and the zero matrices of appropriate dimensions. For symmetric matrices, $A > (\geq) B$ if and only if $A-B$ is positive (semi) definite. To reduce the size of some formulas $\begin{bmatrix} A & * \\ B & C \end{bmatrix}$ stands for the symmetric matrix $\begin{bmatrix} A & B^T \\ B & C \end{bmatrix}$ and $\langle A \rangle$ stands for the symmetric matrix $\langle A \rangle = A + A^T$.

2 Control strategies

The paper considers passification of uncertain LTI systems described in state-space as:

$$\begin{array}{rlll} \dot{x}(t) = & A(\Delta)x(t) & +B(\Delta)u(t) & +B_L(\Delta)w_L(t) \\ y(t) = & C(\Delta)x(t) & & \\ z_L(t) = & C_L(\Delta)x(t) & & +D_L(\Delta)w_L(t) \end{array} \qquad (1)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control, $y \in \mathbb{R}^l$ is the measurement output, $w_L \in \mathbb{R}^{m_w}$ is a disturbance and $z_L \in \mathbb{R}^{l_2}$ is a performance evaluation output. Δ stands for parametric uncertainty which is supposed to belong to a given set $\mathbf{\Delta}$ and is assumed to be constant (or varying sufficiently slowly in comparison with the system dynamics). Considered control objectives are to achieve some passivity property which includes asymptotic stability of the system state x and to guarantee some L_2 gain attenuation of the disturbance signal measured on the output z_L:

$$\int_0^\infty \|z_L(t)\|^2 dt \leq \gamma^2 \int_0^\infty \|w_L(t)\|^2 dt \ . \tag{2}$$

Two control strategies are adopted and compared. One is parameter-dependent static output-feedback with bounded gain

$$u(t) = F(\Delta)y(t) + w_P(t) \quad , \quad \mathrm{Tr}(F^T(\Delta)F(\Delta)) \leq \alpha \tag{3}$$

and supposes measurement or estimation of the uncertain parameters Δ. The second, is adaptive control defined as

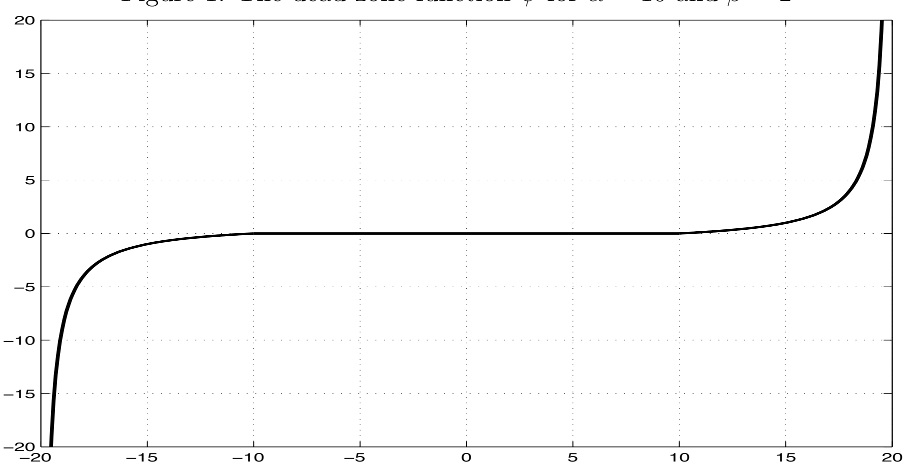
$$\begin{array}{l} u(t) = K(t)y(t) + w_P(t) \\ \dot{K}(t) = -Gy(t)y^T(t)\Gamma - \phi(K(t))\Gamma \end{array} \tag{4}$$

where $\phi(K) = \psi(\mathrm{Tr}(K^TK)) \cdot K$ and ψ is a scalar dead-zone type function such that:

$$\left\{ \begin{array}{ll} \psi(k) = 0 & \text{if } \ k \leq \alpha \\ \psi(k) = \frac{k-\alpha}{\alpha\beta - k} & \text{if } \ \alpha \leq k < \alpha\beta \end{array} \right.$$

where $\alpha > 0$ and $\beta > 1$. For the scalar case, the function is as represented in Figure 1. Throughout the paper, closed-loop properties will be studied for any valid initial conditions such that $(x(0), K(0)) \in \mathcal{XK}$ where $\mathcal{XK} = \{(x, K) \in \mathbb{R}^n \times \mathbb{R}^{m \times l} \ : \ \mathrm{Tr}(K^TK) < \alpha\beta\}$.

Figure 1: The dead-zone function ψ for $\alpha = 10$ and $\beta = 2$



The adaptive control law is conceived such that the term $-Gy(t)y^T(t)\Gamma$ tunes in real time the feedback gain $K(t)$ in order, hopefully, to converge towards passifying values. Conditions for that property to hold are given in the next section. The second term $-\phi(K(t))\Gamma$ is intended to guarantee bounded values for $K(t)$. In practice it acts on convergence of gain $K(t)$ to the bounded region $\mathrm{Tr}(K^T K) \leq \alpha$ and the logarithmic-type barrier forces for all instants the bound $\mathrm{Tr}(K^T K) \leq \alpha\beta$. As shown in [1, 11] such term as $-\phi(K(t))\Gamma$ also contributes to robustness with respect to external disturbances, in particular to noise on the measurements.

Some properties of ϕ are now given.

