Ellipsoidal Output-Feedback Sets for Robust Multi-Performance Synthesis

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Abstract

The paper deals with static output-feedback design. It adopts a new framework based on the synthesis of ellipsoidal sets of controllers. The contribution is to formulate conditions for robust multi-performance design. The considered performance levels are defined as H_∞ and/or H_2 norms on possibly distinct linear time invariant systems. The numerical resolution of the obtained formulae is done with the help of a cone complementarity algorithm and validated on an illustrative example.

Keywords

Static Output-Feedback, H_2, H_∞, robust, dissipative uncertainty

I. INTRODUCTION

The Static Output Feedback (SOF) design is a central problem in control engineering and is still open [BER 92], [BLO 95], [SYR 97]. It has a most simple formulation. Consider an LTI system with the state-space representation:

$$\Sigma \ : \ \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input control vector and $y \in \mathbb{R}^p$ is the output measure vector. A SOF control law is defined by a constant gain matrix K, such that:

$$u(t) = Ky(t) \tag{2}$$

The closed-loop system is composed of two interconnected operators Σ and K. In the sequel, such an interconnection is denoted $\Sigma \overset{u,y}{\star} K$. By definition, the LTI system Σ is said to be *stabilisable* via static output-feedback if and only if there exists a gain matrix K such that $\Sigma \overset{u,y}{\star} K$ is stable. The design problem is to find such a stabilising gain K.

The framework adopted in this paper is based on the Lyapunov theory and uses matrix inequality based formulations. A matrix inequality, such as $A > B$, is defined using the Loëner partial order on symmetric matrices and reads for example as: $A - B$ is symmetric positive definite. Such matrix inequality formulations proved to be most effective to derive valuable results in the past years. In particular, it has been shown that linear matrix inequalities (LMIs) for which decision variables enter affinely in the formulae lead to convex optimisation problems that can be solved with efficient semi-definite programming tools (see [BOY 94], [ELG 00] for surveys on this topic).

Within the adopted framework, the stabilising SOF design writes as:

Theorem 1:
The LTI system Σ is stabilisable via static output-feedback if and only if there exist two matrices $\mathbf{P} \in \mathbb{S}^n$ and $\mathbf{K} \in \mathbb{R}^{m \times p}$ such that:

$$\begin{cases} \mathbf{P} > \mathbb{0} \\ (A + B\mathbf{K}(\mathbb{1} - D\mathbf{K})^{-1}C)'\mathbf{P} + \mathbf{P}(A + B\mathbf{K}(\mathbb{1} - D\mathbf{K})^{-1}C) < \mathbb{0} \end{cases}$$

Theorem 1 implies to solve non-linear matrix inequalities with respect to the variables written in bold-face. At our knowledge, there does not exist any exact solution to this problem. Perhaps one of the first papers dealing with this problem is [LEV 70] where a non-linear programming approach was proposed. Another well-known necessary and sufficient condition for static output-feedback stabilisability is given in [IWA 94]:

Theorem 2:
The LTI system Σ (with $D = \mathbb{0}$) is stabilisable via static output-feedback if and only if there exist two matrices $\mathbf{P} \in \mathbb{S}^n$ and $\mathbf{Q} \in \mathbb{S}^n$ such that:

$$\left\{ \begin{array}{ll} \mathbf{P} > \mathbb{0} & C'^{\perp}(A'\mathbf{P} + \mathbf{P}A)C^{\perp} < \mathbb{0} \\ \mathbf{Q} > \mathbb{0} & B^{\perp}(\mathbf{Q}A' + A\mathbf{Q})B'^{\perp} < \mathbb{0} \\ \mathbf{PQ} = \mathbb{1} & \end{array} \right.$$

where the rows of $B^{\perp}$ and $C^{\perp}$ form a basis for the null space of B' and C' respectively.

The difficulty holds in the non-linear elements, $\mathbf{PQ} = \mathbb{1}$. In [IWA 95], [GRI 96] different numerical approaches are proposed to address this difficulty.

Yet another SOF synthesis condition was published in [PEA 02] and was associated to a sub-optimal algorithm inspired from [ELG 97]. Let the two matrices:

$$L_1 = \left[\begin{array}{cc} \mathbb{1} & \mathbb{0} \\ A & B \end{array} \right] \qquad L_2 = \left[\begin{array}{cc} C & D \\ \mathbb{0} & \mathbb{1} \end{array} \right]$$

Theorem 3:
The LTI system Σ is stabilisable via static output-feedback if and only if there exist four matrices $\mathbf{P} \in \mathbb{S}^n$, $\mathbf{X} \in \mathbb{S}^p$, $\mathbf{Y} \in \mathbb{R}^{p \times m}$ and $\mathbf{Z} \in \mathbb{S}^m$ that simultaneously satisfy the following matrix inequalities:

$$\left\{ \begin{array}{l} \mathbf{X} \leq \mathbf{Y}\mathbf{Z}^{-1}\mathbf{Y}' \qquad \mathbf{Z} > \mathbb{0} \qquad \mathbf{P} > \mathbb{0} \\ \\ L_1' \left[\begin{array}{cc} \mathbb{0} & \mathbf{P} \\ \mathbf{P} & \mathbb{0} \end{array} \right] L_1 < L_2' \left[\begin{array}{cc} \mathbf{X} & \mathbf{Y} \\ \mathbf{Y}' & \mathbf{Z} \end{array} \right] L_2 \end{array} \right. \tag{3}$$

Although the results of theorems 1, 2 and 3 seem quite equivalent since they all write as matrix inequalities involving one particular non-linear element, it appears that the last formulation has major theoretical and practical features.

