Quadratic Stabilisability and Disk Pole Assignment for Generalised Uncertainty Models - An $\mathcal{LMI}$ Approach

D. Peaucelle - D. Arzelier - G. Garcia[1]

LAAS - CNRS
7, Avenue du Colonel Roche
31077 Toulouse cedex - France
Tél. 05 61 33 62 86 - email: peaucell@laas.fr

Abstract

A new framework for the modelling of uncertainties affecting the dynamical matrix of a L.T.I. system is proposed in this paper. This modelling extends the ones developed in the small gain and passivity context. We also define a new setup for L.T.I. systems which generalises passivity setup, [11] or β-bounded setup, [9]. In such a framework, a transfer function is supposed to verify a generalised sector constraint associated with a symmetric matrix. A state-space characterisation is proposed and it is shown that it is equivalent to a condition of quadratic stability of the uncertain model with generalised uncertainty. Finally, the problems of robust state-feedback synthesis and robust disk pole location by state feedback are considered in the quadratic setup. Convex optimisation problems based on necessary and sufficient conditions in terms of linear matrix inequalities are developed. A numerical example illustrates these results.

Keywords: Quadratic stabilisability, $\mathcal{LMI}$, generalised uncertainty.

1 Introduction

The new modelling of uncertainties affecting the dynamical matrix of a Linear Time Invariant, L.T.I. system, proposed in this paper extends the ones developed in the small gain and passivity context to which a considerable amount of work has been devoted, [18], [22], [23], [10], [11], [12], [8], to cite just a few. It is well-known that positive real modelling of system uncertainty can be less conservative than small gain modelling because of its exploitation of phase information which is ignored in small gain modelling. As mentioned in [11] and in [19], the MIMO version of the circle criterion provides the means for the gain and phase characteristics to be accounted for. Moreover, the circle theorem yields the small gain theorem and the positivity theorem as special cases. In [9], another framework for L.T.I. systems referred to as β-bounded systems is developed. The main advantage of this modelling is that we get a parameter β to choose a bounded sector for the uncertainties in a set that goes from the norm-bounded to the positive real domains. This enables to take into account phase and gain constraints simultaneously. The alternative model for real parameter uncertainty proposed in this paper, unifies the above points of view in a generalised framework. Moreover this modelling allows to describe with more accuracy the characteristics of the uncertainties via a matrix inequality relating the input and the output of the perturbations acting on the nominal system.

The results of this paper are based upon the quadratic stability concept which consists in searchin a single Lyapunov function over all the domain of uncertainties. Necessary and sufficient conditions of quadratic stability and quadratic stabilisability are then derived in terms of the existence of solutions to Linear Matrix Inequalities, $\mathcal{LMI}$'s. Robust stability is of course a minimal requirement and we are interested in finding a robust control law ensuring robust performance. One way to do so is to consider robust controllers forcing the closed loop poles to lie in some region of

[1] Also with INSA , Complexe Scientifique de Rangueil , 31077 Toulouse Cedex

the left-half plane. As we are working in a quadratic setup, it seems natural to use the quadratic d-stability concept, developed in [8]. So we are looking for a robust controller ensuring quadratic stability against generalised uncertainties and disk pole placement of the closed-loop poles. A necessary and sufficient condition of disk pole location is obtained through an $\mathcal{LMI}$ formulation. Simultaneously, a new setup for LTI systems which generalises passivity, [11] or β-bounded setups, [9] is also defined. In such a framework, a transfer function is supposed to verify a generalised sector constraint associated with a symmetric matrix. A state-space characterisation is proposed and equivalence with quadratic stability of the uncertain generalised model is established.
The paper is organised as follows. The first section presents the generalised sector constraint, the generalised uncertainty model and the state-space characterisation. Section 3 is devoted to the development of quadratic stability conditions and discusses the links between quadratic stability and generalised sector constraints. Then the quadratic stabilisation by state feedback and the disk pole location by state feedback are considered. The control conditions are illustrated by a motivating numerical example. All the results are expressed through necessary and sufficient conditions written in terms of the existence of solutions to $\mathcal{LMI}$'s.
Notation is standard. The transpose of a matrix A is denoted A' and $*$ reads for complex conjugate transpose. For symmetric matrices A and B, $A < (\leq) B$ means that the matrix $B - A$ is positive definite, (non negative). $\mathbf{1}$ stands for the identity matrix of the appropriate dimensions.

2 Generalised uncertainty and generalised sector constraints

2.1 Generalised uncertainty model and well-posedness

We consider the system:

$$\left\{ \begin{array}{ll} \dot{x}(t) = Ax(t) + Bw(t) & x(0) = 0 \\ z(t) = Cx(t) + Dw(t) & \\ w(t) = -\Delta(t) z(t) & \end{array} \right. \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times q}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times q}$ and the matrix H is the symmetric matrix given by:

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{12}' & H_{22} \end{bmatrix} \quad H_{11} = H_{11}' = MM' \in \mathbb{R}^{p\times p},\; H_{22} = H_{22}' = -L'L \in \mathbb{R}^{q\times q}$$

$w(t)$ and $z(t)$ are respectively the input and output of the perturbations of the system and $\Delta(t)$ is a time-varying uncertainty matrix belonging to some uncertainty set $\mathcal{F}_H$ defined by:

$$\mathcal{F}_H = \left\{ \Delta(t) \in \mathbb{R}^{q\times p} \;:\; [\mathbf{1} \;\; \Delta(t)'] \, H \begin{bmatrix} \mathbf{1} \\ \Delta(t) \end{bmatrix} \geq 0 \right\} \tag{2}$$

The model can be viewed as an upper linear fractional transformation, $\mathcal{L}_u(\Sigma, \Delta)$, [17], on the linear system $\Sigma \sim [A, B, C, D]$ by $-\Delta(t)$ provided the inverse $(\mathbf{1} + D\Delta(t))^{-1}$ exists for all $\Delta(t) \in \mathcal{F}_H$. The model (1) is equivalent to the following model:

$$\dot{x}(t) = [A \underbrace{-B\Delta(t)(\mathbf{1} + D\Delta(t))^{-1}C}_{\Delta A(t)}]x(t) \quad \Delta(t) \in \mathcal{F}_H \tag{3}$$

The modelling of structured uncertainty in matrix A in (3) is carried out via a fictitious linear fractional transformation as in [10] and [13]. (2) covers the principal cases encountered in the literature, bounded real uncertainty, [11], [22], positive real uncertainty, [11], [19], sector-bounded uncertainty, [11], [19] and β-bounded uncertainty, [9], for which the choice for H are respectively:

$$H_{b.r.} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{bmatrix} \quad H_{s.b.} = \begin{bmatrix} -K_1'K_2 & \frac{(K_2+K_1)'}{2} \\ \frac{(K_2+K_1)}{2} & -\mathbf{1} \end{bmatrix}$$

$$H_{p.r.} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} \quad H_{\beta-b.} = \begin{bmatrix} \beta\mathbf{1} & \frac{1}{2}(1-\beta^2)\mathbf{1} \\ \frac{1}{2}(1-\beta^2)\mathbf{1} & -\beta\mathbf{1} \end{bmatrix}$$

Existence of the inverse $(\mathbf{1} + D\Delta)^{-1}$ is related to the well-posedness of $\mathcal{L}_u(\Sigma, \Delta)$.

Definition 1 *: [17]*
$\mathcal{L}_u(\Sigma, \Delta)$ is well-posed if and only if

$$det(\mathbf{1} + D\Delta(t)) \neq 0 \quad \forall \ \Delta(t) \in \mathcal{F}_H$$

A sufficient condition of well-posedness can be easily deduced.

Lemma 1 *:*
$\mathcal{L}_u(\Sigma, \Delta)$ is well-posed if:

$$[-D' \quad \mathbf{1}]\, H \begin{bmatrix} -D \\ \mathbf{1} \end{bmatrix} < 0 \tag{4}$$

The proof is elementary and is obtained similarly as the sufficient conditions given in [11] for the bounded real or positive real cases. Well-posedness of uncertain models $\mathcal{L}_u(\Sigma, \Delta)$, $\Delta(t) \in \mathcal{F}_H$ is assumed in the sequel.

2.2 Generalised sector constraints

We consider in this section, a stable, linear time-invariant, (L.T.I.), model, $\Sigma \sim [A, B, C, D]$. The transfer function matrix of this model is $G(s) = C(s\mathbf{1} - A)^{-1}B + D$. First is given a definition of generalised sector constraints in terms of a frequency domain inequalities to be verified by $G(s)$.