Lemma 1 *If $y(t)$ is bounded for all $t \geq 0$, then $Tr(K^T(t)K(t)) < \alpha\beta$ for all $t \geq 0$.*

Proof. Consider the following Lyapunov function $V(K) = \mathrm{Tr}(KK^T)$. Its derivatives along the trajectories of (1) with feedback (4) writes for $\mathrm{Tr}(K^T K) \geq \alpha$ as

$$\begin{aligned} \dot{V}(K) &= -2\mathrm{Tr}(Gyy^T\Gamma K^T + \phi(K)\Gamma K^T) \\ &= -y^T(\Gamma K^T G + G^T K\Gamma)y \\ &\qquad -2\tfrac{\mathrm{Tr}(K^TK)-\alpha}{\alpha\beta-\mathrm{Tr}(K^TK)}\mathrm{Tr}(K\Gamma K^T)\ . \end{aligned}$$

The last term in this formula goes to infinity as $\mathrm{Tr}(K^T K)$ goes to $\alpha\beta$. Therefore, for bounded values of y and K there exists a scalar k such that $\dot{V}(K) < 0$ for all K such that $k \leq \mathrm{Tr}(K^T K) < \alpha\beta$, *i.e.* the trajectories are decreasing and $\mathrm{Tr}(K^T K)$ cannot exceed $\alpha\beta$. □

Lemma 2 *For all (F, K) satisfying $Tr(F^T F) \leq \alpha$ and $Tr(K^T K) < \alpha\beta$, the inequality $Tr(\phi(K)(K - F)^T) \geq 0$ holds.*

Proof. If $\mathrm{Tr}(K^T K) \leq \alpha$ then $\phi(K) = \mathbf{0}$ and the result is trivial. Let T such that $\mathrm{Tr}(T) \leq \alpha$ with $F^T F \leq T$. A Schur complement argument gives $\begin{bmatrix} T & F^T \\ F & \mathbf{1} \end{bmatrix} \geq \mathbf{0}$. Pre and post multiply this inequality by $\begin{bmatrix} \mathbf{1} & -K^T \end{bmatrix}$ and by its transpose respectively to get $T - F^T K - K^T F + K^T K \geq \mathbf{0}$. Add and substract $K^T K$ in this expression to get $(K - F)^T K + K^T(K - F) \geq K^T K - T$ which, taking the trace of the inequality, implies

$$\begin{aligned} 2\mathrm{Tr}((K - F)^T K) = 2\mathrm{Tr}(K(K - F)^T) &\geq \mathrm{Tr}(K^T K - T) \\ &\geq \mathrm{Tr}(K^T K) - \alpha. \end{aligned}$$

Due to this inequality one gets for all K such that $\alpha \leq \mathrm{Tr}(K^T K) < \alpha\beta$:

$$\mathrm{Tr}(\phi(K)(K - F)^T) \geq \tfrac{(\mathrm{Tr}(K^TK)-\alpha)^2}{2(\alpha\beta-\mathrm{Tr}(K^TK))} \geq 0$$

and concludes the proof. □

Remark 1 *All results are provided assuming given values of G, α, β and Γ.*

- *The a priori knowledge of the G matrix is the major assumption because it renders linear the matrix inequalities of the design problem. For a given system choosing an appropriate matrix G may sometimes be done based on known physical properties. In other cases, some heuristics are available in [2, 11]. These certainly need to be developed further and this will be done in future work.*

- *The parameter α characterizes a bound on the feedback gains. Such bound may be seen as a restriction due to control implementation possibilities. The value α is therefore chosen a priori based on engineering information. Yet, if not limitation is known, α could be chosen arbitrarily large.*

- *β is, as exposed upper, a parameter that tunes the logarithmic barrier which prevents the adaptive control gains to exit too far from the desired bounded domain. It should be strictly greater than 1 and, due to possible implementation difficulties, should not be too close to 1. The provided experiments show that $\beta = 1.1$ or $\beta = 2$ are appropriate choices. For β larger than 2, the LMI conditions may be too conservative.*

- *The matrix Γ has to be chosen positive definite. It has an influence on the adaptation dynamics but does not change the passivity and L_2 gain attenuation properties. For the following, assume it is given, whatever its value.*

3 Closed-loop passivity

This section considers the case of systems without uncertainties

$$\begin{array}{l} \dot{x} = Ax + Bu + B_L w_L \ , \\ y = Cx \ \ , \ \ z_L = C_L x + D_L w_L . \end{array} \tag{5}$$

The goal is to derive constructive LMI conditions for that case. These will allow in the next section to produce robustness results.

Passification is looked for by means of either static feedback $u = Fy + w$ or non-linear adaptive feedback (4). In both cases the closed-loop system enters the format

$$\begin{array}{l} \dot{\eta} = f(\eta) + \hat{B}w + \hat{B}_L w_L \ , \\ y = \hat{C}\eta \ \ , \ \ z_L = C_L x + D_L w_L \end{array} \tag{6}$$

where $\eta = x$ in case of static feedback and $\eta = (x^T \ k^T)^T$ in case of adaptive feedback where k is the vector build by concatenation of all columns of K. In this last case f is a non-linear function. In the latter case $f(\eta) = A(F)x$ where $A(F) = A + BFC$. For such state-space model, consider the following inequality

$$V(\eta(t)) \leq V(\eta(0)) + \int_0^t \left[w_P(\theta)^T z_P(\theta) - \rho(\eta(\theta)) \right] \, d\theta \ . \tag{7}$$

Based on this inequality and assuming zero disturbances $w_L = 0$, define:

Definition 1 *The system is said to be globally passive with respect to the signals w_P, z_P if there exists a nonnegative scalar function $V(\eta)$ (storage function) and $\rho = 0$ such that (7) holds for all $t \geq 0$ and all valid initial conditions $(x(0), K(0)) \in \mathcal{X}K$.*

Definition 2 *The system is said to be globally strictly passive with respect to the signals w_P, z_P if there exists a nonnegative scalar function $V(\eta)$ and a nonnegative scalar function $\rho(\eta)$, strictly positive for all $\eta \neq 0$, such that (7) holds for all $t \geq 0$ and all valid initial conditions $(x(0), K(0)) \in \mathcal{X}K$.*

Definition 3 *According to the upper defined partitioning of the state $\eta = (x^T \; k^T)^T$, the system is said to be globally x-strictly passive with respect to the signals w_P, z_P if there exists a nonnegative scalar function $V(\eta)$ and a nonnegative scalar function $\rho(\eta)$, strictly positive for all $x \neq 0$, such that (7) holds for all $t \geq 0$ and all valid initial conditions $(x(0), K(0)) \in \mathcal{X}K$.*

All three definitions are global, *i.e.* they hold for all admissible initial conditions $(x(0), K(0)) \in \mathcal{X}K$. The adjective "global" is eluded in the remaining to alleviate the text. All passivity conditions imply system stability. Additionally, strict passivity implies asymptotic stability and x-strict passivity implies that for zero inputs $w_P = 0$ and $w_L = 0$, the part of the state x converges to zero.

