# AN EFFICIENT NUMERICAL SOLUTION FOR $\mathcal{H}_2$ STATIC OUTPUT FEEDBACK SYNTHESIS

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## Abstract

This paper addresses the problem of static output feedback synthesis and focuses on $\mathcal{H}_2$ optimisation. The bilinear problem of finding a control feedback gain $\mathbf{K}$ and a Lyapunov matrix $\mathbf{P}$ is shown to be equivalent to a BMI problem that involves slack variables and a state feedback gain. This BMI condition is a promising theoretical result that links the two state and output feedback questions in a unified formulation. Based on this new expression, an efficient numerical procedure is derived. Some telling examples show that it is quite competitive compared to other algorithms proposed in the literature. Extensions to robust $\mathcal{H}_2$ optimal synthesis are derived. They show that unlike other BMI approaches, the method is of the same complexity if the system is certain or uncertain.

## 1 Introduction

The static or reduced fixed-order dynamic output feedback problem is always an active research area in the control literature. Given a Linear Time-Invariant, (LTI), system, the problem of finding a stabilising static output feedback is still open, [19]. In the recent years, many attempts have been made to give efficient numerical procedures to solve related problems, [1, 5, 8, 9, 10, 11]. In [16], a numerical comparison is performed and some classification is proposed.

In this paper, the problem of $\mathcal{H}_2$ static output feedback stabilisation is revisited. This problem which has been dealt with for the first time in [14], cannot be transformed in a convex problem on the contrary of the state feedback case, [3]. We therefore propose a new parameterisation of all stabilising static output feedback gains based on the introduction of additional variables. Even though, this parameterisation is nonconvex, it is however possible to propose an efficient numerical procedure based on cross-decomposition. Moreover, not only precisely known systems may be tackled with such an algorithm but also uncertain models with the same complexity.

The outline of the paper is the following. First, the problem is stated and well-known conditions from the literature are recalled. In the third part, all the results are presented. First, a new necessary and sufficient condition of static output feedback stabilisability is presented. The associated numerical procedure is then developed. The final section is devoted to some telling numerical examples from the literature.

<u>Notations:</u> In some equations, for conciseness reasons, the following notations are used

- $sym\{A\} = A + A'$
- $\bigstar A = A'A$ and $\bigstar BA = A'BA$
- $\begin{bmatrix} A & * \\ B & C \end{bmatrix} = \begin{bmatrix} A & B' \\ B & C \end{bmatrix}$

For symmetric matrices, $>$ $(\geq)$ is the Loëner partial order, i.e., $A > (\geq) B$ iff $A - B$ is positive (semi) definite. $CO.\{A^1 \ \dots \ A^N\}$ is the convex hull of the given vertex matrices.

## 2 Problem statement

Consider an LTI system defined as:

$$\left\{ \begin{array}{rlll} \dot{x}(t) = & Ax(t) & +B_2 w(t) & +Bu(t) \\ z(t) = & C_2 x(t) & & +D_2 u(t) \\ y(t) = & Cx(t) & & \end{array} \right. \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control vector, $y \in \mathbb{R}^r$ is the output vector, $w \in \mathbb{R}^q$ is the disturbance vector and $z \in \mathbb{R}^p$ is the controlled output vector. All matrices are assumed to be of appropriate dimensions and we assume that $rank(B) = m$ and $rank(C) = r$.

By definition, the system is stabilisable by static output feedback if there exists a gain matrix $\mathbf{K}$ such that the closed loop system $\dot{x}(t) = (A + B\mathbf{K}C)x(t)$ is stable. Moreover, the system is stabilisable by state feedback if there exists a gain matrix $\mathbf{K_s}$ such that $\dot{x}(t) = (A + B\mathbf{K_s})x(t)$ is stable. This last property is a special case of the former and corresponds to full state information output feedback ($C = \mathbb{1}$). Let $\mathcal{K}$ and $\mathcal{K}_s$ be respectively the sets of all static output feedback stabilising gains and all state feedback stabilising gains.