First, it is closely related to topological separation [SAF 80], [GOH 95]. The SOF design is shown to be equivalent to the design of a quadratic separator that defines a whole ellipsoidal set of controllers. A detailed investigation of the theoretical implications can be found in [PEA 03].

Second, the stabilisability result can be easily extended for important related applicative problems. In [PEA 02] fragility, bounded controller and pole location issues are exposed. Here, we focus on new contributions of the *ellipsoidal output-feedback sets* with two orientations:

- Robustness with respect to parametric uncertainties.

Dissipative non-structured uncertainties Δ are considered. The system's model is assumed to be a rational function of the uncertain parameters. This dependency is represented via a Linear Fractional Transform (LFT) interconnection. The contribution of the paper is to give design methods such that the obtained controller guarantees the closed-loop performances whatever the uncertainty realisation.

- H_∞ and H_2 performances.

Both the H_∞ and the H_2 induced norms of LTI systems are considered. These criteria prove to be important tools to characterise input/output performances such as noise and perturbation rejection as well as for loop shaping. Often, these two criteria are applied to independent input/output signals that may enter the model via weighting functions. Therefore, the multi-performance synthesis can be recast as the design of a common controller that guarantees H_∞ and/or H_2 closed-loop performances for various distinct systems. Such design specification, that goes beyond the multi-objective problem of [SCH 97], cannot be tackled with existing techniques. The new adopted framework on the contrary proves to be well adapted for the robust multi-performance synthesis. The result's formulation is similar to that of theorem 3.

The paper is organised as follows. First, some notations are defined and the uncertain system modelling is given. The third section is then devoted to the theoretical aspects; In turn the SOF design formulae are given for robust H_∞ and H_2 performances; The section ends with the formulation of the global multi-performance problem. The fourth section is devoted to numerical results that illustrate the contributions. At last, a conclusion suggests directions for prospective work.

II. Preliminaries

A. Notations

$\mathbb{R}^{m\times n}$ is the set of m-by-n real matrices and $\mathbb{S}^n$ is the subset of symmetric matrices in $\mathbb{R}^{n\times n}$. A' is the transpose of the matrix A. $\mathbb{1}$ and $\mathbb{0}$ are respectively the identity and the zero matrices of appropriate dimensions.

Throughout this paper, a particular set of matrices is used. Due to the notations and by extension of the notion of $\mathbb{R}^n$ ellipsoids, these sets are referred to as matrix ellipsoids of $\mathbb{R}^{m\times p}$:

Given three matrices $X \in \mathbb{S}^p$, $Y \in \mathbb{R}^{p\times m}$ and $Z \in \mathbb{S}^m$,
the $\{X,Y,Z\}$–ellipsoid of $\mathbb{R}^{m\times p}$ is the set of matrices $\mathbf{K}$ satisfying the following matrix inequalities:

$$Z > \mathbb{0} \qquad \begin{bmatrix} \mathbb{1} & \mathbf{K}' \end{bmatrix} \begin{bmatrix} X & Y \\ Y' & Z \end{bmatrix} \begin{bmatrix} \mathbb{1} \\ \mathbf{K} \end{bmatrix} \leq \mathbb{0} \tag{4}$$

By definition, $K_o \stackrel{\Delta}{=} -Z^{-1}Y'$ is the centre of the ellipsoid and $R \stackrel{\Delta}{=} K_o' Z K_o - X$ is the radius. A matrix ellipsoid is a compact convex set. An ellipsoid is non-empty if and only if the radius ($R \geq \mathbb{0}$) is positive semi-definite. More details can be found in [PEA 02].

B. Robustness with respect to dissipative uncertainty

Consider a continuous-time LTI system such as:

$$\begin{pmatrix} z \\ g \\ y \end{pmatrix} = \Sigma(s) \begin{pmatrix} w \\ v \\ u \end{pmatrix} \tag{5}$$

The measure output and control input are respectively $y \in \mathbb{R}^p$ and $u \in \mathbb{R}^m$. The input/output performances are defined in the sequel for the signals $g \in \mathbb{R}^{m_g}$ and $v \in \mathbb{R}^{p_v}$. The input $w \in \mathbb{R}^{m_w}$ and output $z \in \mathbb{R}^{p_z}$ define an exogenous feedback of an uncertainty matrix Δ defined by:

$$w(t) = \Delta z(t) \tag{6}$$

For any admissible uncertainty Δ, the uncertain model is an LTI system obtained through the inter-connexion $\Sigma_{\text{lft}}(\Delta) = \Sigma_{\text{lft}} \overset{w,z}{\star} \Delta$. The resulting state-space matrices are rational in the uncertain parameters. The inter-connexion defines a Linear Fractional Transformation (LFT).