Theorem 1 *The closed-loop system (5) with zero disturbance $w_L = 0$ and with static feedback $u = Fy + w$ is strictly passive with respect to the signals $w_P, z_P = Gy + Dw_P$ if and only if there exist a symmetric positive-definite matrix $P > \mathbf{0}$ and a positive scalar $\epsilon > 0$ such that the following LMI conditions hold:*

$$\left[\begin{array}{cc} A^T(F)P + PA(F) & PB \\ B^TP & \mathbf{0} \end{array} \right] \leq \left[\begin{array}{cc} -\epsilon\mathbf{1} & C^TG^T \\ GC & D + D^T \end{array} \right] \tag{8}$$

Proof. The theorem corresponds to the positive-real lemma [4] applied to the closed-loop system $\dot{x} = A(F)x + Bw_P, \quad z_P = GCx + Dw_P$. It is also an LMI version of the SPR conditions recalled in [7]. □

Note that if $D = \mathbf{0}$, the inequality (8) become that of [6]

$$A^T(F)P + PA(F) \leq -\epsilon\mathbf{1} \quad , \quad PB = C^TG^T \; .$$

Any robustness tests based on this last formula have the disadvantage that B and C matrices need to be exactly known. Indeed, except for very special situations, the equality constraint $PB = C^TG^T$ is impossible to maintain even when assuming parameter-dependent matrices P. Hence, the feed-through gain D is an opportunity for producing robustness results with respect to uncertain systems with uncertainties affecting B and C.

Theorem 2 *If there exist two positive scalars $\epsilon_1 > 0$, $\epsilon_2 > 0$, two matrices $\hat{F}$, $\hat{D}$, and three symmetric positive-definite matrices $\hat{P} > \mathbf{0}$, $\hat{R} > \mathbf{0}$, $\hat{T} > \mathbf{0}$ such that the following LMI conditions hold*

$$\left[\begin{array}{cc} \hat{R} & \hat{P}B - \epsilon_2 C^TG^T \\ B^T\hat{P} - \epsilon_2 GC & \epsilon_2\mathbf{1} \end{array} \right] \geq \mathbf{0} \tag{9}$$

$$\left[\begin{array}{cc} \hat{T} & \hat{F}^T \\ \hat{F} & \epsilon_2\mathbf{1} \end{array} \right] \geq \mathbf{0} \quad , \quad Tr(\hat{T}) \leq \epsilon_2\alpha \tag{10}$$

$$\begin{bmatrix} \hat{L}+\hat{R}+C_L^TC_L & * & * \\ B^T\hat{P}-\epsilon_2 GC & -\hat{D}-\hat{D}^T & \mathbf{0} \\ B_L^T\hat{P}+D_LC_L & \mathbf{0} & D_L^TD_L-\gamma^2\mathbf{1} \end{bmatrix} \leq \mathbf{0} \tag{11}$$

where

$$\hat{L} = A^T\hat{P}+\hat{P}A+\epsilon_2\alpha\beta C^TC+\epsilon_1\mathbf{1}+C^T(G^T\hat{F}+\hat{F}^TG)C,$$

then let $F=\frac{1}{\epsilon_2}\hat{F}$ *and* $D=\frac{1}{\epsilon_2}\hat{D}$*, both control laws (3) and (4)*

- *x-strictly passify the system with respect to the signals $w_P, z_P = Gy+Dw_P$ in case of zero disturbance $w_L=0$*
- *and guarantee L_2-gain attenuation level of γ for zero initial conditions and zero reference signal $w_P=0$.*

Proof. As a start, multiply all inequalities of Theorem 2 by $\tau = 1/\epsilon_2$ to get

$$\begin{bmatrix} R & PB-C^TG^T \\ B^TP-GC & \mathbf{1} \end{bmatrix} \geq \mathbf{0} \tag{12}$$

$$\begin{bmatrix} T & F^T \\ F & \mathbf{1} \end{bmatrix} \geq \mathbf{0} \quad , \quad \mathrm{Tr}(T) \leq \alpha \tag{13}$$

$$\begin{bmatrix} L+R+\tau C_L^TC_L & * & * \\ B^TP-GC & -D-D^T & \mathbf{0} \\ B_L^TP+\tau D_LC_L & \mathbf{0} & \tau(D_L^TD_L-\gamma^2\mathbf{1}) \end{bmatrix} \leq \mathbf{0} \tag{14}$$

where $L = A^TP+PA+\alpha\beta C^TC+\epsilon\mathbf{1}+C^T(G^TF+F^TG)C$, $\epsilon=\tau\epsilon_1$, $P=\tau\hat{P}$, $R=\tau\hat{R}$ and $T=\tau\hat{T}$.