For $K \in \mathcal{K}$ and $K_s \in \mathcal{K}_s$ consider the following transfer functions:

$$\begin{array}{l} T(s,K) = (C_2 + D_2 KC)(s\mathbb{1} - (A + BKC))^{-1} B_2 \\ T_s(s,K_s) = (C_2 + D_2 K_s)(s\mathbb{1} - (A + BK_s))^{-1} B_2 \end{array} \tag{2}$$

Let $\|T(s,K)\|_2 = \Gamma(K)$ be the $\mathcal{H}_2$ norm of the transfer matrix $T(s,K)$. The problem to be addressed is stated as follows :

*Find $K^{opt} \in \mathcal{K}$ such that $\Gamma(K)$ is minimum*, i.e.

$$K^{opt} = arg \left\{ \min_{\mathbf{K}\in\mathcal{K}} \; ||T(s,\mathbf{K})||_2 \right\} \tag{3}$$

Note that the problem of static output feedback is of major importance since it encompasses the problem of fixed order controller design, [18]. Moreover, we have chosen to address $\mathcal{H}_2$ optimal synthesis but other questions such as $\mathcal{H}_\infty$ control or pole assignment can be considered with the same methods (see [2, 17] for unified formulations of these problems).

In [16], a numerical comparison of several recent output feedback design techniques is provided and the available methods are classified. The method proposed in our paper may be classified as a parametric optimisation method. For this class of techniques, the optimisation problem (3) is recast as the following minimisation over Bilinear Matrix Inequality (BMI) constraints:

$$\min_{\mathbf{P}\,,\,\mathbf{K}} \; trace(B_2'\mathbf{P}B_2) \;\; s.t. \quad \left\{ \begin{array}{l} \mathbf{P} > 0 \\ sym\{\mathbf{P}(A+B\mathbf{K}C)\} + \bigstar(C_2 + D_2\mathbf{K}C) < \mathbb{0} \end{array} \right. \tag{4}$$

In [9, 18], the idea is then to apply the elimination lemma on (4) to get an equivalent optimisation problem involving two Linear Matrix Inequality (LMI) constraints related by a nonconvex algebraic constraint :

$$\min_{\mathbf{P}\,,\,\mathbf{X}} \; trace(B_2'\mathbf{P}B_2) \;\; s.t. \quad \left\{ \begin{array}{ll} \mathbf{P} = \mathbf{X}^{-1} > 0 & (A) \\ \bigstar \begin{bmatrix} A'\mathbf{P} + \mathbf{P}A & C_2' \\ C_2 & -\mathbb{1} \end{bmatrix} \begin{bmatrix} C' \\ \mathbb{0} \end{bmatrix}^{\perp'} < \mathbb{0} & (B) \\ \bigstar \begin{bmatrix} A\mathbf{X} + \mathbf{X}A' & \mathbf{X}C_2' \\ C_2\mathbf{X} & -\mathbb{1} \end{bmatrix} \begin{bmatrix} B \\ D_2 \end{bmatrix}^{\perp'} < \mathbb{0} & (C) \end{array} \right.$$

Then, three different approaches may be proposed to handle the non convex constraint (A), [16]. Loosely speaking, it always consists in performing an iterative procedure composed of two steps, that is expected to converge to a solution such that (A) is verified.

In this paper, we also propose a two step iterative algorithm but the constraints are handled differently. The main difference is that while in [9, 18] the elimination lemma is used to get rid off the feedback gain, it is used here to introduce slack variables that add some degree of freedom and lead to an efficient numerical procedure.

## 3 Optimal $\mathcal{H}_2$ synthesis

This section is devoted to the main result of the paper and to the associated numerical procedures. A necessary and sufficient condition of static output feedback stabilisability based on a new parameterisation is given. Even if a bilinear matrix inequality is involved, an efficient numerical algorithm may be put forward.