Definition 2 *:*

1- *A stable L.T.I. system, with transfer function matrix $G(s)$ verifies a strict generalised sector constraint, (s.g.s.c.), with respect to the symmetric matrix $H \in \mathbb{R}^{(p+q)\times(p+q)}$ if*

$$[-G^*(j\omega) \quad \mathbf{1}]\, H \begin{bmatrix} -G(j\omega) \\ \mathbf{1} \end{bmatrix} < 0 \tag{5}$$

for all ω.

2- *A stable L.T.I. system, with transfer function matrix $G(s)$ verifies an extended strict generalised sector constraint, (e.s.g.s.c.), with respect to the symmetric matrix $H \in \mathbb{R}^{(p+q)\times(p+q)}$ if (5) is verified and*

$$[-D' \quad \mathbf{1}]\, H \begin{bmatrix} -D \\ \mathbf{1} \end{bmatrix} < 0 \tag{6}$$

where $D = G(j\infty)$.

Before introducing a state-space characterisation of this notion, a useful transformation is defined.

$$\begin{aligned} T(s) &= \begin{bmatrix} 2H'_{12}D - H_{22} & \mathbf{0} \\ 2M'D & \mathbf{1} \end{bmatrix} + \begin{bmatrix} 2H'_{12}C \\ 2M'C \end{bmatrix} (s\mathbf{1} - A)^{-1} [B \;\; \mathbf{0}] \\ &= \tilde{D} + \tilde{C}(s\mathbf{1} - A)^{-1}\tilde{B} \end{aligned} \qquad (7)$$

Lemma 2 *:*

1- $G(s)$ verifies a s.g.s.c. with respect to the symmetric matrix $H \in \mathbb{R}^{(p+q)\times(p+q)}$ if and only if $T(s)$ defined in (7) is strongly positive real.

2- $G(s)$ verifies an e.s.g.s.c. with respect to the symmetric matrix $H \in \mathbb{R}^{(p+q)\times(p+q)}$ if and only if $T(s)$ defined in (7) is extended strongly positive real.

Proof :

1- $T(s)$ is strongly positive real if it is analytic in $Re[s] \geq 0$ and $T(j\omega) + T^*(j\omega) > 0$ for all ω. Writing the last condition and applying a Schur complement argument leads to

$$-D'H_{12} - H'_{12}D + H_{22} - H'_{12}C\Phi(j\omega)B - B'\Phi^*(j\omega)C'H_{12} + D'H_{11}D$$

$$+D'H_{11}C\Phi(j\omega)B + B'\Phi^*(j\omega)C'H_{11}D + B'\Phi^*(j\omega)C'H_{11}C\Phi(j\omega)B < 0$$

for all ω and where $\Phi(j\omega) = (j\omega\mathbf{1} - A)^{-1}$. This is clearly equivalent to the condition (5).

2- The second point is easy to complete since we have:

$$[-D' \;\; \mathbf{1}]\, H \begin{bmatrix} -D \\ \mathbf{1} \end{bmatrix} < 0 \Leftrightarrow \tilde{D} + \tilde{D}' > 0 \qquad \square$$

State-space characterisation of systems verifying an e.s.g.s.c. with respect to H as defined in the first section can be given in terms of an $\mathcal{LMI}$ similar to the $\mathcal{LMI}$ given in the positive real lemma.

Theorem 1 *:*
A stable L.T.I. system, with square transfer function matrix $G(s) \sim [A, B, C, D]$ verifies an e.s.g.s.c. with respect to the symmetric matrix $H \in \mathbb{R}^{(p+q)\times(p+q)}$ if and only if there exists $X = X' > 0$ such that:

$$\begin{bmatrix} A'X + XA & XB - C'H_{12} & -C'M \\ B'X - H'_{12}C & -H'_{12}D - D'H_{12} + H_{22} & -D'M \\ -M'C & -M'D & -\mathbf{1} \end{bmatrix} < 0 \qquad (8)$$

Proof :

$G(s) \sim [A, B, C, D]$ verifies an e.s.g.s.c. with respect to H if and only if $T(s) \sim [\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}]$ defined by (7) is extended strictly positive real if and only if there exists $X = X' > 0$ such that, [3],

$$\begin{bmatrix} \tilde{A}'X + X\tilde{A} & X\tilde{B} - \tilde{C}' \\ \tilde{B}'X - \tilde{C} & -(\tilde{D} + \tilde{D}') \end{bmatrix} < 0 \tag{9}$$

where:

$$\begin{cases} \tilde{A} = A \quad \tilde{B} = [B \ \ 0_{n\times p}] \\ \tilde{C} = \begin{bmatrix} 2H_{12}'C \\ 2M'C \end{bmatrix} \\ \tilde{D} = \begin{bmatrix} 2H_{12}'D - H_{22} & 0_{q\times p} \\ 2M'D & \mathbf{1}_{p\times p} \end{bmatrix} \end{cases} \tag{10}$$

Writing down equation (9) leads to (8) □

Equation (8) is a time-domain characterisation of L.T.I. systems verifying an e.s.g.s.c. with respect to H. An alternative time-domain characterisation can be given in terms of an input-output property. Let us define:

$$D_H = H_{11}D - H_{12}$$

$$R_H = H_{12}'D + D'H_{12} - D'H_{11}D - H_{22}$$

By theorem 1 and with a simple algebraic manipulation, e.s.g.s.c. property is equivalent to the existence of a symmetric matrix $X = X' > 0$ such that:

$$\begin{bmatrix} A'X + XA + C'H_{11}C & XB + C'D_H \\ B'X + D_H'C & -R_H \end{bmatrix} < 0 \tag{11}$$

Considering now, a positive definite function defined from the matrix X by $v(x) = x'Xx$, we get that:

$$\dot{v}(x) < -[z'(t) \quad -w'(t)]H \begin{bmatrix} z(t) \\ -w(t) \end{bmatrix} \leq 0 \leq [z'(t) \quad -w'(t)]H \begin{bmatrix} z(t) \\ -w(t) \end{bmatrix} \tag{12}$$

Integrating over an arbitrary interval $[0\ ,\ T]$ leads to:

$$0 < v(x(T)) < \int_0^T [z' \quad -w'(t)]H \begin{bmatrix} z(t) \\ -w(t) \end{bmatrix} dt \tag{13}$$

for all $T \in [0\ ,\ \infty)$, all $w(t) \in \mathbb{R}^q$ and for zero initial conditions. Reciprocally, applying Parseval's theorem, the frequency-domain condition (5) can be deduced from time-domain inequality (13).

Corollary 1 *:*

1- *A stable L.T.I. system (1) verifies a strict generalised sector constraint, (s.g.s.c.), with respect to the symmetric matrix $H \in \mathbb{R}^{(p+q)\times(p+q)}$ if and only if*

$$\int_0^T [z'(t) \quad -w'(t)]H \begin{bmatrix} z(t) \\ -w(t) \end{bmatrix} dt > 0 \tag{14}$$

for all $T \in [0\ ,\ \infty)$ and all $(w(t),\ z(t)) \in \mathbb{R}^q \times \mathbb{R}^p$.

2- *A stable L.T.I. system (1) verifies an extended strict generalised sector constraint, (e.s.g.s.c.), with respect to the symmetric matrix $H \in \mathbb{R}^{(p+q)\times(p+q)}$ if (14) is verified and $R_H > 0$.*

3 Quadratic stabilisability and generalised uncertainty

In this section, preliminary results concerning quadratic stability of uncertain linear model (1), (2) are proposed. First of all, the definition of quadratic stability is recalled.

Definition 3 *:*
The uncertain model (1), (2) is said to be quadratically stable if $\forall\ \Delta \in\ \mathcal{F}_H$*, there exists a quadratic Lyapunov function* $V(x) = x'Px$ *and a scalar* $\alpha\ > 0$ *such that:*

$$2x'[A + \Delta A]'Px \leq -\alpha||x||^2$$

Quadratic stability is a sufficient condition of robust stability of system (1), (2). However, the necessary and sufficient conditions of quadratic stability can generally be solved efficiently by using properties of parameter-dependent Riccati equations, [7], [8] or by using convex programming, [3], [4]. A necessary and sufficient condition is derived in terms of $\mathcal{LMI}$ conditions, in the next theorem.