The uncertain parameters are all gathered in a unique matrix Δ. They are assumed to be constant parametric uncertainties and the uncertainty set is a matrix ellipsoid of $\mathbb{R}^{m_w\times p_z}$ defined by:

$$\mathbb{\Delta}_{\text{lft}} = \{X_{\text{lft}}, Y_{\text{lft}}, Z_{\text{lft}}\}\text{-ellipsoid}$$

Such uncertainty sets are also known as $\{X_{\text{lft}}, Y_{\text{lft}}, Z_{\text{lft}}\}$-dissipative uncertainties. As reported in [MEG 97], [PEA 98], [SCO 98], [XIE 98], this modelling of uncertainties contains the well-known norm-bounded uncertainties ($\{-\mathbb{1}, \mathbb{0},$

$\mathbb{1}\}$-dissipative) and positive real uncertainties ($\{\mathbb{0},\ -\mathbb{1},\ \mathbb{0}\}$-dissipative) which respectively lead to the small gain and passivity frameworks.

In order to guarantee that the nominal system $\Sigma_{\text{lft}}(\mathbb{0})$ is included in the set of realisations $\Sigma_{\text{lft}}(\Delta)$, the matrix X_{lft} is assumed to be negative semi-definite ($X_{\text{lft}} \leq \mathbb{0}$).

Let $\Sigma(\Delta)$ be a generic uncertain model and $\mathbb{\Delta}$ any uncertainty set. The general robust stabilisability problem is defined as follows:

Find a gain $\mathbf{K}$ *such that the system* $\Sigma(\Delta) \overset{u,y}{\star} \mathbf{K}$ *is stable for all uncertainties* $\Delta \in \mathbb{\Delta}$.

In the assumed case of parametric constant uncertainty, it is achieved by exhibiting for each uncertainty $\Delta \in \mathbb{\Delta}$ a parameter-dependent Lyapunov function $V_r(x,\Delta) = x'\mathbf{P_r}(\Delta)x$ that proves the stability of the closed-loop $\Sigma(\Delta) \overset{u,y}{\star} \mathbf{K}$.

The quadratic stabilisability problem is defined as follows:

Find a gain $\mathbf{K}$ *and a quadratic Lyapunov function* $V_q(x) = x'\mathbf{P_q}x$ *such that* V_q *proves the stability of the system* $\Sigma(\Delta) \overset{u,y}{\star} \mathbf{K}$ *for all uncertainties* $\Delta \in \mathbb{\Delta}$.

Quadratic stabilisability is a particular instance of robust stabilisability where the Lyapunov matrix is unique over all the set of uncertain parameters $\mathbf{P_r}(\Delta) = \mathbf{P_q}$. To be more precise, quadratic stabilisability is a conservative (sufficient) condition for robust stabilisability. It has nevertheless, major advantages as attested by the considerable and valuable work devoted to this notion.

III. Performance levels

A. H_∞ performance

A common way of measuring robust performance and disturbance rejection is to use the L_2-induced operator norm. The H_∞ norm characterises input/output properties in terms of energy to energy, power to power and spectrum to spectrum relationships, [ZHO 94]. It can also be used for loop-shaping purpose by introducing weighting transfer functions.

Let the following state-space representation of a system such as (5):

$$\Sigma \ : \ \left\{ \begin{array}{rl} \dot{x}(t) = & Ax(t) + B_w w(t) + B_v v(t) + Bu(t) \\ z(t) = & C_z x(t) + D_{zw} w(t) + D_{zv} v(t) + D_{zu} u(t) \\ g(t) = & C_g x(t) + D_{gw} w(t) + D_{gv} v(t) + D_{gu} u(t) \\ y(t) = & Cx(t) + D_{yw} w(t) + D_{yv} v(t) + Du(t) \end{array} \right. \tag{7}$$

The matrix dimensions are such that $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$. The input w and the output z define the uncertainty exogenous feedback as in (6). The uncertain system is given by $\Sigma(\Delta) = \Sigma \overset{w,z}{\star} \Delta$. The guaranteed robust H_∞ synthesis problem is formulated as follows:

Find a stabilising gain $\mathbf{K}$ *such that for all uncertainties the closed-loop transfer from* v *to* g *has an* H_∞ *norm less than some specified level* γ_∞*:* $\quad \forall \Delta \in \mathbb{\Delta} \ , \ ||\Sigma(\Delta) \overset{u,y}{\star} \mathbf{K}||_\infty < \gamma_\infty$.

Let the four matrices:

$$M_1 = \begin{bmatrix} \mathbb{1} & \mathbb{0} & \mathbb{0} & \mathbb{0} \\ A & B_w & B_v & B \end{bmatrix} \qquad M_2 = \begin{bmatrix} C_z & D_{zw} & D_{zv} & D_{zu} \\ \mathbb{0} & \mathbb{1} & \mathbb{0} & \mathbb{0} \end{bmatrix}$$

$$M_3 = \begin{bmatrix} C_g & D_{gw} & D_{gv} & D_{gu} \\ \mathbb{0} & \mathbb{0} & \mathbb{1} & \mathbb{0} \end{bmatrix} \quad M_4 = \begin{bmatrix} C & D_{yw} & D_{yv} & D \\ \mathbb{0} & \mathbb{0} & \mathbb{0} & \mathbb{1} \end{bmatrix}$$