### 3.1 A new parameterisation

Let $M(\mathbf{P})$ be the following linear matrix function:

$$M(\mathbf{P}) = \begin{bmatrix} A'\mathbf{P} + \mathbf{P}A + C_2'C_2 & \mathbf{P}B + C_2'D_2 \\ B'\mathbf{P} + D_2'C_2 & D_2'D_2 \end{bmatrix}$$

**Theorem 1**
*The optimisation problem (3) is equivalent to:*

$$\min_{\mathbf{P}\,,\,\mathbf{K_s}\,,\,\mathbf{F}\,,\,\mathbf{R}} \; trace(B_2'\mathbf{P}B_2) \;\; s.t. \quad \left\{ \begin{array}{l} \mathbf{P} > \mathbb{0} \\ M(\mathbf{P}) + sym\left\{ \begin{bmatrix} \mathbf{K_s}' \\ -\mathbb{1} \end{bmatrix} \begin{bmatrix} \mathbf{R}C & -\mathbf{F} \end{bmatrix} \right\} < \mathbb{0} \end{array} \right. \tag{5}$$

*At the optimum, the static output feedback gain is given by: $K^{opt} = \mathbf{F}^{-1}\mathbf{R}$*

<u>**Proof**</u> Starting from the formulation (4), first note that:

$$sym\{\mathbf{P}(A+B\mathbf{K}C)\} + \bigstar(C_2 + D_2\mathbf{K}C) = \begin{bmatrix} \mathbb{1} & C'\mathbf{K}' \end{bmatrix} M(\mathbf{P}) \begin{bmatrix} \mathbb{1} \\ \mathbf{K}C \end{bmatrix}$$

Applying the elimination lemma [18], the inequality (4) is satisfied if and only if there exists a symmetric positive definite matrix $\mathbf{P}$ and a matrix $\mathbf{E}$ such that:

$$M(\mathbf{P}) + sym\{\mathbf{E}\begin{bmatrix} \mathbf{K}C & -\mathbb{1} \end{bmatrix}\} < \mathbb{0} \tag{6}$$

Choosing the following block decomposition for $\mathbf{E}' = \begin{bmatrix} \mathbf{F_s}' & -\mathbf{F}' \end{bmatrix}$, note that the bottom right block of inequality (6) is:

$$D_2'D_2 + \mathbf{F} + \mathbf{F}' < \mathbb{0}$$

Therefore, $\mathbf{F}$ is necessarily non singular. Choosing $\mathbf{K_s} = (\mathbf{F_s}\mathbf{F}^{-1})'$ and $\mathbf{R} = \mathbf{F}\mathbf{K}$ leads to (5). At the optimum, the stabilising static output feedback gain is reconstructed as $K^{opt} = F^{-1}R$. ■

On the basis of theorem 1 a numerical procedure for solving the optimal $\mathcal{H}_2$ synthesis is derived in the sequel. Before that, let us state some important facts that highlight the properties of the inequality (5) and clarify the links between the different variables.

**Remark 1**
*From the proof of theorem 1 it comes that the two following conditions are equivalent:*

*i) There exist $\mathbf{P}$ and $\mathbf{K}$ such that (4).*

*ii) There exists $\mathbf{P}$, $\mathbf{K_s}$, $\mathbf{F}$ and $\mathbf{R}$ such that (5).*

*Moreover, the inequalities are related by the invertible change of variables:*

$$\mathbf{F}\mathbf{K} = \mathbf{R} \qquad det(\mathbf{F}) \neq 0 \tag{7}$$

This remark points out that inequality (5) is equivalent to the usual Lyapunov inequality. The main difference is that additive variables are introduced. The following lemma clarifies the properties of $\mathbf{K_s}$ that appears to be a stabilising state feedback gain.

**Lemma 1**
*The variable $K_s$ of inequality (5) is a stabilising state feedback gain such that the Lyapunov function $V(x) = x'Px$ proves simultaneously the stability of both closed loop systems defined by $A + BKC$ and $A + BK_s$. Moreover, the inequality (5) implies that:*

$$||T(s,K)||_2^2 < trace(B_2' P B_2)$$
$$||T_s(s,K_s)||_2^2 < trace(B_2' P B_2)$$

**Proof** Multiplying the inequality in (5) on the left by $\begin{bmatrix} \mathbb{1} & K_s' \end{bmatrix}$ and on the right by its transpose, one gets:

$$sym\{P(A+BK_s)\} + \bigstar(C_2 + D_2 K_s) < \mathbb{0} \qquad (8)$$

Hence, $V(x) = x'Px$ is a Lyapunov function that proves the stability of $\dot{x}(t) = (A+BK_s)x(t)$ and $P > W_c(K_s)$ where $W_c(K_s)$ is the controllability grammian of the transfer $T_s(s, K_s)$.