Let us first prove the result for the static feedback control. Pre and post multiply inequality (12) by $\begin{bmatrix} \mathbf{1} & -C^TF^T \end{bmatrix}$ and by its transpose respectively to get:

$$\begin{aligned} & C^T(G^TF+F^TG)C+R \\ & \qquad \geq PBFC+C^TF^TB^TP-C^TF^TFC \ . \end{aligned}$$

Due to this inequality, (14) implies that

$$\begin{bmatrix} \tilde{L}+\tau C_L^TC_L & * & * \\ B^TP-GC & -D-D^T & \mathbf{0} \\ B_L^TP+\tau D_LC_L & \mathbf{0} & \tau(D_L^TD_L-\gamma^2\mathbf{1}) \end{bmatrix} \leq \mathbf{0} \tag{15}$$

where $\tilde{L} = A^T(F)P+PA(F)+\epsilon\mathbf{1}-C^T(F^TF-\alpha\beta\mathbf{1})C$. A Schur complement argument on (13) gives that $\mathrm{Tr}(F^TF)\leq\mathrm{Tr}(T)\leq\alpha$. It implies that $F^TF\leq\alpha\mathbf{1}$. As $\beta>1$ one gets $F^TF-\alpha\beta\mathbf{1}\leq\mathbf{0}$. Let $N_1 = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix}$, $N_2 = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$ with appropriate block dimensions. Pre and post multiply (15) by N_1 and its transpose respectively, it implies inequality of Theorem 1 which proves x-strict passivity. Pre and post multiply (15) by N_2 and its transpose respectively to get

$$\left[\begin{array}{cc} A^T(F)P + PA(F) + \tau C_L^T C_L & PB_L + \tau C_L^T D_L \\ B_L^T P + \tau D_L C_L & \tau(D_L^T D_L - \gamma^2 \mathbf{1}) \end{array}\right] < \mathbf{0}$$

which is the classical H_∞ performance LMI constraint for the closed-loop LTI system [4] and proves L_2-gain attenuation level of γ.

Now, prove the result for the adaptive control law. Let any Hermitian positive definite matrix $\Gamma > 0$ and let the output-feedback law (4). Consider the following storage function $V(\eta) = \frac{1}{2}x^T P x + \frac{1}{2}\mathrm{Tr}\left((K-F)\Gamma^{-1}(K-F)^T\right)$. Along the trajectories of (5) with the control law (4) the derivatives of $V(\eta)$ write

$$\begin{array}{rl} \dot{V}(\eta) = & x^T P(Ax + BKy + Bw_P + B_L w_L) \\ & +\mathrm{Tr}\left(\dot{K}\Gamma^{-1}(K-F)^T\right) . \end{array}$$

Pre and post multiply the matrix inequality (14) by $\left(\begin{array}{ccc} x^T & w_P^T & w_L^T \end{array}\right)$ and its transpose respectively, to get

$$\begin{array}{l} 2x^T P(Ax + Bw_P + B_L w_L) \\ \qquad \leq 2z_P^T w_P - \epsilon\|x\|^2 + \tau(\gamma^2\|w_L\|^2 - \|z_L\|^2) \\ \qquad\quad -\alpha\beta y^T y - x^T R x - 2y^T F^T G y \end{array}$$

and hence

$$\begin{array}{rl} \dot{V}(\eta) \leq & z_P^T w_P - \frac{\epsilon}{2}\|x\|^2 + \frac{\tau}{2}(\gamma^2\|w_L\|^2 - \|z_L\|^2) \\ & -\frac{\alpha\beta}{2} y^T y - \frac{1}{2}x^T R x - y^T F^T G y + x^T PBKy \\ & + \mathrm{Tr}\left(\dot{K}\Gamma^{-1}(K-F)^T\right) . \end{array}$$

Pre and post multiply (12) by $\left(\begin{array}{cc} x^T & -y^T K^T \end{array}\right)$ and its transpose respectively to get

$$2x^T PBKy - x^T R x \leq 2y^T K^T G y + y^T K^T K y .$$

Combining these two last inequalities, the derivatives of $V(\eta)$ satisfy the inequality

$$\begin{array}{rl} \dot{V}(\eta) \leq & z_P^T w_P - \frac{\epsilon}{2}\|x\|^2 + \frac{\tau}{2}(\gamma^2\|w_L\|^2 - \|z_L\|^2) \\ & + \frac{1}{2}y^T(K^T K - \beta\alpha\mathbf{1})y + y^T(K-F)^T G y \\ & + \mathrm{Tr}\left(\dot{K}\Gamma^{-1}(K-F)^T\right) . \end{array}$$

Note the following result based on the fact that $\mathrm{Tr}(M_1 M_2) = \mathrm{Tr}(M_2 M_1)$:

$$\begin{array}{rl} y^T(K-F)^T G y & = \mathrm{Tr}(y^T(K-F)^T G y) \\ & = \mathrm{Tr}(G y y^T (K-F)^T). \end{array}$$

Therefore, replacing $\dot{K}$ by its value, one obtains

$$\begin{array}{l} y^T(K-F)^T G y + \mathrm{Tr}\left(\dot{K}\Gamma^{-1}(K-F)^T\right) \\ \qquad = -\mathrm{Tr}\left(\phi(K)(K-F)^T\right) \end{array}$$

which is negative due to Lemma 2. Moreover, Lemma 1 guarantees that $\mathrm{Tr}(K^TK) \leq \alpha\beta$ and hence $K^TK - \alpha\beta\mathbf{1} \leq \mathbf{0}$. The derivative of the Lyapunov function along the closed-loop trajectories is therefore, for all $t \geq 0$, bounded by:

$$\begin{array}{l} \dot{V}(\eta(t)) \leq z_P^T(t) w_P(t) - \frac{\epsilon}{2}\|x(t)\|^2 \\ \qquad\qquad + \frac{\tau}{2}(\gamma^2\|w_L(t)\|^2 - \|z_L(t)\|^2) \ . \end{array}$$

For $w_L = 0$, taking the integral over time one gets (7) with $\rho(\eta) = \frac{1}{2}\epsilon x^T x$ which proves x-strict passivity. For $w_P = 0$ and zero initial conditions, taking the integral over time one gets (2), *i.e.* L_2-gain attenuation level of γ. □

Note that γ^2 enters linearly in the inequalities of Theorem 2. Therefore γ can be minimized while solving the LMIs. The result is alternatively an optimized static output-feedback gain F with respect to L_2-gain attenuation or an upper bound γ on the L_2-gain attenuation level that adaptive control achieves. The adaptive control has therefore always a better (at least not worse) L_2 performance level than that of a static control law solution to inequalities of Theorem 2. As Theorem 2 is conservative this fact does not prove that adaptive control is always better than any static control, yet it is always better than any of those one can compute with the LMIs.