Theorem 4:
If there exist four matrices $\mathbf{P_\infty} \in \mathbb{S}^n$, $\mathbf{X} \in \mathbb{S}^p$, $\mathbf{Y} \in \mathbb{R}^{p\times m}$, $\mathbf{Z} \in \mathbb{S}^m$ and two scalars $\boldsymbol{\tau_\infty}$, $\boldsymbol{\tau_{\mathrm{lft}}}$ that simultaneously satisfy the constraints:

$$\begin{array}{c} \mathbf{X} \leq \mathbf{Y}\mathbf{Z}^{-1}\mathbf{Y}' \\ \boldsymbol{\tau_{\mathrm{lft}}} > 0 \qquad \boldsymbol{\tau_\infty} > 0 \qquad \mathbf{Z} > \mathbb{0} \qquad \mathbf{P_\infty} > \mathbb{0} \\ M_1' \begin{bmatrix} \mathbb{0} & \mathbf{P_\infty} \\ \mathbf{P_\infty} & \mathbb{0} \end{bmatrix} M_1 < \boldsymbol{\tau_{\mathrm{lft}}} M_2' \begin{bmatrix} X_{\mathrm{lft}} & Y_{\mathrm{lft}} \\ Y_{\mathrm{lft}}' & Z_{\mathrm{lft}} \end{bmatrix} M_2 + \boldsymbol{\tau_\infty} M_3' \begin{bmatrix} -\mathbb{1} & \mathbb{0} \\ \mathbb{0} & \gamma_\infty^2 \mathbb{1} \end{bmatrix} M_3 + M_4' \begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Y}' & \mathbf{Z} \end{bmatrix} M_4 \end{array} \tag{8}$$

then the $\{X, Y, Z\}$-ellipsoid is a set of quadratically stabilising gains such that $||\Sigma(\Delta) \overset{u,y}{\star} K||_\infty < \gamma_\infty$ for all $\Delta \in \mathbb{\Delta}_{\mathrm{lft}}$.

Proof : Take any matrix K in the $\{X, Y, Z\}$-ellipsoid and any uncertainty $\Delta \in \mathbb{\Delta}_{\mathrm{lft}}$. Multiply the left hand side of inequality (8) by vector $\begin{pmatrix} x' & w' & v' & u' \end{pmatrix}$ from and the right hand side by it's transpose. Due to system equations (2), (6) and (7), it writes:

$$xP_\infty\dot{x} + \dot{x}P_\infty x < \tau_{\mathrm{lft}} z' \begin{bmatrix} \mathbb{1} & \Delta \end{bmatrix} \begin{bmatrix} X_{\mathrm{lft}} & Y_{\mathrm{lft}} \\ Y_{\mathrm{lft}}' & Z_{\mathrm{lft}} \end{bmatrix} \begin{bmatrix} \mathbb{1} \\ \Delta \end{bmatrix} z - \tau_\infty g'g + \tau_\infty\gamma_\infty^2 v'v + y' \begin{bmatrix} \mathbb{1} & K \end{bmatrix} \begin{bmatrix} X & Y \\ Y' & Z \end{bmatrix} \begin{bmatrix} \mathbb{1} \\ K \end{bmatrix} y$$

By definition of the uncertainties and the controller matrix gain, the Δ and K dependent terms are negative, therefore:

$$xP_\infty\dot{x} + \dot{x}P_\infty x < -\tau_\infty g'g + \tau_\infty\gamma_\infty^2 v'v$$

Taking the perturbation-free system $v = 0$ one gets that the Lyapunov function $V(x) = x'P_\infty x$ proves the stability for all the uncertainties (quadratic stability). Moreover, taking the time average over the last inequality one gets that:

$$\tau_\infty ||g||^2 < \tau_\infty\gamma_\infty^2 ||v||^2$$

which proves the bound on the H_∞ norm. ■

Corollary 1: Theorem 4 can be particularised as follows.

- Take $v \in \mathbb{R}^0$ and $g \in \mathbb{R}^0$, then the conditions correspond to the synthesis of a robustly stabilising SOF gain without performance specifications.
- Take $w \in \mathbb{R}^0$ and $z \in \mathbb{R}^0$, then the conditions correspond to the synthesis of a SOF gain with H_∞ performance without robustness characteristics.
- Take both, then the condition resumes to that of theorem 3.
- Take $u \in \mathbb{R}^0$ and $y \in \mathbb{R}^0$, then the conditions correspond to the analysis of robust H_∞ performance. In that case it is purely LMI. Moreover, with simple manipulations (divide all variables by $\boldsymbol{\tau_\infty}$) the conditions are also linear in γ_∞. Hence, it can be formulated as an LMI optimisation problem that gives an upper-bound on the robust H_∞ performance of a given quadratically stable system.

B. H_2 performance

The input/output performance may also be expressed in terms of H_2 norm specifications. It allows to characterise white noise response [PAG 97], [PAG 00] and is frequently used in practical situations. Let an LTI system given by its state-space representation:

$$\tilde{\Sigma} \ : \ \begin{cases} \dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}_w\tilde{w}(t) + \tilde{B}_v\tilde{v}(t) + \tilde{B}u(t) \\ \tilde{z}(t) = \tilde{C}_z\tilde{x}(t) + \tilde{D}_{zw}\tilde{w}(t) + \tilde{D}_{zv}\tilde{v}(t) + \tilde{D}_{zu}u(t) \\ \tilde{g}(t) = \tilde{C}_g\tilde{x}(t) + \tilde{D}_{gw}\tilde{w}(t) + \tilde{D}_{gv}\tilde{v}(t) + \tilde{D}_{gu}u(t) \\ y(t) = \tilde{C}\tilde{x}(t) + \tilde{D}_{yw}\tilde{w}(t) + \tilde{D}_{yv}\tilde{v}(t) + \tilde{D}u(t) \end{cases} \tag{9}$$