From remark 1, one has that (4) is satisfied simultaneously for the same $P$ with the invertible change of variables (7). Therefore, $V(x) = x'Px$ is also a Lyapunov function that proves the stability of $\dot{x}(t) = (A+BKC)x(t)$ and $P > W_c(K)$ where $W_c(K)$ is the controllability grammian of the transfer $T(s,K)$. Reminding that:

$$||T(s,K)||_2^2 = trace(B_2' W_c(K) B_2)$$
$$||T_s(s,K_s)||_2^2 = trace(B_2' W_c(K_s) B_2)$$

the lemma is proved. ■

This lemma is of major importance since it relates the state and output feedback synthesis problems. On this basis, the method proposed in this paper takes the advantage of the existing results on state feedback synthesis that are known to be convex, [4].

**Remark 2**
*The optimisation problem in theorem 1 does not have a unique solution $(P^{opt},\ K_s^{opt},\ F^{opt},\ R^{opt})$. In fact, the set of all admissible solutions is convex and parametrised by the following linear equations:*

$$||T(s,K^{opt})||_2^2 = trace(B_2' \mathbf{P} B_2)$$
$$\mathbf{P} > \mathbb{0}$$
$$M(\mathbf{P}) + sym\left\{ \begin{bmatrix} \mathbf{F_s} \\ -\mathbf{F} \end{bmatrix} \begin{bmatrix} K^{opt}C & -\mathbb{1} \end{bmatrix} \right\} \leq \mathbb{0}$$

*where the solutions to (5) are given by the invertible change of variables: $\mathbf{K_s} = (\mathbf{F_s}\mathbf{F}^{-1})'$ , $\mathbf{R} = \mathbf{F}K^{opt}$.*

This remark brings to the fore that equation (5) involves additive slack variables $\mathbf{F}$ and $\mathbf{F_s}$ leading to a new parameterisation of the problem with extra degrees of freedom.

### 3.2 A coordinate descent-type algorithm

The optimisation problem in theorem 1 involves the solution of a bilinear matrix inequality. Our aim is to derive a numerically tractable iterative procedure based on LMI optimisation subproblems that converges to a local optimal solution. Here, the chosen procedure takes advantage of the slack variables introduced in (5). Our algorithm is related to the ones presented in [7, 20] and may be looked on as a coordinate descent-type algorithm, [15]. A solution to (5) is obtained by splitting in two the set of variables. At each iteration, an LMI optimisation problem over a set of variables is solved while the other set of variables is frozen.

Consider the following algorithm decomposed in four operations :

**Algorithm 1**

1. *(Initialisation step - k=1), choose a stabilising state feedback gain $K_s$.*
2. *(Step k - first part), for this choice of $K_s$, solve the following LMI minimisation problem:*

   $$\Gamma_{k,1}^2 = \min_{\mathbf{P},\ \mathbf{F},\ \mathbf{R}} trace(B_2' \mathbf{P} B_2) \quad \textit{s.t.} \quad \textit{(5)}$$

   *At the optimum freeze $R$ and $F$.*
3. *(Step k - second part), for this choice of $R$ and $F$, solve the following LMI minimisation problem:*

   $$\Gamma_{k,2}^2 = \min_{\mathbf{P},\ \mathbf{K_s}} trace(B_2' \mathbf{P} B_2) \quad \textit{s.t.} \quad \textit{(5)}$$

   *At the optimum, freeze $K_s$.*
4. *(Termination step), if $\Gamma_{k,1} - \Gamma_{k,2} < \epsilon$, then stop, $K = F^{-1}R$, otherwise $k \leftarrow k+1$ and go to step 2.*

Due to theorem 1 and remark 1, the above algorithm provides a sequence of feasible solutions and a nonincreasing criteria

**Theorem 2** *:*
*The algorithm 1 generates a sequence of feasible solutions and nonincreasing criteria such that $\forall k \geq 0$ :*

$$\Gamma_{k+1,1} \leq \Gamma_{k,2} \leq \Gamma_{k,1}$$

Only convergence towards a local minimum for which $\Gamma_{k,2}$ is the $\mathcal{H}_2$ norm of the closed-loop system is ensured by such an algorithm. Note that the Lyapunov matrix $P$ is part of the optimisation variables at each substep of the procedure and is never frozen. The initialisation step is crucial for the procedure to succeed. This point is now carefully studied.