The adaptive law does not depend of the model, nor does it depend of the solution to the LMIs, but its properties are proved by the feasibility of these LMIs. Thus, extending the LMIs to the case of uncertain systems may prove, without complexifying the adaptive control formulas, robustness properties. This is the aim of the next section.

4 Guaranteed Robustness

The uncertain system $\Sigma(\Delta)$ of (1) is assumed of polytopic type. The dependency with respect to the uncertain parameters $\Delta = (\ \zeta_1\ ,\ \dots\ ,\ \zeta_N\ )$ is affine such that

$$\left[\begin{array}{ccc} A(\Delta) & B(\Delta) & B_L(\Delta) \\ C(\Delta) & \mathbf{0} & \mathbf{0} \\ C_L(\Delta) & \mathbf{0} & D_L(\Delta) \end{array}\right] = \sum_{i=1}^{N} \zeta_i \left[\begin{array}{ccc} A_i & B_i & B_{Li} \\ C_i & \mathbf{0} & \mathbf{0} \\ C_{Li} & \mathbf{0} & D_{Li} \end{array}\right]$$

$$\zeta_i \geq 0 \quad , \quad \sum_{i=1}^{N} \zeta_i = 1$$

where the matrices A_i, B_i etc. define the vertices of the polytope.

Theorem 3 *If there exists two matrices H_1 and H_2, a positive scalar ϵ_2 and $6N$ elements $\hat{P}_i > \mathbf{0}$, $\epsilon_{1i} > 0$, $\hat{F}_i$, $\hat{D}_i$, $\hat{R}_i > \mathbf{0}$, $\hat{T}_i > \mathbf{0}$ such that the following LMI conditions hold for all $i = 1 \dots N$*

$$\left[\begin{array}{ccc} \hat{R}_i & -\epsilon_2 C_i^T G^T & \hat{P}_i \\ -\epsilon_2 G C_i & \epsilon_2 \mathbf{1} & \mathbf{0} \\ \hat{P}_i & \mathbf{0} & \mathbf{0} \end{array}\right] + \left\langle \left[\begin{array}{c} \mathbf{0} \\ B_i^T \\ -\mathbf{1} \end{array}\right] H_1^T \right\rangle \geq \mathbf{0} \tag{16}$$

$$\begin{bmatrix} \hat{T}_i & \hat{F}_i^T \\ \hat{F}_i & \epsilon_2 \mathbf{1} \end{bmatrix} \geq \mathbf{0} \quad , \quad Tr(\hat{T}_i) \leq \epsilon_2 \alpha \tag{17}$$

$$\hat{L}_i + \left\langle \begin{bmatrix} A_i & B_i & B_{Li} & -\mathbf{1} & \mathbf{0} & \mathbf{0} \\ C_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{1} & \mathbf{0} \\ C_{Li} & \mathbf{0} & D_{Li} & \mathbf{0} & \mathbf{0} & -\mathbf{1} \end{bmatrix}^T H_2^T \right\rangle \leq \mathbf{0} \tag{18}$$

where $\hat{L}_i =$

$$\begin{bmatrix} \hat{R}_i + \epsilon_{1i}\mathbf{1} & -\epsilon_2 C_i^T G^T & \mathbf{0} & \hat{P}_i & \mathbf{0} & \mathbf{0} \\ -\epsilon_2 G C_i & -\hat{D}_i - \hat{D}_i^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\gamma^2 \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hat{P}_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \epsilon_2 \alpha \beta \mathbf{1} + \left\langle G^T \hat{F}_i \right\rangle & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix},$$

then let $F(\Delta) = \frac{1}{\epsilon_2} \sum_{i=1}^N \zeta_i \hat{F}_i$ *and* $D(\Delta) = \frac{1}{\epsilon_2} \sum_{i=1}^N \zeta_i \hat{D}_i$*, both controls (3) and (4)*

- *robustly* x*-strictly passify the system with respect to the signals* $w_P, z_P = Gy + D(\Delta) w_P$ *in case of zero disturbance* $w_L = 0$
- *and guarantee* L_2*-gain attenuation level of* γ *for zero initial conditions and zero reference signal* $w_P = 0$*.*

Proof. Define $\hat{P}(\Delta) = \sum_{i=1}^N \zeta_i \hat{P}_i$, $\epsilon_1(\Delta) = \sum_{i=1}^N \zeta_i \epsilon_{1i}$, $\hat{R}(\Delta) = \sum_{i=1}^N \zeta_i \hat{R}_i$ and $\hat{T}(\Delta) = \sum_{i=1}^N \zeta_i \hat{T}_i$. Note that the LMIs (16), (17) and (18) are all linear with respect to the vertex matrices (with indices i). Therefore if the LMIs hold for all vertices then they also hold for all the elements of their convex hull. Denote (16-Δ), (17-Δ) and (18-Δ) the obtained parameter-dependent inequalities. Let the matrices

$$M_1(\Delta) = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & B^T(\Delta) \end{bmatrix},$$
$$M_2(\Delta) = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & A^T(\Delta) & C^T(\Delta) & C_L^T(\Delta) \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & B^T(\Delta) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & B_L^T(\Delta) & \mathbf{0} & D_L^T(\Delta) \end{bmatrix}.$$

Pre and post multiply (16-Δ) by $M_1(\Delta)$ and its transpose respectively. The operation eliminates the slack variable H_1 and the result happens to be exactly the parameter-dependent version of (9). (17-Δ) is exactly the parameter-dependent version of (10). Pre and post multiply (18-Δ) by $M_2(\Delta)$ and its transpose respectively. The operation eliminates the slack variable H_2 and the result happens to be exactly the parameter-dependent version of (11). Hence, it is shown that the conditions of Theorem 2 hold for all admissible uncertainties. □