Theorem 2 *:*
The system (1), (2) is quadratically stable if and only if there exists a symmetric positive definite matrix $X \in \mathbb{R}^{n\times n}$ *such that:*

$$\left\{ \begin{array}{l} X > 0 \\ \\ \left[\begin{array}{cc} A'X + XA + C'H_{11}C & XB + C'D_H \\ B'X + D_H'C & -R_H \end{array} \right] < 0 \end{array} \right. \tag{15}$$

Proof :

The quadratic stability of system (1), (2) is equivalent to the existence of a symmetric positive definite matrix $P = P' > 0$ such that: œOB

$$[x' \;\; w'] \left[\begin{array}{cc} A'P + PA & PB \\ B'P & 0 \end{array} \right] \left[\begin{array}{c} x \\ w \end{array} \right] < 0 \tag{16}$$

for all $x(t) \neq 0 \in \mathbb{R}^n$ and all $w(t) \in \mathbb{R}^q$ such that:

$$[x' \;\; w'] \left[\begin{array}{cc} C'H_{11}C & C'D_H \\ D_H'C & -R_H \end{array} \right] \left[\begin{array}{c} x \\ w \end{array} \right] \geq 0 \tag{17}$$

Stating $w = -\Delta(\mathbf{1} + D\Delta)^{-1}Cx$ this last matrix inequality is deduced from $\Delta \in \mathcal{F}_H$. Applying the $\mathcal{S}$-procedure, [3], it is then equivalent to the existence of a symmetric positive definite matrix $P = P' > 0$ and a scalar $\tau\ \geq 0$ such that:

$$\left[\begin{array}{cc} A'P + PA + \tau C'H_{11}C & PB + \tau C'D_H \\ B'P + \tau D_H'C & -\tau R_H \end{array} \right] < 0 \tag{18}$$

Keeping in mind $R_H > 0$ for well-posedness of $\mathcal{L}_u(\Sigma, \Delta)$, $\mathcal{LMI}$ (18) implies $\tau > 0$ and by defining $X = \tau^{-1}P$, we get an equivalent condition:

$$\left\{ \begin{array}{l} X > 0 \\ \\ \left[\begin{array}{cc} A'X + XA + C'H_{11}C & XB + C'D_H \\ B'X + D_H'C & -R_H \end{array} \right] < 0 \end{array} \right. \tag{19}$$

□

The condition (19) is very interesting because it is very close to the condition of quadratic stability of uncertain models affected by positive real uncertainty, [3]. However, it is to be noticed that, using the transformation defined by (10), we deduce the following corollary as a by-product.

Corollary 2 *:*
$\mathcal{L}_u(\Sigma, \Delta)$, $\Delta \in \mathcal{F}_H$, is quadratically stable if and only if $\mathcal{L}_u(\tilde{\Sigma}, \tilde{\Delta})$, $\tilde{\Delta} \in \mathcal{F}_{H_{p.r.}}$ is quadratically stable, where $\tilde{\Sigma}$ is defined by (10) and

$$\tilde{\Delta} = \left[\begin{array}{cc} \mathbf{1}_{q\times q} & 2L'\Delta \\ \mathbf{0}_{p\times q} & 2H_{12}\Delta + H_{11} \end{array} \right] \tag{20}$$

Proof :

Remembering that a sufficient condition for well-posedness of $\mathcal{L}_u(\tilde{\Sigma}, \tilde{\Delta})$, $\tilde{\Delta} \in \mathcal{F}_{H_{p.r.}}$, [11], is $\tilde{D} + \tilde{D}' > 0$, we get

$$R_H > 0 \Leftrightarrow \tilde{D} + \tilde{D} > 0 \tag{21}$$

and so $\mathcal{L}_u(\Sigma, \Delta)$, $\Delta \in \mathcal{F}_H$ is well-posed if and only if $\mathcal{L}_u(\tilde{\Sigma}, \tilde{\Delta})$, $\tilde{\Delta} \in \mathcal{F}_{H_{p.r.}}$ is well-posed. Moreover, writing down the necessary and sufficient condition of extended strict positive realness of $\tilde{\Sigma}$, [19], we get the equivalent condition in terms of the existence of a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that:

$$\left[\begin{array}{cc} \tilde{A}'P + P\tilde{A} & P\tilde{B} - \tilde{C}' \\ \tilde{B}'P - \tilde{C} & -(\tilde{D} + \tilde{D}') \end{array} \right] < 0 \tag{22}$$

which is nothing but (15) □

Recently, Shim has proved the equivalence between extended strict positive realness of (1) and quadratic stability of (1), (2) with $H = H_{p.r.}$. Looking at theorem 2 and at equation (11), the following corollary is easily deduced.

Corollary 3 *:*
A stable L.T.I. system Σ verifies an e.s.g.s.c. with respect to the symmetric matrix $H \in \mathbb{R}^{(p+q)\times(p+q)}$ if and only if $\mathcal{L}_u(\Sigma, \Delta)$, $\Delta \in \mathcal{F}_H$, is quadratically stable.

4 Robust controller synthesis via state-feedback

The robust controler synthesis is considered via the quadratic stabilisability approach that gives a sufficient condition of robust stability of the closed-loop system, [18].

Let us consider the uncertain model $\mathcal{L}_u(\Sigma, \Delta)$ closed by the state-feedback gain K, (see figure 1), and given by:

$$\left\{ \begin{array}{l} \dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t) \\ z(t) = Cx(t) + Dw(t) \\ y(t) = x(t) \\ w(t) = -\Delta(t) z(t) \quad \Delta(t) \in \mathcal{F}_H \end{array} \right. \tag{23}$$

where $u(t) \in \mathbb{R}^m$ is the control vector.

The model (23) is equivalent to:

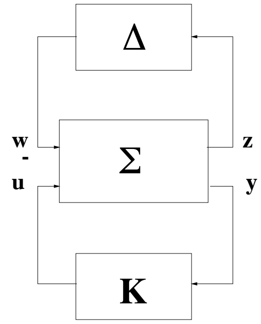

Figure 1: uncertainty model $\mathcal{L}_u(\Sigma, \Delta)$

$$\dot{x}(t) = [A \underbrace{-B_1\Delta(t)(\mathbf{1} + D\Delta(t))^{-1}C}_{\Delta A(t)}]x(t) + B_2u(t) \qquad \Delta \in \mathcal{F}_H \tag{24}$$

Let us first recall the well-known definition of quadratic stabilisability.

Definition 4 *: [22]*
The system (19) is quadratically stabilizable by state feedback if and only if there exist a matrix $K \in \mathbb{R}^{m\times n}$, a symmetric matrix $P \in \mathbb{R}^{n\times n}$ and a strictly positive scalar α such that:

$$2x'\left[(A + \Delta A) + B_2K\right]' Px \leq -\alpha||x||^2 \tag{25}$$

Following the approach of section 2, a necessary and sufficient condition of quadratic stabilisability in terms of linear matrix inequalities is proposed below.

Theorem 3 *:*
The system (24) is quadratically stabilisable by state feedback if and only if there exist a symmetric positive definite matrix $X \in \mathbb{R}^{n\times n}$ and a matrix $S \in \mathbb{R}^{m\times n}$ such that:

$$\left\{ \begin{array}{l} X > 0 \\ \\ \left[\begin{array}{ccc} AX + XA' + B_2S + S'B_2' & XC'M & B_1 + XC'D_H \\ M'CX & -\mathbf{1} & \mathbf{0} \\ B_1' + D_H'CX & \mathbf{0} & -R_H \end{array} \right] < 0 \end{array} \right. \tag{26}$$

A control gain is then given by:

$$K = SX^{-1} \tag{27}$$

Proof:

The quadratic stabilisability of (24) via a linear state-feedback $u(t) = Kx(t)$ is equivalent to the quadratic stability of

$$\left\{ \begin{array}{l} \dot{x}(t) = (A + B_2K)x(t) + B_1w(t) \\ z(t) = Cx(t) + Dw(t) \\ w(t) = -\Delta(t)z(t) \quad \Delta(t) \in \mathcal{F}_H \end{array} \right. \tag{28}$$

Applying theorem 2, it is equivalent to the existence of a symmetric matrix $P \in \mathbb{R}^{n\times n}$ such that:

$$\left\{\begin{array}{l} P > 0 \\ \\ \left[\begin{array}{cc} (A+B_2K)'P + P(A+B_2K) + C'H_{11}C & PB_1 + C'D_H \\ B_1'P + D_H'C & -R_H \end{array}\right] < 0 \end{array}\right. \tag{29}$$

Which can be rewritten, stating $X = P^{-1}$ and recalling that $H_{11} = MM'$,

$$\left\{\begin{array}{l} X > 0 \\ \\ \left[\begin{array}{ccc} X(A+B_2K)' + (A+B_2K)X & XC'M & B_1 + XC'D_H \\ M'CX & -\mathbf{1} & \mathbf{0} \\ B_1' + D_H'CX & \mathbf{0} & -R_H \end{array}\right] < 0 \end{array}\right. \tag{30}$$

Noting $S = KX$, we get the equivalent condition in terms of the existence of a symmetric matrix $X \in \mathbb{R}^{n\times n}$ and a matrix $S \in \mathbb{R}^{m\times n}$ such that:

$$\left\{\begin{array}{l} X > 0 \\ \\ \left[\begin{array}{ccc} AX + XA' + B_2S + S'B_2' & XC'M & B_1 + XC'D_H \\ M'CX & -\mathbf{1} & \mathbf{0} \\ B_1' + D_H'CX & \mathbf{0} & -R_H \end{array}\right] < 0 \quad \square \end{array}\right. \tag{31}$$