### 3.3 Initialisation of the algorithm

In the algorithm 1, we are supposed to choose an initial stabilising state feedback gain $K_s$. This point is now clarified . Different possible initialisations, based on LMI conditions, are exposed. Even though, no theoretical result has been proved, the proposed initialisation have proved to be very successful in practice as it is illustrated by statistical examples presented at the end of this section. First, define three LMI-based conditions for finding a state feedback gain, $K_s$, that may initialise the algorithm 1, [3]:

(**I1**) Let $X$ and $S$ be a solution to:

$$\mathbf{X} > \mathbb{0} \qquad A\mathbf{X} + \mathbf{X}A' + B\mathbf{S} + \mathbf{S}'B' < \mathbb{0} \quad (9)$$

then $K_s = SX^{-1}$ is a stabilising state feedback.

(**I2**) Let $X$ and $S$ be a solution to:

$$\begin{cases} \mathbf{X} > \mathbb{0} \\ \begin{bmatrix} A\mathbf{X} + \mathbf{X}A' + B\mathbf{S} + \mathbf{S}'B' & * \\ C_2\mathbf{X} + D_2\mathbf{S} & -\mathbb{1} \end{bmatrix} < \mathbb{0} \end{cases} \quad (10)$$

then $K_s = SX^{-1}$ is a stabilising state feedback and $||T_s(s, K_s)||_2^2 = trace(B_2'\mathbf{X}^{-1}B_2)$

(**I3**) Let $X$, $S$ and $T$ be the optimal solution to:

$$\min_{\mathbf{P},\mathbf{S},\mathbf{T}} \; trace(\mathbf{T}) \;\; s.t. \;\; (10) \;\; , \;\; \begin{bmatrix} -\mathbf{T} & B_2' \\ B_2 & -\mathbf{X} \end{bmatrix} < \mathbb{0}$$

then $K_s = SX^{-1}$ is the optimal $\mathcal{H}_2$ stabilising state feedback gain and $||T_s(s, K_s)||_2^2 = trace(T)$.

These three LMI conditions are tested on numerical examples in the sequel as initialisations for algorithm 1. We want to point out that the initialisation may fail in some cases due to the fact that the inequality (5) in not feasible for all state feedback gains $K_s \in \mathcal{K}_s$. Fortunately it appears that the initialisation is nevertheless efficient in most cases. Consider the following corollary derived from theorem 1:

**Corollary 1**
*The system (1) is stabilisable by static output feedback* **iff** *there exists a symmetric positive definite matrix* $\mathbf{P}$*, a state feedback gain* $\mathbf{K_s}$ *and two matrices* $\mathbf{F}$ *and* $\mathbf{R}$ *such that:*

$$\begin{bmatrix} A'\mathbf{P} + \mathbf{P}A & * \\ B'\mathbf{P} & \mathbb{0} \end{bmatrix} + sym\left\{ \begin{bmatrix} \mathbf{K_s}' \\ -\mathbb{1} \end{bmatrix} \begin{bmatrix} \mathbf{R}C & -\mathbf{F} \end{bmatrix} \right\} < \mathbb{0} \quad (11)$$

*A stabilising static output feedback gain is then obtained by:* $K = F^{-1}R$

Based on this corollary, a procedure to test the efficiency of the initialisation is proposed. This procedure (**Proc. 1**) is composed of two LMI feasibility tests:

1. Solve LMI problem (9) and get $K_s = SX^{-1} \in \mathcal{K}_s$.
2. Solve LMI problem (11) for this matrix $K_s$.

When the procedure succeeds this implies that the gain $K_s$ makes the output feedback synthesis problem (5) feasible for some $P$, $R$ and $F$. The initialisation step is then admissible.