Remark 2

- *To apply the control law $u = F(\Delta)y + w_P$ it supposes to have a measurement of the uncertain parameters or at least an estimate of these parameters. This is not the case for the adaptive control law (4).*

- *The control laws make the closed-loop passive with respect to a parameter-dependent output signal $z_P = Gy + D(\Delta)w_P$. If, for some backstepping design procedure for example, the output signal z_P is needed to be reconstructed exactly whatever the uncertain parameters, then the LMI conditions may be solved for a unique matrix D by constraining $D_i = D$ identical for all vertices.*

- *Note that if the LMI conditions hold for two different sets of vertices then the adaptive control law (4) proves to passify the system for the union of the two convex hulls. A gain scheduling strategy of the type $u = F(\Delta)y + w_P$ would in this case be even more complex to perform because it would need to switch form one polytopic description of $F(\Delta)$ to another depending on the convex hull in which the parameters lie.*

- *As exposed in the previous section, for any value of the uncertainty, the adaptive control has lower L_2-gain attenuation level than any parameter-dependent static output-feedback gain computed on the basis of Theorem 2. Theorem 3 makes it possible to guarantee an upper bound on that attenuation level for all uncertainties.*

5 Example

Let the following data $\left[\begin{array}{ccc} A(\Delta) & B(\Delta) & B_L(\Delta) \end{array}\right] =$

$$\left[\begin{array}{cccc|c|c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 12-7.5\delta_1 & -0.6+0.7\delta_1 & 5-4.5\delta_1 & 0 & 1+3\delta_1 \\ 0 & 0 & 0 & -20+\delta_2 & 20-\delta_2 & 0 \end{array}\right]$$

$$C(\Delta) = \left[\begin{array}{cccc} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1+0.1\delta_2 \end{array}\right] , \quad \begin{array}{c} C_L(\Delta) = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \end{array}\right] , \\ D_L(\Delta) = 0 , \\ G = \left[\begin{array}{ccc} 400 & 300 & 200 \end{array}\right] \end{array}$$

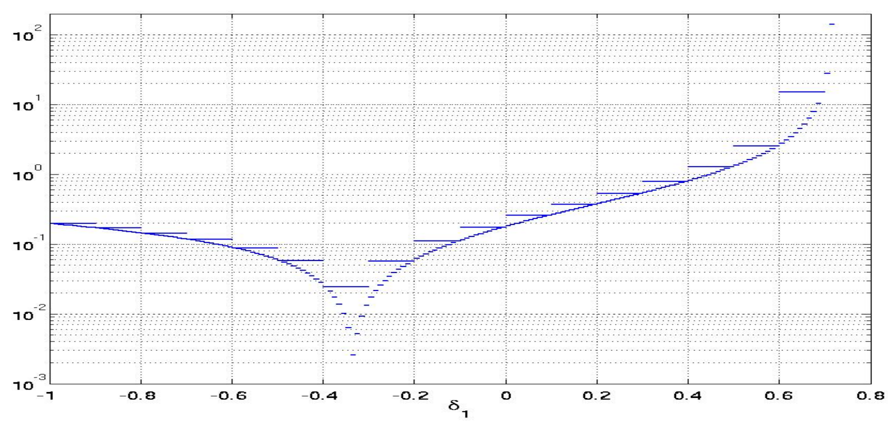
where $\delta_2 \in [\ 0\ \ 2.5\ ]$ and $\delta_1 \geq -1$. The objective is to prove closed-loop stability for the largest interval on δ_1. The tests are made considering $\alpha = 10$ and $\beta = 2$. The adopted procedure is to check whether the LMIs of Theorem 3 are feasible for various intervals. The results are given in Table 1. For some intervals the LMIs are found infeasible although the following tests prove the existence of some solution to the control problem. This is due to conservatism of the LMI conditions that fail to find a polytopic parameter-dependent static control gain for large intervals where such control may not exist. The conservatism is reduced as the uncertainty set is reduced. Globally, taking the union of all feasible intervals, allows to conclude that the adaptive control law (4) stabilizes the system for the entire uncertainty set $(\delta_1, \delta_2) \in [\ -1 \quad 0.722\ ] \times [\ 0 \quad 2.5\ ]$. Yet the L_2 gain attenuation is not acceptable in practice for all the interval $\delta_1 \in [\ -1 \quad 0.722\ ]$.

Table 1: LMI tests for $\delta_2 \in [\ 0 \quad 2.5\ ]$

δ_1	min γ	δ_1	min γ	δ_1	min γ
$[\ -1 \quad 0\ ]$	0.2	$[\ 0.7 \quad 0.72\ ]$	141	$[\ 0.72 \quad 0.722\ ]$	1001
$[\ -1 \quad 0.7\ ]$	24	$[\ 0.7 \quad 0.73\ ]$	infeas.	0.723	infeas.
$[\ -1 \quad 0.72\ ]$	infeas.				

To evaluate more accurately the L_2 gain attenuation, smaller intervals on δ_1 are tested. The results are given in Figure 2. The example being of relatively small dimensions, rather small intervals can be tested. First we have taken intervals of 0.1 length and then of 0.01 length. Results indicate that for all $\delta_1 \in [\ -1 \quad 0.45\ ]$ the L_2-gain of the output signal is smaller than that of the input.

Figure 2: Robust L_2 gain attenuation levels for $\delta_2 \in [\ 0 \quad 2.5\ ]$ and various ranges of δ_1



The numerical burden of the LMI problems solved in this example is characterized by 270 scalar decision variables and 129 rows in the LMI constraints. The latest version of SeDuMi [12] is used along with the parser YALMIP [9]. The computation time of each individual LMI optimization problem is less than 2 seconds (Linux PC computer with i686 processor and 2GB memory).