Note that $v(x) = x'X^{-1}x$ is a single quadratic Lyapunov function ensuring quadratic stability of $\mathcal{L}_u(\Sigma,\Delta)$, closed by $K = SX^{-1}$, for all $\Delta \in \mathcal{F}_H$.
The main advantage of the necessary and sufficient condition given in theorem 3, is its $\mathcal{LMI}$ formulation which guarantees the existence of efficient numerical tools to solve it, [16], [5]. An alternative necessary and sufficient condition could be given in terms of a parameter-dependent Riccati equation, [1]. The condition (31) defines a convex set for the pair (X,S) and therefore is easily testable. To find an admissible pair (X_{adm}, S_{adm}) implies to solve the following convex optimisation problem by the projective method of Nesterov and Nemirovski, [16]:

$$\left\{\begin{array}{ll} \min\limits_{(X,S)} & (t) \\ \\ \text{under} & \left[\begin{array}{cc} \mathcal{H}(X,S) & 0 \\ 0 & -X \end{array}\right] < t\mathbf{1} \end{array}\right. \tag{32}$$

where:

$$\mathcal{H}(X,S) = \left[\begin{array}{ccc} AX + XA' + B_2S + S'B_2' & XC'M & B_1 + XC'D_H \\ M'CX & -\mathbf{1} & \mathbf{0} \\ B_1' + D_H'CX & \mathbf{0} & -R_H \end{array}\right]$$

Note that the convex set defined by (31) is not empty if the optimal solution for t is negative. This will be illustrated in section 6.

5 Robust disk pole assignment via state-feedback

Let us consider the uncertain model $\mathcal{L}(\Sigma,\Delta)$ described by:

$$\dot{x}(t) = (A + \Delta A)x(t) + B_2u(t) \tag{33}$$

where the uncertainty is time-invariant and satisfies:

$$\Delta A = -B_1 \Delta (\mathbf{1} + D\Delta)^{-1} C \qquad \Delta \in \mathcal{F}_H \tag{34}$$

Note that here, the uncertainty term is no more time-varying. The problem to solve is to determine a state feedback $u(t) = Kx(t)$ such that the closed-loop system is quadraticaly stable and the closed loop poles lie in the disk $\mathcal{D}(\alpha, r)$. $\mathcal{D}(\alpha, r)$ is the disk centred at $\alpha + j0$ with the radius r. We introduce the following notations:

$$\begin{gathered} A_r = (A - \alpha\mathbf{1})/r \quad B_{2r} = B_2/r \quad B_{1r} = B_1/\sqrt{r} \quad C_r = C/\sqrt{r} \\ A_c = (A_r + B_{2r}K) \quad \Delta A_r = \Delta A/r \end{gathered} \tag{35}$$

Lemma 3 *:*
Let $A \in \mathbb{R}^{n\times n}$ be a given matrix. The eigenvalues of A belong to $\mathcal{D}(\alpha, r)$ if and only if there exists a symmetric matrix $P \in \mathbb{R}^{n\times n}$ such that:

$$\begin{pmatrix} -P^{-1} & A_r \\ A_r' & -P \end{pmatrix} < \mathbf{0} \tag{36}$$

The proof is obtained from the standard Schur complement result and from [15].
From now on, only the stable $\mathcal{D}(\alpha, r)$ location cases are considered, i.e. the ones where $\mathcal{D}(\alpha, r)$ belongs to the left-half complex plane. We recall the definition of quadratic d-stabilisability that we'll use extensively in the sequel.

Definition 5 *: [8]*
The system (33) is quadratically d-stabilisable in $\mathcal{D}(\alpha, r)$ by state feedback if and only if there exists a matrix $K \in \mathbb{R}^{m\times n}$ and a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that for all $\Delta \in \mathcal{F}_H$:

$$\begin{pmatrix} -P^{-1} & A_c + \Delta A_r \\ (A_c + \Delta A_r)' & -P \end{pmatrix} < \mathbf{0} \tag{37}$$

The state feedback is $u(t) = Kx(t)$.

It is interesting to relate the P matrix to a Lyapunov matrix. The existence of a single Lyapunov matrix for all the system realisations in the uncertainty domain ensures the closed-loop quadratic stability.

Lemma 4 *:*
If the system (33) is quadratically d-stabilisable in $\mathcal{D}(\alpha, r)$ by state feedback then the system is quadratically stabilisable, P being a Lyapunov matrix for all the realisations of the system in the uncertainty domain and $u(t) = Kx(t)$ is a state feedback that quadratically stabilises the system (33).

The proof of this lemma can be found in [6].

Considering the definition of quadratic d-stabilisability we obtain a necessary and sufficient $\mathcal{LMI}$ condition:

Theorem 4 *:*
The system (33) is quadratically d-stabilisable in $\mathcal{D}(\alpha, r)$ by state feedback if and only if there

exists a symmetric positive definite matrix $X \in \mathbb{R}^{n\times n}$ and a matrix $S \in \mathbb{R}^{m\times n}$ such that:

$$\left\{\begin{array}{l} X > \mathbf{0} \\ \begin{bmatrix} -X & XC_r'D_H & XA_r' + S'B_{2r}' & XC_r'M \\ D_H'C_rX & -R_H & B_{1r}' & \mathbf{0} \\ A_rX + B_{2r}S & B_{1r} & -X & \mathbf{0} \\ M'C_rX & \mathbf{0} & \mathbf{0} & \mathbf{-1} \end{bmatrix} < \mathbf{0} \end{array}\right. \tag{38}$$

A state feedback is then given by $u(t) = Kx(t)$ with:

$$K = SX^{-1} \tag{39}$$

One can notice that the $\mathcal{LMI}$ condition (38) implies that $-R_H + B_{1r}'X^{-1}B_{1r} < \mathbf{0}$ which is a sufficient condition for well-posedness of $\mathcal{L}(\Sigma, \Delta)$.

Proof of the necessity

If (33) is quadratically d-stabilisable in $\mathcal{D}(\alpha, r)$, there exist $K \in \mathbb{R}^{m\times n}$ and a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that (37) holds.
This means that for all $x \neq 0 \in \mathbb{R}^n$:

$$x'(-P + (A_c + \Delta A_r)'P(A_c + \Delta A_r))x < 0 \tag{40}$$

Stating $w_r = -\Delta(\mathbf{1} + D\Delta)^{-1}C_r x$, this implies that:

$$\begin{pmatrix} x' & w_r' \end{pmatrix} \begin{bmatrix} A_c'PA_c - P & A_c'PB_{1r} \\ B_{1r}'PA_c & B_{1r}'PB_{1r} \end{bmatrix} \begin{pmatrix} x \\ w_r \end{pmatrix} < 0 \tag{41}$$

for all $x \neq 0 \in \mathbb{R}^n$ and all $w_r \in \mathbb{R}^q$ such that:

$$\begin{pmatrix} x' & w_r' \end{pmatrix} \begin{bmatrix} C_r'H_{11}C_r & C_r'D_H \\ D_H'C_r & -R_H \end{bmatrix} \begin{pmatrix} x \\ w_r \end{pmatrix} \geq 0 \tag{42}$$

Applying the $\mathcal{S}$-procedure, [3], it is then equivalent to the existence of a symmetric positive definite matrix $P = P' > 0$ and a scalar $\tau \geq 0$ such that:

$$\begin{bmatrix} A_c'PA_c - P & A_c'PB_{1r} \\ B_{1r}'PA_c & B_{1r}'PB_{1r} \end{bmatrix} + \tau \begin{bmatrix} C_r'H_{11}C_r & C_r'D_H \\ D_H'C_r & -R_H \end{bmatrix} < \mathbf{0} \tag{43}$$

Here, $\tau \neq 0$ because $B_{1r}'PB_{1r} \geq \mathbf{0}$

This can be rewritten as:

$$\begin{bmatrix} -P + \tau C_r'H_{11}C_r & \tau C_r'D_H \\ \tau D_H'C_r & -\tau R_H \end{bmatrix} + \begin{bmatrix} A_c' \\ B_{1r}' \end{bmatrix} P \begin{bmatrix} A_c & B_{1r} \end{bmatrix} < \mathbf{0} \tag{44}$$

Applying a Schur complement argument we obtain:

$$\begin{bmatrix} -P + \tau C_r'H_{11}C_r & \tau C_r'D_H & A_c' \\ \tau D_H'C_r & -\tau R_H & B_{1r}' \\ A_c & B_{1r} & -P^{-1} \end{bmatrix} < \mathbf{0} \tag{45}$$