The matrix dimensions are such that $\tilde{x} \in \mathbb{R}^{\tilde{n}}$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$. The input $\tilde{w}$ and output $\tilde{z}$ define the uncertainty exogenous feedback $\tilde{w} = \tilde{\Delta}\tilde{z}$ where $\tilde{\Delta}$ belongs to the $\{\tilde{X}_{\text{lft}}, \tilde{Y}_{\text{lft}}, \tilde{Z}_{\text{lft}}\}$-ellipsoid ($\tilde{\mathbb{\Delta}}_{\text{lft}}$). The uncertain system is given by $\tilde{\Sigma}(\tilde{\Delta}) = \tilde{\Sigma} \overset{\tilde{w},\tilde{z}}{\star} \tilde{\Delta}$.

The guaranteed robust H_2 synthesis problem is formulated as follows:

Find a stabilising gain $\mathbf{K}$ *such that for all uncertainties the closed-loop transfer from* $\tilde{v}$ *to* $\tilde{g}$ *has an* H_2 *norm less than some specified level* γ_2*:* $\quad \forall \tilde{\Delta} \in \tilde{\mathbb{\Delta}}\ ,\ ||\tilde{\Sigma}(\tilde{\Delta}) \overset{u,y}{\star} \mathbf{K}||_2 < \gamma_2.$

Assumption

The H_2 norm of some continuous-time transfer matrix is only defined if the feed-through matrix is zero. For the system (9), the closed-loop feed-through matrix of $\tilde{\Sigma}(\tilde{\Delta}) \overset{u,y}{\star} \mathbf{K}$ depends on $\tilde{\Delta}$ and $\mathbf{K}$. Some assumptions are required for $\tilde{D}_{gv}(\tilde{\Delta}, \mathbf{K})$ be zero whatever the controller $\mathbf{K}$ and the uncertainty $\tilde{\Delta}$ are. In this paper, the results are given only for the case when $\tilde{D}_{gv} = \mathbb{0}$, $\tilde{D}_{zv} = \mathbb{0}$ and $\tilde{D}_{yv} = \mathbb{0}$. Obviously, other assumptions are possible but they lead to tedious formulations that we chose to avoid here.

Let the four matrices:

$$N_1 = \begin{bmatrix} \mathbb{1} & \mathbb{0} & \mathbb{0} \\ \tilde{A} & \tilde{B}_w & \tilde{B} \end{bmatrix} \qquad N_2 = \begin{bmatrix} \tilde{C}_z & \tilde{D}_{zw} & \tilde{D}_{zu} \\ \mathbb{0} & \mathbb{1} & \mathbb{0} \end{bmatrix}$$

$$N_3 = \begin{bmatrix} \tilde{C}_g & \tilde{D}_{gw} & \tilde{D}_{gu} \end{bmatrix} \qquad N_4 = \begin{bmatrix} \tilde{C} & \tilde{D}_{yw} & \tilde{D} \\ \mathbb{0} & \mathbb{0} & \mathbb{1} \end{bmatrix}$$

Theorem 5:

If there exist four matrices $\mathbf{P_2} \in \mathbb{S}^{\tilde{n}}$, $\mathbf{X} \in \mathbb{S}^p$, $\mathbf{Y} \in \mathbb{R}^{p \times m}$, $\mathbf{Z} \in \mathbb{S}^m$ and two scalars $\boldsymbol{\tau_2}$, $\tilde{\boldsymbol{\tau}}_{\mathbf{lft}}$ that simultaneously satisfy the constraints:

$$\begin{gathered} \mathbf{X} \leq \mathbf{Y}\mathbf{Z}^{-1}\mathbf{Y}' \\ \tilde{\boldsymbol{\tau}}_{\mathbf{lft}} > 0 \qquad \boldsymbol{\tau_2} > 0 \qquad \mathbf{Z} > \mathbb{0} \qquad \mathbf{P_2} > \mathbb{0} \\ Trace(\tilde{B}_v'\mathbf{P_2}\tilde{B}_v) \leq \boldsymbol{\tau_2}\gamma_2^2 \\ N_1' \begin{bmatrix} \mathbb{0} & \mathbf{P_2} \\ \mathbf{P_2} & \mathbb{0} \end{bmatrix} N_1 < \tilde{\boldsymbol{\tau}}_{\mathbf{lft}} N_2' \begin{bmatrix} \tilde{X}_{\text{lft}} & \tilde{Y}_{\text{lft}} \\ \tilde{Y}_{\text{lft}}' & \tilde{Z}_{\text{lft}} \end{bmatrix} N_2 - \boldsymbol{\tau_2} N_3' N_3 + N_4' \begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Y}' & \mathbf{Z} \end{bmatrix} N_4 \end{gathered} \tag{10}$$

then the $\{X, Y, Z\}$-ellipsoid is a set of quadratically stabilising gains such that $||\,\tilde{\Sigma}(\tilde{\Delta}) \overset{u,y}{\star} K||_2 < \gamma_2$ for all $\tilde{\Delta} \in \tilde{\mathbb{\Delta}}_{\text{lft}}$.