To illustrate the efficiency of the adaptive control law, simulations are performed. $\delta_2 = 2.5$ is taken constant while δ_1 is taken slowly time varying. It grows linearly from -1 to 0.75 from time $t = 0$ till time $t = 500$ then decreases with same speed till $t = 1000$. This sequence is then repeated a second time. To evaluate the L_2-gain attenuation, every 20 seconds a short triangular signal is applied during 2 seconds on input w_L such that $\int_{20k}^{20(k+1)} w_L^2(t)dt = 2$ whatever integer k. To make the simulations realistic, the control input is saturated between values ± 20 and a white noise is added to the measured signal (noise power 10^{-6}, sample time $0.1s$). The initial control gain is taken equal to zero, $K(0) = [\ 0\ \ 0\ \ 0\ ]$ and $\Gamma = \mathbf{1}$ is chosen.

Figure 3: $\delta_1(t)$ and $z_L(t)$ histories

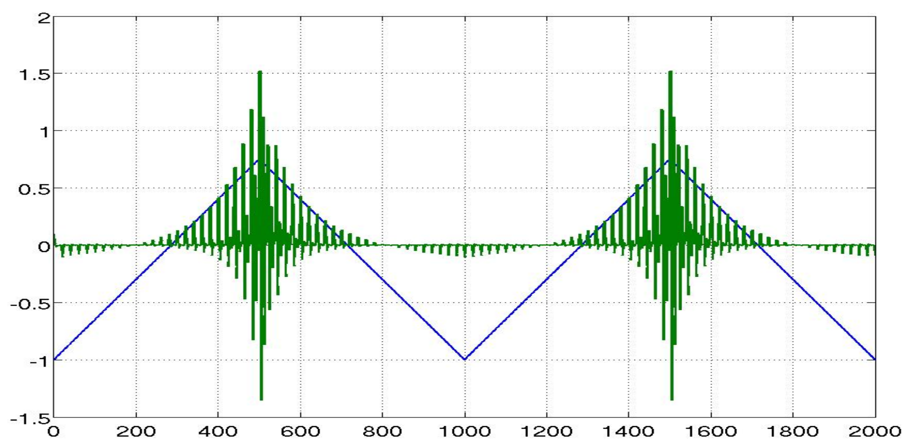
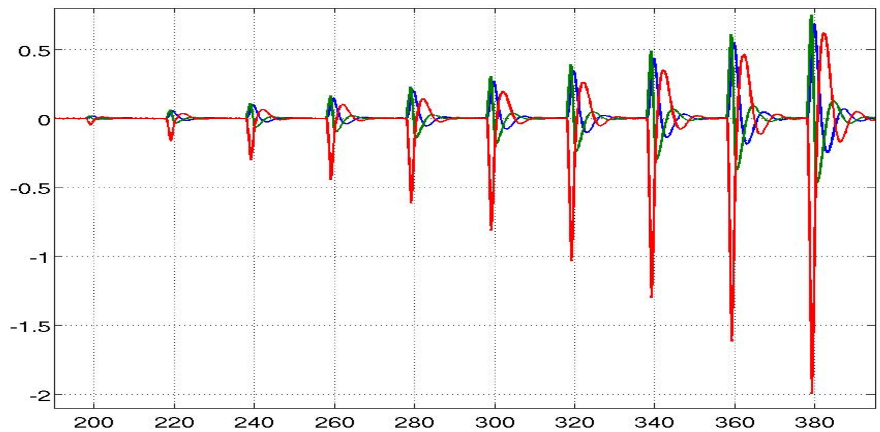


Figure 3 represents the time history of the signal $\delta_1(t)$ that is applied together with the time history of performance output $z_L(t)$. This plot clearly shows the relevance of the L_2-gain attenuation levels given in Figure 2. Figure 4 represents a zoom on the time histories of the measured outputs $y(t)$. It illustrates the quick convergence of the system in response to the disturbance inputs which is deteriorated as the parameter δ_1 grows staring from values close to which the L_2 gain attenuation is minimal. Figure 5 shows time histories of the control gain $K(t)$ parameters. It illustrates adaptation to moving $\delta_1(t)$ values and quick convergence to valid values starting from zero initial conditions. Figure 6 shows time histories of $\mathrm{Tr}(K^T(t)K(t))$. It demonstrates the bounded behavior of gains $K(t)$. Note that although the parameter $\delta_1(t)$ temporarily goes beyond the stability margin 0.723, the system does not have time to diverge. This behavior is possible thanks to temporary growth of $\mathrm{Tr}(K^T(t)K(t))$ beyond the value $\alpha = 10$.

Remark 3 *About the choice of parameters G, α, β, Γ :*

- *To illustrate the impact of the choice of G, the same example has been tested for $G = \left[\ 40\ \ \ 30\ \ \ 20\ \right]$, $\alpha = 10$, $\beta = 2$. In that case the maximal $\bar{\delta}_1$ such that the system is closed-loop passive for all $(\delta_1, \delta_2) \in [\ -1\ \ \bar{\delta}_1\ ] \times [\ 0\ \ 2.5\ ]$ is $\bar{\delta}_1 = 0.031$. The choice of G is thus a critical question.*