Multiplying both sides of (45) by $\begin{bmatrix} \tau^{1/2}P^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tau^{-1/2}\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \tau^{1/2}\mathbf{1} \end{bmatrix}$ and stating $X = \tau P^{-1}$, leads to the existence of a symmetric positive definite matrix $X \in \mathbb{R}^{n\times n}$ such that:

$$\begin{bmatrix} -X + XC_r'H_{11}C_rX & XC_r'D_H & XA_c' \\ D_H'C_rX & -R_H & B_{1r}' \\ A_cX & B_{1r} & -X \end{bmatrix} < \mathbf{0} \tag{46}$$

This can also be written as:

$$\begin{bmatrix} -X & XC_r'D_H & XA_c' \\ D_H'C_rX & -R_H & B_{1r}' \\ A_cX & B_{1r} & -X \end{bmatrix} + \begin{bmatrix} XC_r'M \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} M'C_rX & \mathbf{0} & \mathbf{0} \end{bmatrix} < \mathbf{0} \tag{47}$$

Applying a Schur complement argument we obtain:

$$\begin{bmatrix} -X & XC_r'D_H & XA_c' & XC_r'M \\ D_H'C_rX & -R_H & B_{1r}' & \mathbf{0} \\ A_cX & B_{1r} & -X & \mathbf{0} \\ M'C_rX & \mathbf{0} & \mathbf{0} & -\mathbf{1} \end{bmatrix} < \mathbf{0} \tag{48}$$

Stating $KX = S$, we get $A_cX = A_rX + B_{2r}S$ and recover the $\mathcal{LMI}$ (38) □

<u>Proof of the sufficiency</u>

We suppose that there exists $X = X' > \mathbf{0} \in \mathbb{R}^{n\times n}$ and $S \in \mathbb{R}^{m\times n}$ such that (38) with $K = SX^{-1}$ the quadratic stabilising gain. (48) is easily rewritten as (46) which can be developed as:.

$$\begin{bmatrix} -X + XC_r'H_{11}C_rX & XC_r'D_H \\ D_H'C_rX & -R_H \end{bmatrix} + \begin{bmatrix} XA_c' \\ B_{1r}' \end{bmatrix} X^{-1} \begin{bmatrix} A_cX & B_{1r} \end{bmatrix} < \mathbf{0} \tag{49}$$

Multipling both sides of (49) by $\begin{bmatrix} X^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$ and stating $P = X^{-1}$, we have:

$$\begin{bmatrix} A_c'PA_c - P & A_c'PB_{1r} \\ B_{1r}'PA_c & B_{1r}'PB_{1r} \end{bmatrix} < \begin{bmatrix} -C_r'H_{11}C_r & -C_r'D_H \\ -D_H'C_r & R_H \end{bmatrix} \tag{50}$$

Stating $w_r = -\Delta(\mathbf{1} + D\Delta)^{-1}C_rx$ we know that $\forall x \in \mathbb{R}^n, \forall \Delta \in \mathcal{F}_H$ we have (42).
So $\forall x \in \mathbb{R}^n, \forall \Delta \in \mathcal{F}_H$:

$$x'(-P + A_c'PA_c + A_c'P\Delta A_r + \Delta A_r'PA_r + \Delta A_r'PA_r)x < \mathbf{0} \tag{51}$$

The condition of quadratic d-stability is verified. □

As for the quadratic state feedback synthesis case, one can notice that the condition (38) defines a convex set for the pair (X, S). To find an admissible pair (X_{adm}, S_{adm}) implies to solve the following convex optimisation problem by the method of Nesterov and Nemirovski, [16]:

$$\begin{cases} \min\limits_{(X,S)} & (t) \\ \text{under} & \begin{bmatrix} \mathcal{H}_r(X,S) & \mathbf{0} \\ \mathbf{0} & -X \end{bmatrix} < t\mathbf{1} \end{cases} \tag{52}$$

Where:

$$\mathcal{H}_r(X,S) = \begin{bmatrix} -X & XC_r'D_H & XA_r' + S'B_{2r}' & XC_r'M \\ D_H'C_rX & -R_H & B_{1r}' & \mathbf{0} \\ A_rX + B_{2r}S & B_{1r} & -X & \mathbf{0} \\ M'C_rX & \mathbf{0} & \mathbf{0} & -1 \end{bmatrix} \tag{53}$$

The system is therefore quadratically d-stabilisable if and only if the optimal solution of this problem is for t negative. This will be illustrated in the foregoing section.

<u>**Nota:**</u>

For an adequat choice of $\mathcal{D}(\alpha, r)$, $(\alpha = 0,\ r = 1)$, the $\mathcal{LMI}$ (38) is nothing but a necessary and sufficient condition of quadratic stability in the discrete-time case:

$$\left\{ \begin{array}{rcl} x_{k+1} & = & Ax_k + B_1w_k + B_2u_k \\ z_k & = & Cx_k + Dw_k \\ w_k & = & -\Delta z_k \qquad \Delta \in \mathcal{F}_{\mathcal{H}}. \end{array} \right. \tag{54}$$

For a choice of $\mathcal{D}(\alpha, r)$ included in $\mathcal{D}(0,1)$, the $\mathcal{LMI}$ (38) is a necessary and sufficient condition of quadratic d-stability in the discrete-time case.

6 A numerical example

The most common uncertainties have been the norm bounded and the positive real uncertainties. This is why in the following example we study the advantages and disadvantages of these two ways of modelling the uncertainties and we show that by a generalised modelling approach the stability results can be improved using the algorithms described in the first sections of this paper.

6.1 Description of the system

Let us consider the two-mass spring system of figure 2 inspired from the benchmark [21], which is a generic model of an uncertain dynamical system with a rigid-body mode and one vibration mode.

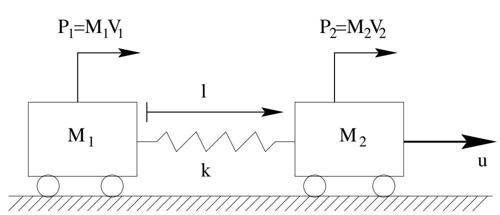

Figure 2: Two-mass Spring System

The state vector of the system can be reduced to a three entries vector: the inertia of each body P_1 and P_2, and the distance between the two bodies l. The input is a control force u that acts on the second body. The state-space model is given by:

$$\begin{bmatrix} \dot{P}_1 \\ \dot{P}_2 \\ \dot{l} \end{bmatrix} = \begin{bmatrix} 0 & 0 & k \\ 0 & 0 & -k \\ -1/M_1 & 1/M_2 & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ l \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u \tag{55}$$

The masses of the two bodies of the system are considered to be uncertain. We assume that the experimentation has shown that the uncertainties are not totally independent one from the other. The extra masses added to the nominal mass of each body are almost equal (see figure

3). We propose four different modellings of these uncertainties, two norm bounded modellings, a positive real modelling and a generalised modelling. These modellings will be compared in terms of quadratic stabilisability by state feedback, in the case of time-varying uncertainties, and of quadratic d-stabilisability by state feedback in the case of time-invariant uncertainties.

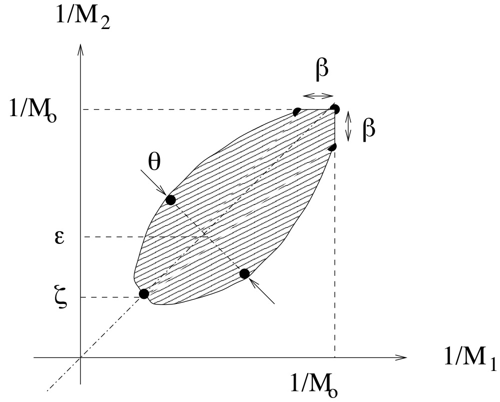

Figure 3: Domains of parameters obtained by experimentation

The uncertain domain has no particular mathematical equation. It is defined by five parameters that define the six extremal points, as presented in figure 3. The problem is then to find domains defined by norm bounded models, positive real model or generalised model including the six points and therefore including the identified domain of figure 3.