The proof follows the lines of theorem's 4 proof. It is omitted for conciseness.

Corollary 2: Theorem 5 can be particularised as follows.

- Take $\tilde{w} \in \mathbb{R}^0$ and $\tilde{z} \in \mathbb{R}^0$, then the conditions correspond to the synthesis a SOF gain with H_2 performance without robustness characteristics.

- Take $u \in \mathbb{R}^0$ and $y \in \mathbb{R}^0$, then the conditions correspond to the analysis of robust H_2 performance. In that case it is purely LMI. Moreover, with simple manipulations (divide all variables by $\boldsymbol{\tau_2}$) the conditions are also linear in γ_2. Hence, it can be formulated as an LMI optimisation problem that gives an upper-bound on the robust H_2 performance of a given quadratically stable system.

C. Robust multi-performance synthesis

The multi-performance synthesis problem amounts to a collection of H_∞ and H_2 specifications, each of which are defined for possibly distinct uncertain LTI systems. All the uncertain models should have common control input / measure output dimensions. The design objective is then to find a common controller that satisfies all the specifications.

In order to alleviate the notations, consider only two such specifications. One is a robust H_∞ bound specification on a system $\Sigma(\Delta)$ and the second is a robust H_2 bound on a system $\tilde{\Sigma}(\tilde{\Delta})$. The robust multi-performance synthesis problem writes as:

For two given levels on the H_∞ and H_2 norms, γ_∞ and γ_2 respectively, find a stabilising gain $\mathbf{K}$ such that:

$$\forall \Delta \in \mathbb{\Delta} \ , \ ||\Sigma(\Delta) \overset{u,y}{\star} \mathbf{K}||_\infty < \gamma_\infty$$
$$\forall \tilde{\Delta} \in \tilde{\mathbb{\Delta}} \ , \ ||\tilde{\Sigma}(\tilde{\Delta}) \overset{u,y}{\star} \mathbf{K}||_2 < \gamma_2.$$

The result is straightforward. It amounts to the collection of all related matrix inequality constraints.

Theorem 6:
If there exist five matrices $\mathbf{P}_\infty \in \mathbb{S}^n$, $\mathbf{P_2} \in \mathbb{S}^{\tilde{n}}$, $\mathbf{X} \in \mathbb{S}^p$, $\mathbf{Y} \in \mathbb{R}^{p\times m}$, $\mathbf{Z} \in \mathbb{S}^m$ and four scalars $\boldsymbol{\tau}_\infty$, $\boldsymbol{\tau}_{\mathbf{lft}}$, $\boldsymbol{\tau}_2$, $\tilde{\boldsymbol{\tau}}_{\mathbf{lft}}$ that simultaneously satisfy the constraints (8) and (10), then the $\{X, Y, Z\}$-ellipsoid is a set of quadratically stabilising gains for both systems Σ and $\tilde{\Sigma}$ such that the performance levels are robustly satisfied.

The theorem illustrates that the *ellipsoidal output-feedback sets* enable to formulate a wide variety of design problems that may include robust or not specifications such as quadratic stability, H_∞ and H_2 performances. With the help of results in [PEA 02], these specifications can be enriched with closed-loop pole location as well as constraints on the structure of the control law K and resiliency characteristics.

All such SOF design problems write as a feasibility problem over matrix inequalities. More accurately speaking, they are all equivalent to finding an admissible solution $(\mathbf{Q}, \mathbf{X}, \mathbf{Y}, \mathbf{Z})$ to the constraints summarised as:

$$\mathcal{L}(\mathbf{Q}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) < \mathbb{0} \qquad \text{and} \qquad \mathbf{X} \leq \mathbf{Y}\mathbf{Z}^{-1}\mathbf{Y}' \tag{11}$$

where $\mathbf{Q}$ represents all the stacked variables such as the Lyapunov matrices $\mathbf{P}_\bullet$ and other scalars $\boldsymbol{\tau}_\bullet$, and where $\mathcal{L}(\cdot)$ is a linear matrix operator. The first constraint $\mathcal{L}(\mathbf{Q}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) < \mathbb{0}$ is convex and there exist efficient numerical tools to solve such LMI constraints. The main difficulty comes from the non-linear constraint.

IV. Numerical issues and examples

A. Cone complementarity algorithm

The numerical examples are solved using a first order iterative algorithm. It is based on a cone complementarity technique, [ELG 97], that allows to concentrate the non convex constraint in the criterion of some optimisation problem.

Lemma 1:
The problem (11) is feasible if and only if zero is the global optimum of the optimisation problem:

$$\begin{array}{ll} \min & \mathrm{trace}(\mathbf{TS}) \\ \text{s.t.} & \mathcal{L}(\mathbf{Q}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) < \mathbb{0} \\ & \mathbf{X} \leq \hat{\mathbf{X}} \qquad \mathbf{S} = \begin{bmatrix} \hat{\mathbf{X}} & \mathbf{Y} \\ \mathbf{Y}' & \mathbf{Z} \end{bmatrix} \geq \mathbb{0} \\ & \mathbf{T_1} \geq \mathbb{1} \qquad \mathbf{T} = \begin{bmatrix} \mathbf{T_1} & \mathbf{T_2} \\ \mathbf{T_2}' & \mathbf{T_3} \end{bmatrix} \geq \mathbb{0} \end{array} \tag{12}$$