In order to evaluate the initialisation, a large number of stabilisable systems has been randomly generated and the results of our procedure were compared with the results obtained with an alternating projection algorithm given in [10]. The systems were generated in the same way as in this last reference. The tested cases correspond to different choices of the system dimensions $n$, $m$, $r$. The following table summarises the results. The rate is the percentage of systems for which the procedures succeed and the time is the average computation time. Note that since the experiments were performed on different computers the average time is produced as a clue on the computational complexity of both methods.

| | **Proc. 1** | | [10] | |
|---|---|---|---|---|
| | time | rate | time | rate |
| Case a | 1.32s | 95 | 18.71s | 97.5 |
| Case b | 1.67s | 99.5 | 15.93s | 100 |
| Case c | 3.26s | 99.5 | 26.15s | 99.5 |
| Case d | 1.44s | 100 | 1.17s | 100 |
| Case e | 2.43s | 100 | 3.19s | 100 |
| Case f | 5.65s | 100 | 11.18s | 100 |

These tests clearly show that the efficiency of this method may be compared with the existing ones.

## 4 Optimal $\mathcal{H}_2$ robust synthesis

An extension of theorem 1 to robust synthesis is now provided. The aim is to compute a static output feedback gain that stabilises uncertain systems and guarantees a bound on the $\mathcal{H}_2$ norm of the transfer function over all admissible uncertainties. The results are given in the quadratic stability framework [12], i.e., a single Lyapunov function assesses the stability over all the uncertainty set. Consider the uncertain LTI model such that (1) where the matrices belong to a polytope defined as the convex hull of $N$ vertices :

$$\begin{bmatrix} A(\zeta) & B_2(\zeta) & B(\zeta) \\ C_2(\zeta) & \mathbb{0} & D_2(\zeta) \\ C(\zeta) & \mathbb{0} & \mathbb{0} \end{bmatrix} \in$$
$$CO.\left\{ \begin{bmatrix} A^1 & B_2^1 & B^1 \\ C_2^1 & \mathbb{0} & D_2^1 \\ C^1 & \mathbb{0} & \mathbb{0} \end{bmatrix} \cdots \begin{bmatrix} A^N & B_2^N & B^N \\ C_2^N & \mathbb{0} & D_2^N \\ C^N & \mathbb{0} & \mathbb{0} \end{bmatrix} \right\}$$

The constant parametric uncertainty $\zeta$ is defined as the barycentric coordinate:

$$\Xi = \left\{ \zeta = [\zeta_1, \ldots, \zeta_N] \; : \; \zeta_i \geq 0 \;\; , \;\; \sum_1^N \zeta_i = 1 \right\}$$

and all the admissible models of the polytope are exactly described by $\zeta \in \Xi$,

$$\begin{bmatrix} A(\zeta) & B_2(\zeta) & B(\zeta) \\ C_2(\zeta) & \mathbb{0} & D_2(\zeta) \\ C(\zeta) & \mathbb{0} & \mathbb{0} \end{bmatrix} = \sum_1^N \zeta_i \begin{bmatrix} A^i & B_2^i & B^i \\ C_2^i & \mathbb{0} & D_2^i \\ C^i & \mathbb{0} & \mathbb{0} \end{bmatrix}$$

For a given value of the uncertainties $\zeta \in \Xi$ and a given stabilising gain $K \in \mathcal{K}$ let $T(s,\zeta,K)$ be the closed-loop transfer function. The optimal $\mathcal{H}_2$ robust stabilisability problem by static output feedback writes as:

$$K^{r.opt} = arg \left\{ \min_{\mathbf{K}\in\mathcal{K}} \max_{\zeta\in\Xi} \| T(s,\zeta,\mathbf{K}) \|_2 \right\}$$

and corresponds to the minimisation of the worst-case $\mathcal{H}_2$ norm, $\Gamma_{w.c.}(K)$.