6.2 The different modellings

6.2.1 Norm bounded modelling

We decide to include the identified domain into a rectangular domain described by a norm bounded modelling. In this section the two uncertain parameters are δ_1, δ_2 and satisfy the norm bounded inequalities:

$$\delta_i^2 \leq 1 \tag{56}$$

The uncertain masses are defined by:

$$\begin{aligned} \frac{1}{M_1} &= \frac{1/M_o+\zeta}{2} + \frac{1/M_o-\zeta}{2}\delta_1 + \frac{\theta}{2\sqrt{2}}\delta_2 \\ \frac{1}{M_2} &= \frac{1/M_o+\zeta}{2} + \frac{1/M_o-\zeta}{2}\delta_1 - \frac{\theta}{2\sqrt{2}}\delta_2 \end{aligned} \tag{57}$$

Considering the norm bounded model:

$$\mathcal{F}_{n.b.} = \left\{ \Delta(t) \in \mathbb{R}^{2\times 2} \;:\; [\mathbf{1} \;\; \Delta(t)'] \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \Delta(t) \end{bmatrix} \geq 0 \right\} \tag{58}$$

The matrices of the form $\Delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}$ where each δ_i satisfies (56) are included in this modelling. It is important to note here, that the diagonal matrices are not the only ones that satisfy (56). This implies an extra conservatism in the calculation of a robust control. It would be interesting to take into account the diagonal structure of Δ.
This modelling gives the following matrices for the system:

$$A_{n.b.} = \begin{bmatrix} 0 & 0 & k \\ 0 & 0 & -k \\ -(1/M_o+\zeta)/2 & (1/M_o+\zeta)/2 & 0 \end{bmatrix} \quad B_{1\ n.b.} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$C_{n.b.} = \begin{bmatrix} (1/M_o-\zeta)/2 & -(1/M_o-\zeta)/2 & 0 \\ \theta/(2\sqrt{2}) & \theta/(2\sqrt{2}) & 0 \end{bmatrix} \qquad D_{n.b.} = \mathbf{0} \tag{59}$$

Here D is taken as zero for simplicity and it satisfies the well-posedness condition.
The figure 4 represents the domain of evolution of the parameters $1/M_i$.

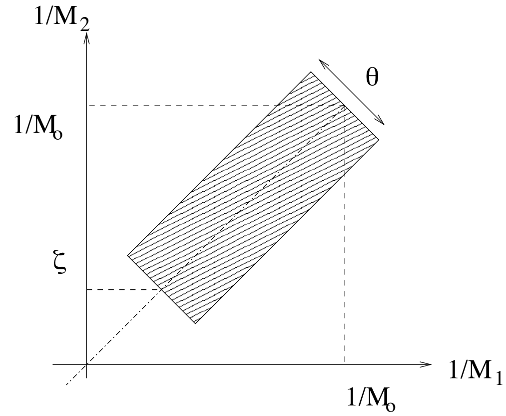

Figure 4: Domains of parameters - norm bounded modelling

6.2.2 Positive real modelling

Via a positive real modelling we consider now the same rectangular domain but we exclude from it two of its sides. The uncertainty parameters are μ_1 and μ_2, they are positive real and as the uncertainties can't be physically infinite, we give a limit to their influence for high values.
For each uncertainty:

$$\mu_i \geq 0 \tag{60}$$

The masses satisfy:

$$\begin{aligned} \frac{1}{M_1} &= \frac{1}{M_o} - \frac{\mu_1}{1+(1/M_o-\zeta)^{-1}\mu_1} - \frac{\theta}{2\sqrt{2}} + \frac{\mu_2}{1+(\theta/\sqrt{2})^{-1}\mu_2} \\ \frac{1}{M_2} &= \frac{1}{M_o} - \frac{\mu_1}{1+(1/M_o-\zeta)^{-1}\mu_1} + \frac{\theta}{2\sqrt{2}} - \frac{\mu_2}{1+(\theta/\sqrt{2})^{-1}\mu_2} \end{aligned} \tag{61}$$

Considering the positive real model:

$$\mathcal{F}_{p.r.} = \left\{ \Delta(t) \in \mathbb{R}^{2\times 2} \ : \ [\mathbf{1} \ \ \Delta(t)'] \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \Delta(t) \end{bmatrix} \geq 0 \right\} \tag{62}$$

The matrices of the form $\Delta = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}$ where each μ_i satisfies (60) are included in this modelling. The uncertainties are bounded for high values so that $\mu_1 < 1/M_o - \zeta$ and $\mu_2 < \theta/\sqrt{2}$. This modelling gives the following matrices for the system:

$$A_{p.r.} = \begin{bmatrix} 0 & 0 & k \\ 0 & 0 & -k \\ -1/M_o + \theta/(2\sqrt{2}) & 1/M_o + \theta/(2\sqrt{2}) & 0 \end{bmatrix} \quad B_{1\ p.r.} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$D_{p.r.} = \begin{bmatrix} (1/M_o - \zeta)^{-1} & 0 \\ 0 & (\theta/\sqrt{2})^{-1} \end{bmatrix} \quad C_{p.r.} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \tag{63}$$

Here D is not a zero matrix to ensure the asymptotic attitude of the parameters and it satisfies the well-posedness condition.

The domain of evolution of the parameters $1/M_i$ is the same as for the norm bounded modelling although two of the bounds of the domain are not included.

6.2.3 Generalised modelling

In this section, we represent the uncertainties by the interior of a parabola which main axis is $1/M_1 - 1/M_2 = 0$. As the uncertainties cannot be physically infinite we give a limit to the influence of the uncertainty parameters for high values. The domain is then an ellipse but we'll see further on that the modelling via a parabola has its importance.

The uncertainty parameters are η_1 and η_2, they satisfy:

$$\phi(\eta_2 - \eta_1)^2 \quad \leq \quad \eta_1 + \eta_2 + \gamma \tag{64}$$

The uncertain masses are obtained by:

$$\tfrac{1}{M_i} \quad = \quad \tfrac{1}{M_o} - \tfrac{\eta_i}{1+1/2(1/M_o - \zeta)^{-1}(\eta_1+\eta_2)} \tag{65}$$

The parameter ϕ allows to choose the admissible variations of the uncertainty around the $\eta_2 - \eta_1$ axis and the parameter γ allows to chose the variations of each uncertainty when the other is zero. This enables to define the best parabola that fits with the six identified extremal points. We notice that the inequality of the interior of the parabola is equivalent to:

$$\begin{bmatrix} 1 & 0 & \eta_1 \\ 0 & 1 & \eta_2 \end{bmatrix} \begin{bmatrix} \phi\gamma + 1/4 & \phi\gamma - 1/4 & \phi \\ \phi\gamma - 1/4 & \phi\gamma + 1/4 & \phi \\ \phi & \phi & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \eta_1 & \eta_2 \end{bmatrix} \geq \mathbf{0} \tag{66}$$

So the uncertainties can be rewritten in a generalised form:

$$\mathcal{F}_H = \left\{ \Delta(t) \in \mathbb{R}^{1\times 2} \ : \ [\mathbf{1} \ \ \Delta(t)']\, H \begin{bmatrix} \mathbf{1} \\ \Delta(t) \end{bmatrix} \geq 0 \right\} \tag{67}$$

With the matrix H:

$$H = \left[\begin{array}{cc|c} \phi\gamma + 1/4 & \phi\gamma - 1/4 & \phi \\ \phi\gamma - 1/4 & \phi\gamma + 1/4 & \phi \\ \hline \phi & \phi & 0 \end{array} \right] \tag{68}$$

The influence of the uncertainties parameters is bounded for high values so that $1/M_2$ and $1/M_1$ are always strictly greater than ζ. This modelling gives the following matrices for the system:

$$A_H = \begin{bmatrix} 0 & 0 & k \\ 0 & 0 & -k \\ -1/M_o & 1/M_o & 0 \end{bmatrix} \qquad B_{1\ H} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$D_H = \begin{bmatrix} 1/2(1/M_o - \zeta)^{-1} \\ 1/2(1/M_o - \zeta)^{-1} \end{bmatrix} \qquad C_H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \tag{69}$$

Here D is not a zero matrix to take into account the asymptotic influence of the uncertainty parameters when η_1 and η_2 take high values. It satisfies the well-posedness condition while γ satisfies the condition $H_{11} \geq \mathbf{0}$ if:

$$0 \leq \gamma/2(1/M_o - \zeta)^{-1} < \phi \tag{70}$$

We represent in the figure 5 the domain of evolution of the parameters $1/M_i$. It is easy to show that the resultant domain is the interior of an ellipse with one border point excluded.