Proof : The constraints $T \geq \mathbb{0}$ and $S \geq \mathbb{0}$ make that $\mathrm{trace}(TS) = 0$ implies $TS = \mathbb{0}$ and therefore:

$$T_1\hat{X} + T_2Y' = \mathbb{0} \qquad\qquad T_1Y + T_2Z = \mathbb{0}$$

Since both matrices T_1 and Z are non singular under the LMI constraints, it implies:

$$\hat{X} = -T_1^{-1}T_2Y' = -T_1^{-1}(-T_1YZ^{-1})Y' = YZ^{-1}Y'$$

Thus the nonlinear constraint is satisfied: $X \leq \hat{X} = YZ^{-1}Y'$.
The converse implication is proved taking $\hat{X} = YZ^{-1}Y'$ and T such that $TS = \mathbb{0}$. ■

As in [APK 98], [ELG 97], [LEI 01], the optimisation problem (12) can then be solved with a first order conditional gradient algorithm also known as the Frank and Wolfe feasible direction method. Its properties are not reminded here. Note only that the non linear objective trace($\mathbf{TS}$) is relaxed as the linear objective trace($T_k\mathbf{S} + \mathbf{T}S_k$). The obtained LMI optimisation is repeated iteratively with matrices T_k and S_k computed from each previous optimisation step. The obtained sequence, trace(T_kS_k), is strictly decreasing. There is no guarantee that the algorithm converges to the global optimum. It can either stop at a local optimum or due to “plateauing” behaviour.

Remark 1: The stopping criteria of the usual gradient algorithm is either related to slow progress of the optimisation objective or to the achievement of $\text{trace}(TS) = 0$. In the first case, the algorithm fails due to “plateauing” behaviour or because it found a non satisfactory local optimum. The second case corresponds to the expected success of the algorithm. Unfortunately, due to the constraints $\mathbf{T} \geq \mathbb{0}$ and $\mathbf{S} \geq \mathbb{0}$ the algorithm is more often stopped while $\text{trace}(TS) = \varepsilon$ where ε is a chosen accuracy level. The exact non linear constraint may then not be exactly satisfied which is a significant weakness of the algorithm.

As a matter of fact, since the equality constraint involving $\hat{X}$ is not the goal of the original problem (11), we adopted in the numerical examples the following stopping criteria for the conditional gradient algorithm:

- If the progress of the optimisation objective is below a chosen level, then STOP, the algorithm failed.
- As soon as $X \leq YZ^{-1}Y'$, STOP, a stabilising ellipsoid is found.

This allows in all tested examples to avoid several optimisation steps which can be highly valuable for large problems.

B. VTOL Example

The model is taken from [SIN 84]. It characterises the longitudinal motion of a VTOL helicopter. It is composed of four states, two control inputs and one measured output. The linearised uncertain model is the same as in [KEE 88] and additional performance input/output vectors are given following those in [LEI 01].

The robust H_2 performance is defined for a model $\tilde{\Sigma}$ such that:

$$\tilde{A} = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & 0.3681 & -0.7070 & 1.4200 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \tilde{B}_w = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{B}_v = \begin{bmatrix} 0.0468 & 0 \\ 0.0457 & 0.0099 \\ 0.0437 & 0.0011 \\ -0.0218 & 0 \end{bmatrix} \quad \tilde{B} = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.5446 & -7.5922 \\ -5.5200 & 4.4900 \\ 0 & 0 \end{bmatrix}$$

$$\tilde{C}_z = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \tilde{D}_{zw} = \mathbb{0} \quad \tilde{D}_{zv} = \mathbb{0} \quad \tilde{D}_{zu} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\tilde{C}_g = \begin{bmatrix} \frac{2}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix} \quad \tilde{D}_{gw} = \mathbb{0} \quad \tilde{D}_{gv} = \mathbb{0} \quad \tilde{D}_{gu} = \frac{1}{\sqrt{2}}\mathbb{1}$$

$$\tilde{C} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \quad \tilde{D}_{yw} = \mathbb{0} \quad \tilde{D}_{yv} = \mathbb{0} \quad \tilde{D} = \mathbb{0}$$

and the three uncertainties are gathered in a diagonal matrix such that:

$$\Delta = \text{diag}(\Delta_{p_1}, \Delta_{p_2}, \Delta_{p_3})$$

$$|\Delta_{p_1}| \leq \alpha 0.05 \quad |\Delta_{p_2}| \leq \alpha 0.01 \quad |\Delta_{p_3}| \leq \alpha 0.04$$

In [KEE 88] the uncertainties correspond to $\alpha = 1$. Here will be considered more important variations of the uncertain parameters, $\alpha \geq 1$. The chosen modelling of uncertainties does not allow to take into account the structured nature of Δ. It will therefore be embodied into a larger uncertainty domain $\mathbb{\Delta}_{\text{lft}}$ defined as the $\{ \begin{bmatrix} -(\alpha 0.05)^2 & 0 & 0 \\ 0 & -(\alpha 0.01)^2 & 0 \\ 0 & 0 & -(\alpha 0.04)^2 \end{bmatrix}, \mathbb{0}, \mathbb{1} \}$-ellipsoid.