Let $M_\zeta(\mathbf{P})$ be the following parameter-dependent linear matrix function:

$$M_\zeta(\mathbf{P}) = \begin{bmatrix} A'(\zeta)\mathbf{P} + \mathbf{P}A(\zeta) + C_2'(\zeta)C_2(\zeta) & * \\ B'(\zeta)\mathbf{P} + D_2'(\zeta)C_2(\zeta) & D_2'(\zeta)D_2(\zeta) \end{bmatrix}$$

and let $M^i(\mathbf{P})$ be the particular cases of this function for each vertex of the polytope ($i = 1..N$):

$$M^i(\mathbf{P}) = \begin{bmatrix} A^{i\prime}\mathbf{P} + \mathbf{P}A^i + C_2^{i\prime}C_2^i & \mathbf{P}B^i + C_2^{i\prime}D_2^i \\ B^{i\prime}\mathbf{P} + D_2^{i\prime}C_2^i & D_2^{i\prime}D_2^i \end{bmatrix}$$

**Theorem 3**
*Consider the following optimisation problem*

$$\min_{\mathbf{P},\mathbf{K_s},\mathbf{F},\mathbf{R}} \boldsymbol{\gamma} \quad s.t. \quad \forall i = 1..N$$

$$\left\{ \begin{array}{ll} trace(B_2^{i\prime}\mathbf{P}B_2^i) < \boldsymbol{\gamma} & \mathbf{P} > \mathbb{0} \\ M^i(\mathbf{P}) + sym\left\{ \begin{bmatrix} \mathbf{K_s}' \\ -\mathbb{1} \end{bmatrix} \begin{bmatrix} \mathbf{R}C^i & -\mathbf{F} \end{bmatrix} \right\} < \mathbb{0} & \end{array} \right. \tag{12}$$

*At the optimum $K^{q.opt} = F^{-1}R$ is a quadratically stabilising static output feedback gain and $\sqrt{\gamma^{opt}}$ is a robust $\mathcal{H}_2$ guaranteed cost for the closed-loop system.*

**Proof** The inequality (12) is convex with respect to the system matrices. This can be seen when applying Schur complement arguments. For conciseness reasons this part is removed here. The convexity of (12) implies that if the condition is satisfied for all vertices of the polytope, it is also satisfied for all elements of the convex hull. Therefore, one gets that for all $\zeta \in \Xi$:

$$\left\{ \begin{array}{ll} trace(B_2'(\zeta)\mathbf{P}B_2(\zeta)) < \boldsymbol{\gamma} & \mathbf{P} > \mathbb{0} \\ M_\zeta(\mathbf{P}) + sym\left\{ \begin{bmatrix} \mathbf{K_s}' \\ -\mathbb{1} \end{bmatrix} \begin{bmatrix} \mathbf{R}C(\zeta) & -\mathbf{F} \end{bmatrix} \right\} < \mathbb{0} & \end{array} \right.$$

This corresponds to condition (5) in theorem 1 where unique matrices $\mathbf{P}$, $\mathbf{K_s}$ and $\mathbf{F}$ asses the stability of the closed-loop systems over all uncertainties and $\sqrt{\boldsymbol{\gamma}}$ is a bound on the $\mathcal{H}_2$ norms of all transfer functions. Therefore, $K^{q.opt}$ is a quadratically stabilising gain and $\sqrt{\gamma^{opt}}$ is an optimised robust guaranteed cost for the closed-loop. ■

As theorem 1 and theorem 3 are quite similar in their structure, the same remarks, lemma and algorithm can be written for the quadratic stabilisation problem. The fact that the methods extend directly to quadratic stabilisability is an important feature of this paper. As in the state feedback quadratic stabilisability case, it suffices to test the static output stabilisability conditions on the vertices of the polytope.

## 5 Numerical examples

*Example 1:* Algorithm 1 is first evaluated on a very classical example borrowed from [14]. The result is compared with the theoretical optimal solution and with the suboptimal solution given in [9], (example 1). The data may be found in these two last references.