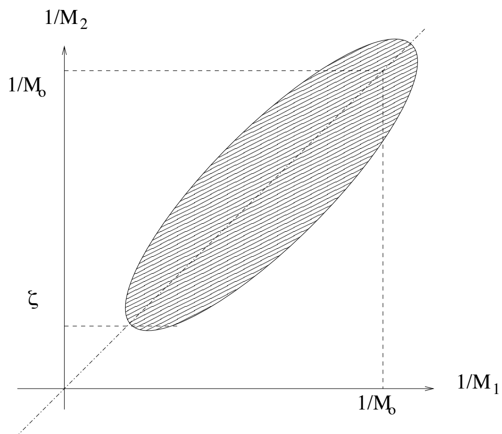

Figure 5: Domains of parameters - generalised modelling

6.2.4 Elliptic norm bounded modelling

Inspired from the generalised modelling we consider an other norm bounded modelling where the uncertainties belong to an elliptic domain which main axis is $1/M_1 - 1/M_2 = 0$. As we saw in the past section the generalised modelling gives in fact, considering the asymptotic influence of the uncertain parameters, an elliptic domain, so that we can compare two identical domains as for the rectangular domains of section 6.2.1 and 6.2.2.
The uncertainty parameters are ψ_1 and ψ_2 and they satisfy a norm bounded inequality:

$$\psi_1^2 + \psi_2^2 \ \leq \ 1 \tag{71}$$

The masses are then written as:

$$\begin{aligned} 1/M_1 &= 1/M_o - c + a/2\ \psi_1 + b/2\ \psi_2 \\ 1/M_2 &= 1/M_o - c + a/2\ \psi_1 - b/2\ \psi_2 \end{aligned} \tag{72}$$

The parameter c gives the centre of the ellipse and the parameters a and b give the dimensions of the ellipse along the axis. This enables to define the best ellipse that fits with the six identified extremal points. We notice that the inequality of the interior of the ellipse is equivalent to:

$$\begin{bmatrix} 1 & 0 & \psi_1 \\ 0 & 1 & \psi_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \psi_1 & \psi_2 \end{bmatrix} \geq \mathbf{0} \tag{73}$$

So the uncertainties can be rewritten in a norm bounded form:

$$\mathcal{F}_{b.n.} = \left\{ \Delta(t) \in \mathbb{R}^{1\times 2} \ : \ [\mathbf{1} \ \ \Delta(t)'] \begin{bmatrix} \mathbf{1}_{2\times 2} & \mathbf{0}_{2\times 1} \\ \mathbf{0}_{1\times 2} & -1 \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \Delta(t) \end{bmatrix} \geq 0 \right\} \tag{74}$$

This modelling gives the following matrices for the system:

$$\begin{array}{ll} A_{ell.} = \begin{bmatrix} 0 & 0 & k \\ 0 & 0 & -k \\ -1/M_o + c & 1/M_o - c & 0 \end{bmatrix} & B_{1\ ell.} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ C_{ell.} = \begin{bmatrix} a/2 & -a/2 & 0 \\ b/2 & b/2 & 0 \end{bmatrix} & D_{ell.} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array} \tag{75}$$

Here D is taken zero and it satisfies the well-posedness condition.

6.3 Numerical results

The values of the parameters defining the uncertain domain are chosen to be:

$$\begin{array}{ccc} M_o = 1 & & k = 1 \\ \beta = 0.05 & \epsilon = 0.35 & \theta = 0.2 \end{array} \tag{76}$$

6.3.1 Quadratic Stabilisability

We desire to be able to stabilise the system for domains defined by values of ζ as small as possible, i.e. for masses as great as possible. By testing the quadratic stabilisability for different values of ζ on each modelling we show the superiority of the generalised modelling.
The following results give an inferior bound for the values of the parameter ζ that enable the quadratic stabilisability for each model. As the quadratic stabilisabilty theorem gives a necessary and sufficient condition, we could find the exact set where ζ enables the quadratic stabilisability but this would need an infinite precision of the calculus. The matrices X, S and K in the following are given only as an indication. Their values are not useful because they are badly conditioned and have extremely high values. This is due to the difficulty of finding a good solution to the $\mathcal{LMI}$ problem when the model is at the limit on quadratic stabilisability.
The results of this study are summarised on figure 6.

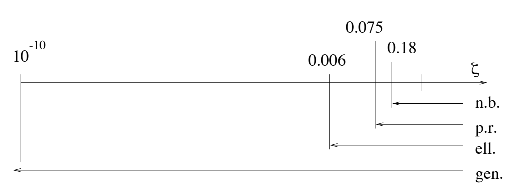

Figure 6: Quadratic stabilisability results

- For $\zeta \geq 0.18$

The norm bounded modelling enables to find a quadratically stabilisable state feedback. For example with $\zeta = 0.18$ the quadratic stabilisation algorithm gives the following solution:

$$X = 10^3 \begin{bmatrix} 0.0023 & 0.0021 & -0.0001 \\ 0.0021 & 2.4993 & -0.0035 \\ -0.0001 & -0.0035 & 0.0001 \end{bmatrix} \quad S = 10^6 \begin{bmatrix} 0.0000 & -1.6449 & -0.0000 \end{bmatrix} \tag{77}$$

$$K = 10^4 \begin{bmatrix} -0.0395 & -0.0704 & -3.3391 \end{bmatrix} \tag{78}$$

- For $\zeta \geq 0.075$

The positive real modelling enables to find a quadratically stabilisable state feedback. For example with $\zeta = 0.075$ the quadratic stabilisation algorithm gives the following solution:

$$X = 10^3 \begin{bmatrix} 0.0002 & 0.0002 & -0.0000 \\ 0.0002 & 2.8271 & -0.0010 \\ -0.0000 & -0.0010 & 0.0000 \end{bmatrix} \quad S = 10^6 \begin{bmatrix} -0.0000 & -6.9530 & 0.0000 \end{bmatrix} \tag{79}$$

$$K = 10^4 \begin{bmatrix} -0.1507 & -0.2491 & -8.8098 \end{bmatrix} \tag{80}$$

- For any $\zeta \geq 0.006$

The elliptic norm bounded modelling with adequately chosen parameters a, b, c enables to find a quadratically stabilisable state feedback. For example with $\zeta = 0.006$ we choose the domain generated with the parameters $a = 1$, $b = 0.155$, $c = 0.494$ that fits with the identified six point. The quadratic stabilisation algorithm gives the following solution:

$$X = 10^4 \begin{bmatrix} 0.0000 & 0.0000 & -0.0000 \\ 0.0000 & 4.2462 & -0.0002 \\ -0.0000 & -0.0002 & 0.0000 \end{bmatrix} \quad S = 10^8 \begin{bmatrix} 0.0000 & -6.5637 & -0.0000 \end{bmatrix} \tag{81}$$

$$K = 10^7 \begin{bmatrix} -0.0002 & -0.0016 & -1.1338 \end{bmatrix} \tag{82}$$

- For any $\zeta > 0$

The generalised modelling with adequately chosen parameters γ, ϕ enables to find a quadratically stabilisable state feedback. For example for $\zeta = 10^{-10}$ we choose the domain generated with the parameters $\gamma = 0.01$, $\phi = 17$ that fits with the identified six point. The quadratic stabilisation algorithm gives the following solution:

$$X = \begin{bmatrix} 0.4263 & 0.3691 & -0.2855 \\ 0.3691 & 4.1143 & -0.4284 \\ -0.2855 & -0.4284 & 0.2425 \end{bmatrix} \quad S = \begin{bmatrix} 1.4309 & -180.5758 & -0.0085 \end{bmatrix} \tag{83}$$

$$K = \begin{bmatrix} -68.7685 & -56.5485 & -180.8412 \end{bmatrix} \tag{84}$$

This stabilisability experimentation clearly shows that the choice of a modelling is most important. An identical uncertain domain (rectangular or elliptic), modelled in different ways, allows or not to find a quadratically stabilisable feedback. This is due to the parasitic terms influencing the system when we don't take into account the structure of the matrix Δ (norm bounded and positive real modellings). But the comparisons of the two rectangular cases and of the two elliptic cases show that independently to the choice of a structure, the choice of a modelling has an extreme importance. An analogous study with a fixed value of ζ, shows that the hierarchy between the modellings isn't modified when one looks for a smaller gain, better conditioned, controller.

6.3.2 pole location

In a second part, we study the pole location of the system. From now on, time-invariant uncertainties are considered. To be able to compare the four modellings we consider the case were $\zeta = 0.3$, which leads to a quadratically stabilisable state feedbacks for the four modellings. It will be possible to find a disk where the poles are located. This disk is, in the worst case, the left-half plane itself ($\alpha = -\infty$, $R = +\infty$). We choose $\gamma = 0.01$, $\phi = 4$, $a = 0.702$, $b = 0.3$, $c = 0.349$ that satisfies the identified constraints for the uncertain domain. See figure 7.

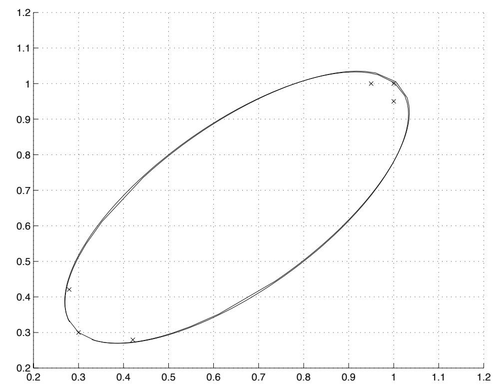

Figure 7: chosen domain of parameters - generalised and elliptic modelling

We impose the following specifications for the state feedback: all the real parts of the poles of system have to be inferior to -1. Simultaneously we desire to locate the poles in the smallest disk satisfying this condition. This ensures a minimum decay rate of 1 and gives a bound on undamped natural frequency of $\omega_d = R$. See figure 8.
The open loop poles of the system are all on the imaginary axis and one is zero. It is well known that pole location depends on how far the open loop poles have to be moved. So the disks leading to an "easy" pole location are the ones with the ceter near to the imaginary axis ($\alpha + R = -1$). See figure 8.