Figure 4: Zoom on $y(t)$ histories



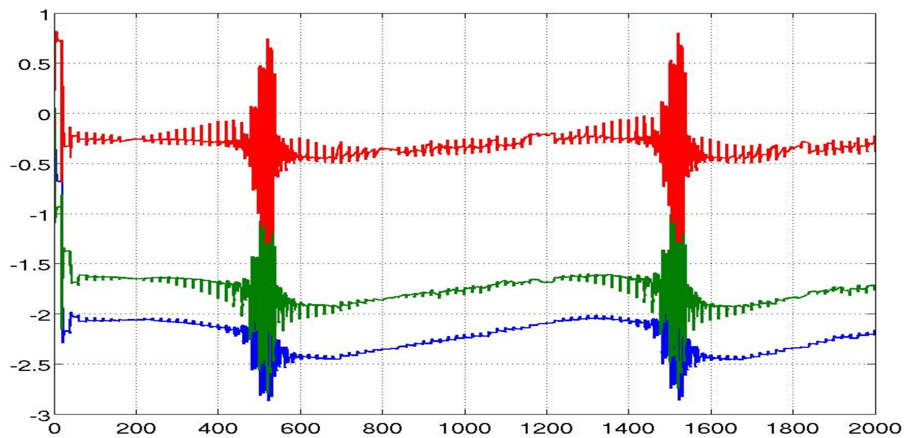
- *To apprehend the influence of the α parameter, the same test is done for $G = \left[\begin{array}{ccc} 400 & 300 & 200 \end{array}\right]$, $\beta = 2$ and $\alpha = 0.1$. The LMIs are then infeasible, whatever the uncertainties. This makes sense because the system is not stabilizable for such small feedback gains. One would therefore expect that when α increases then the stabilization domain also grows. But unfortunately this is not true. For $\alpha = 11$, the maximal $\bar{\delta}_1$ is $\bar{\delta}_1 = 0.708$. The fact that is is untrue is due to conservatism of the LMIs (in particular due to the various over-bounding inequalities used in the proofs).*

- *Related to the upper discussion on α, some tests are also performed to evaluate the influence of β. For $\alpha = 11$ and $\beta = 1.1$ the maximal $\bar{\delta}_1$ is then 0.739. Reducing β tends to reduce the conservatism, but taking β too close to the value 1 may imply difficulties in implementing the adaptive law (strong logarithmic barrier). One would therefore recommend taking $\beta > 1$ as small as admissible for implementation.*

6 Conclusions

New LMI based results are provided for proving robust passification via direct passification-based adaptive control in the case of uncertain polytopic LTI systems. LMI optimization allows as well to evaluate an upper bound on the L_2-gain attenuation level. Proof of the adaptive control properties is done using the existence of a parameter-dependent static control with same properties. But adaptive control is shown to be always not worse than such parameter-dependent strategy, better in terms of L_2 gain attenuation, but also because it avoids doing online estimation and avoids complex switching rules. The efficiency of the results is illustrated on an example.

Figure 5: $K(t)$ histories



Many issues remain still to explore. One of which is to consider other uncertain representations such as LFT modeling as it is done in [11] and to consider the case of possibly fast time-varying uncertain parameters. Other open questions are the evaluation of other performances such as convergence rate. But the main issue to be considered is the design of the G matrix. The choice of G is a critical issue. In the vein of [11], results for G design are expected to be more complex than pure LMI problems.

References

[1] K. Åström, B. Wittenmark, Adaptive Control, Addison-Wesley, 1989.

[2] I. Barkana, M. Teixeira, L. Hsu, Mitigation of symmetry condition in positive realness for adaptive control, Automatica 42 (9) (2006) 1611–1616.

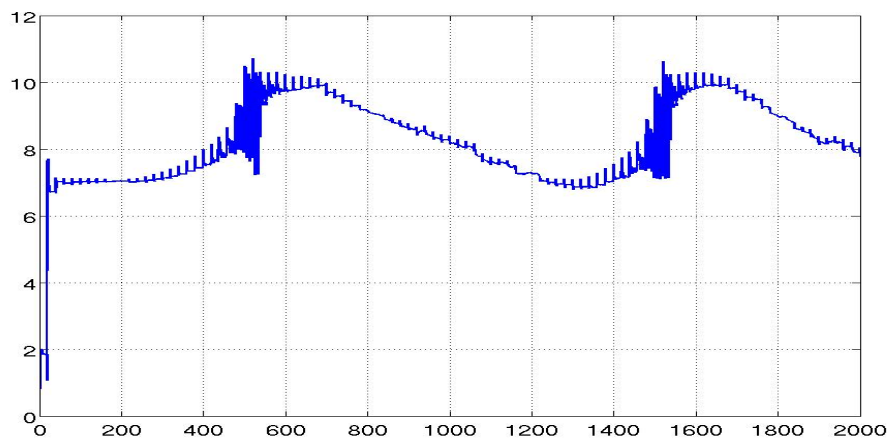
[3] R. Ben Yamin, I. Yaesh, U. Shaked, Simplified adaptive control with guaranteed H_∞ performance, in: IFAC Workshop on Adaptation and Learning in Control and Signal Processing, St. Petersburg, 2007.

[4] S. Boyd, L. E. Ghaoui, E. Feron, V. Balakrishnan, Linear Matrix Inequalities in System and Control Theory, SIAM Studies in Applied Mathematics, Philadelphia, 1994.

[5] A. Fradkov, Adaptive stabilization of an linear dynamic plant, Autom. Remote Contr. 35 (12) (1974) 1960–1966.

[6] A. Fradkov, Passification of non-square linear systems and feedback Yakubovich-Kalman-Popov lemma, European J. of Control 6 (2003) 573–582.

Figure 6: $\mathrm{Tr}(K^T(t)K(t))$ history



[7] P. Ioannou, J. Sun, Robust Adaptive Control, Prentice Hall, Inc, 1996.

[8] H. Kaufman, I. Barkana, K. Sobel, Direct adaptive control algorithms, Springer, New York, 1994.

[9] J. Löfberg, YALMIP : A Toolbox for Modeling and Optimization in MATLAB (2004).
URL `http://control.ee.ethz.ch/ joloef/yalmip.php`

[10] D. Peaucelle, D. Arzelier, O. Bachelier, J. Bernussou, A new robust D-stability condition for real convex polytopic uncertainty, Systems & Control Letters 40 (1) (2000) 21–30.

[11] D. Peaucelle, A. Fradkov, B. Andrievsky, Passification-based adaptive control : Robustness issues, in: IFAC Symposium on Robust Control Design, Toulouse, 2006, paper in an invited session.

[12] J. Sturm, Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones, Optimization Methods and Software 11-12 (1999) 625–653.
URL `http://sedumi.mcmaster.ca`.

[13] I. Yaesh, U. Shaked, Simplified adaptive control via improved robust positive real conditions, Systems & Control Letters 55 (2006) 1033–1040.