The robust H_∞ performance is defined for a slightly different model. It is obtained by considering weighting, first order operators $\frac{1}{s+1}$, applied on the $\tilde{v}$. The resulting model Σ is such that:

$$A = \begin{bmatrix} \tilde{A} & \tilde{B}_v \\ \mathbb{0} & -\mathbb{1} \end{bmatrix} \quad B_w = \begin{bmatrix} \tilde{B}_w \\ \mathbb{0} \end{bmatrix} \quad B_v = \begin{bmatrix} \mathbb{0} \\ \mathbb{1} \end{bmatrix} \quad B = \begin{bmatrix} \tilde{B} \\ \mathbb{0} \end{bmatrix}$$

$$C_z = \begin{bmatrix} \tilde{C}_z & \mathbb{0} \end{bmatrix} \quad D_{zw} = \tilde{D}_{zw} \quad D_{zv} = \tilde{D}_{zv} \quad D_{zu} = \tilde{D}_{zu}$$

$$C_g = \begin{bmatrix} \tilde{C}_g & \mathbb{0} \end{bmatrix} \quad D_{gw} = \tilde{D}_{gw} \quad D_{gv} = \tilde{D}_{gv} \quad D_{gu} = \tilde{D}_{gu}$$

$$C = \begin{bmatrix} \tilde{C} & \mathbb{0} \end{bmatrix} \quad D_{yw} = \tilde{D}_{yw} \quad D_{yv} = \begin{bmatrix} 0.00039 & 0.00174 \end{bmatrix} \quad D_{yu} = \tilde{D}_{gu}$$

Note that the D_{yv} matrix is non zero as in [LEI 01].

For the models described in this way, several numerical experiments are performed using the cone complementarity algorithm. These tests are realised for various specifications on the H_∞ performance (γ_∞), on the H_2 performance (γ_2) as well as for various uncertainty levels (α). Here are presented only few significative cases described in table I where `iter` is the number of the algorithms iterations, `CPU` is the total CPU time (LMIs solved with SeDuMi [STU 99] on a SUN SunBlade100 computer), `Tr(TS)` is the value of the optimisation criteria $\text{trace}(T_k S_k)$ at the step when the algorithm stopped, and K_o is the controller obtained as the centre of the stabilising ellipsoid.

test	γ_∞	γ_2	α	`iter`	`CPU`	`Tr(TS)`	K_o'
(a)	0.5	0.3	3	4	8s	500	[0.014 1.55]
(b)	0.5	0.3	5	4	9s	700	[0.059 2.45]
(c)	0.5	0.3	7	4	9s	400	[0.043 1.93]
(d)	3	3	10	65	167s	0.006	[-0.68 0.94]
(e)	10	10	13	51	122s	0.02	[-0.52 1.21]
(f)	10	10	14	65	166s	0.01	[-0.57 1.27]
(g)	3	3	14	36	85s	4	fails

TABLE I

NUMERICAL EXPERIMENTS

Comments:

- The proposed method is conservative, which means that if the algorithm fails it does not mean that there is no such controller. This can be observed when making the comparison between tests (f) and (g). The last one fails but nevertheless if an analysis step is performed on the solution of test (f) one finds out that:

$$\forall \Delta \in \mathbb{\Delta}_{\text{lft}} \;,\; ||\Sigma(\Delta) \overset{u,y}{\star} K_{o(f)}||_\infty < 0.61 \;,\; ||\tilde{\Sigma}(\Delta) \overset{u,y}{\star} K_{o(f)}||_2 < 0.17$$

which means that the solution to (f) could also be a solution to (g), ignored by the algorithm.

• The synthesis method not only concludes with a stabilising gain but moreover gives a whole set of controllers described by an ellipsoid. All the elements inside the ellipsoid guarantee the same properties. To illustrate this, take figure 1 on which the ellipsoids are those obtained for the six successful cases. These ellipsoidal sets can be used to evaluate the resilience of the closed-loop systems as in [PEA 02].

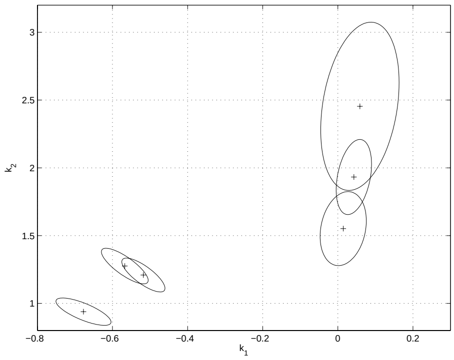

Fig. 1. SOF ellipsoids

V. Conclusion

The design of ellipsoidal sets of controllers is a new framework for output-feedback synthesis. Some aspects of it are exhibited in this paper contributing to the design of robustly stabilising SOF controllers that guarantee bounds on H_∞ and H_2 performances. Treated problems go from the design of SOF stabilising gains for a unique LTI model, to the design of SOF gains satisfying robust performance specifications for multiple distinct models. One would expect that each of these individual problems have different numerical complexities. But in fact, it appears that they all have a similar formulation composed of a unique non-linear inequality and LMI constraints. The sole numerical difference of all these problems is the size of the LMIs and the number of variables.

Prospective work will concern other implications of the ellipsoidal sets of controllers. Extensions will be proposed for robustness with respect to structured uncertainties and dynamic output-feedback. Moreover, we will insist on numerical aspects of the adopted algorithm. Intensive experiments on realistic industrial problems will be described.

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