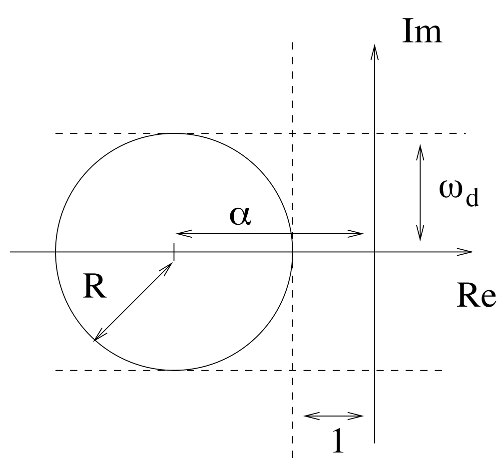

Figure 8: pole location specifications

By testing the modellings with the quadratic d-stabilisation algorithm we find that:

- For $\alpha \leq -17$

Every disk centred in α with the radius $R = -\alpha - 1$ satisfies the requested condition of rapidity and allows to conclude to the quadratic d-stabilisability of the system with the norm bounded modelling. For $\alpha = -17$ and $R = 16$ a solution to the $\mathcal{LMI}$ (38) for the norm bounded modelling is given by:

$$X = \begin{bmatrix} 0.2813 & 0.5408 & -0.2981 \\ 0.5408 & 123.7770 & -5.0527 \\ -0.2981 & -5.0527 & 0.6051 \end{bmatrix} \quad S = 10^3 \begin{bmatrix} 0.0031 & -2.0593 & 0.0105 \end{bmatrix} \tag{85}$$

$$K = \begin{bmatrix} -442.0468 & -34.7249 & -490.4136 \end{bmatrix} \tag{86}$$

- For $\alpha \leq -10$

Every disk centred in α with the radius $R = -\alpha - 1$ satisfies the requested condition of rapidity and allow to conclude to the quadratic d-stabilisability of the system with the positive real modelling. For $\alpha = -10$ and $R = 9$ a solution to the $\mathcal{LMI}$ (38) for the positive real modelling is given by:

$$X = \begin{bmatrix} 0.1062 & 0.2082 & -0.1155 \\ 0.2082 & 21.4465 & -1.4598 \\ -0.1155 & -1.4598 & 0.2448 \end{bmatrix} \quad S = \begin{bmatrix} 0.6545 & -212.3572 & 3.1009 \end{bmatrix} \tag{87}$$

$$K = \begin{bmatrix} -154.6044 & -21.0463 & -185.7730 \end{bmatrix} \tag{88}$$

- For $\alpha \leq -8.5$

Every disk centred in α with the radius $R = -\alpha - 1$ satisfies the requested condition of rapidity and allow to conclude to the quadratic d-stabilisability of the system with the elliptic norm bounded modelling. For $\alpha = -8.5$ and $R = 7.5$ a solution to the $\mathcal{LMI}$ (38) for the norm bounded elliptic modelling is given by:

$$X = \begin{bmatrix} 0.3319 & 0.6881 & -0.3710 \\ 0.6881 & 50.8453 & -4.1457 \\ -0.3710 & -4.1457 & 0.7803 \end{bmatrix} \quad S = \begin{bmatrix} 1.4635 & -421.5356 & 8.8721 \end{bmatrix} \tag{89}$$

$$K = \begin{bmatrix} -111.6720 & -17.9634 & -137.1707 \end{bmatrix} \tag{90}$$

- For $\alpha \leq -3.9$

Every disk centred in α with the radius $R = -\alpha - 1$ satisfies the requested condition of rapidity and allow to conclude to the quadratic d-stabilisability of the system with the generalised modelling. For $\alpha = -3.9$ and $R = 2.9$ a solution to the $\mathcal{LMI}$ (38) for the generalised modelling is given by:

$$X = \begin{bmatrix} 0.1249 & 0.2304 & -0.1559 \\ 0.2304 & 1.9730 & -0.6822 \\ -0.1559 & -0.6822 & 0.3260 \end{bmatrix} \quad S = \begin{bmatrix} -0.2407 & -7.8035 & 1.6464 \end{bmatrix} \tag{91}$$

$$K = \begin{bmatrix} -6.5861 & -9.1513 & -17.2513 \end{bmatrix} \tag{92}$$

The pole location domains for each of the studied cases described above are shown on figure 9. Figure 10 shows the poles for each modelling, the poles are obtained by random calculation of matrices satisfying the uncertainty conditions. We find here again, that the modelling of the uncertainties acting on a system is quite important. The quadratic stability and d-stability are not only more easily found, with more restrictive specifications, in the case of generalised modelling, but the corresponding feedback gain is the better conditioned one and has the smaller norm.

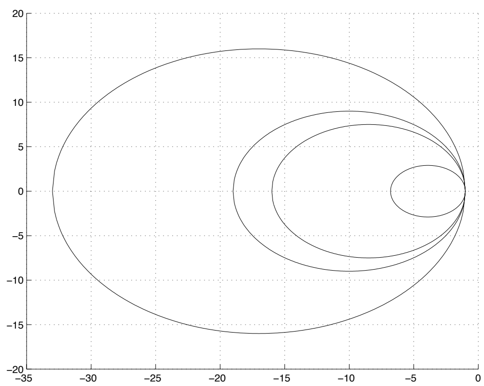

Figure 9: poles location domains

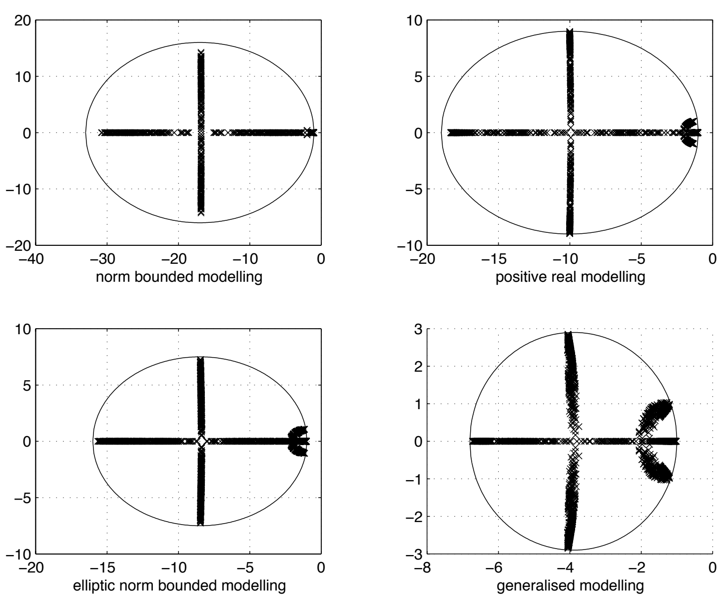

Figure 10: poles location obtained by simulation

7 Conclusions

A new framework for the modelling of uncertainties affecting the dynamical matrix of a L.T.I. system has been proposed in this paper. This modelling extends the ones developed in the small gain and passivity contexts. We hope so to take into account gain and phase constraints simultaneously. Analysis and synthesis results have been obtained using the framework of quadratic stability. This means that a single Lyapunov function has to be found to assess the stability of the closed-loop uncertain model. It is well-known that such an approach is conservative in the sense that time-invariant uncertainty cannot be explicitly considered. However, this approach has proved its efficiency in practice, [14]. The necessary and sufficient conditions derived in this paper are expressed in terms of $\mathcal{LMI}$'s. An alternative characterisation in terms of parameter-dependent Riccati equations could also been proposed. As shown by the numerical example of the previous section, these results seem to be promising and many extensions of the present work will be considered. One of the needs is to study the process of finding the "best" modelling for the uncertainties. In parallel to this, the theory of stabilisability of generalised uncertainty models has to be developed further on. The problem of quadratic stabilisation by output feedback is the natural extension of what has been done via state-feedback. At this point, robust stability via the concept of quadratic stability has been dealt with and the problem of robust performance has been tackled via the pole placement in a disk of the complex plane. An other approach to problems of robust performances is via guaranteed cost controller. These two robust performance approaches can also be jointly performed. This is a future subject of research.

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