In [9], the min/max algorithm converges in one iteration to a suboptimal solution $\Gamma(K)^2 = 2.4525$. The exact optimal solution of the $\mathcal{H}_2$ static output feedback problem given in [14], is such that $K^{opt} = -0.8165$ and $\Gamma^2(K^{opt}) = 2.4495$. Our algorithm converges exactly to this optimum for all tested initialisations. The number of iterations is one for (**I1**) and (**I3**), 13 iterations are needed in the last case, (**I2**). At the optimum the Lyapunov function is the same for all initialisations:

$$P^{opt} = \begin{bmatrix} 1.4287 & 0.5 \\ 0.5 & 1.0207 \end{bmatrix}$$

and as expected by remark 2 $K_s$ is not unique. Here, the three initialisations converge to different gains:

$$(\mathbf{I1}) : \quad K_s = \begin{bmatrix} -0.3125 & -0.9441 \end{bmatrix}$$
$$(\mathbf{I2}) : \quad K_s = \begin{bmatrix} -0.7488 & -1.1227 \end{bmatrix}$$
$$(\mathbf{I3}) : \quad K_s = \begin{bmatrix} -0.4141 & -0.9856 \end{bmatrix}$$

*Example 2 :* The algorithm 1 has also been tested on an example borrowed from [13] and for which the min/max algorithm in [9] (example 3) converges to a suboptimal solution such that $\Gamma^2(K) = 15.3191$. For the initialisation (**I2**), we get in 10 iterations $K = \begin{bmatrix} -1.6277 & 6.5099 \end{bmatrix}'$ and $\Gamma^2(K) = 13.3115$. Note that this point is not the global optimum but gives a good suboptimal solution. Defining a grid on static output feedback entries, $K \in [-10,10] \times [-10,10]$, the minimum $\mathcal{H}_2$ norm is found to be $\Gamma^2(K^{opt}) = 12.2747$.

From these two examples, it appears that theorem 1 is a promising result in view of output stabilisability and corollary 1 is a good associated numerical solution.

*Example 3 :* An example borrowed from [6] is now considered for robust $\mathcal{H}_2$ static output feedback stabilisation. The result of the optimal $\mathcal{H}_2$ robust stabilisation algorithm is compared with the results in [8], (example 2). The uncertain model is not reminded here, (see the references for details).

In [8] a suboptimal solution is found such that $\Gamma^2_{w.c.}(K) \leq 110.8312$. Applying the method exposed in this paper, for all three initialisations, the same optimum is achieved such that $K = \begin{bmatrix} 1.5004 & 1.6336 \end{bmatrix}$ and $\Gamma^2_{w.c.}(K) = 75.6821$. The convergence of the algorithm is illustrated on figure 1 for the three different initialisations. Note the low number of iterations needed to reach this suboptimal solution.

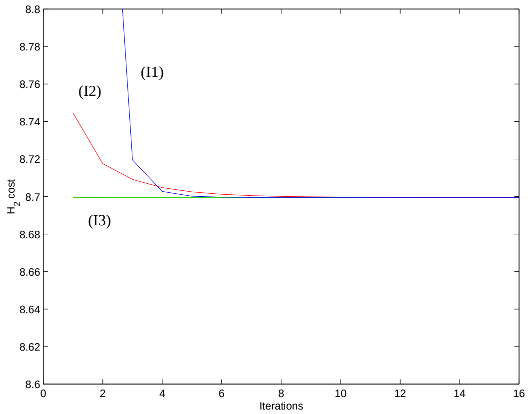


Figure 1: Convergence of the algorithm

## 6 Conclusions

This paper studies output feedback stabilisation and puts forward a new inequality condition that can be solved by efficient LMI-based algorithm. Both stabilisation and $\mathcal{H}_2$ optimal synthesis are performed for certain continuous-time LTI models as well as for polytopic uncertain models. The result shows up to be of comparable complexity for all these problems. Alike most bilinear inequality problems, the algorithm has no proof of global convergence, nevertheless it shows to be quite competitive.

Prospective work will focus on the extensions of these results to other types of uncertain systems. A remarkable feature of the proposed parameterisation is that it may be easily extended to consider the problem of robust synthesis via parameter-dependent Lyapunov functions. From a numerical point of view, the algorithmic solutions may be improved and more elaborated algorithms than coordinate descent-type methods may be tested.

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