Quadratic separation for uncertain descriptor system analysis, strict LMI conditions

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Abstract

The past ten years have witnessed the emergence of many techniques for solving parameter-dependent linear matrix inequality problems relative to robust analysis of dynamical systems. In these, a polynomial parameter-dependent structure of the Lyapunov function is chosen a priori for proving stability and conservative results are proposed to compute the coefficients of the matrix polynomial. Among such results some demonstrate that the polynomial structure of the Lyapunov function is related to an artificially augmented model in descriptor form. These contributions thus justify a renewed interest for robust analysis results in the descriptor context. Linear matrix inequality based formulas are proposed in the quadratic separation framework. Compared with previously derived results they have better numerical behavior, in particular because there is no need for equality constraints and most inequalities are strict.

Keywords: Robustness, descriptor LTI systems, Stability, LMI, dissipative uncertainties, quadratic separation.

1 Introduction

Papers such as [16, 17, 4, 5, 22] demonstrate in various ways the direct relationship between augmented descriptor modeling of systems and the possibility to provide numerically tractable results for their robust analysis. Not only the descriptor modeling allows to render linear some polynomial or rational dependencies, but artificial augmentation of the system (for example by introducing further derivatives of the state vector) is proved to lead to formulas with reduced conservatism. Providing theoretical results for the robust analysis of descriptor systems happens thus to be a complementary approach to parameter-dependent Lyapunov results such as [1, 3, 20, 24, 25] and needs to be further developed.

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In [21] this orientation was taken within the quadratic separation framework [23, 14]. It has proved the relevance of the approach not only for robust stability analysis of descriptor linear time-invariant (LTI) systems but for time-delay systems as well. Unfortunately, the results were formulated combining linear matrix inequalities (LMIs) and linear equality constraints. Moreover, some of the LMIs were non-strict (semi-definite) which is a drawback for numerical efficiency.

Following the example of [27, 13, 2] that investigated strict LMI conditions for stability analysis of $E\dot{x} = Ax$ thus avoiding the usual constraint $E^T P^T = PE \geq \mathbf{0}$, the paper current revisits the results of [21] in order to avoid equality constraints and limit as much as possible the non-strict LMIs. A significant side effect of the improved conditions is that they apply easier to systems with highly structured uncertainties. A quite general modeling of such systems is proposed. Inspired by [15, 25] it has the property to include in a unified formulation all scalar real or complex and full-block uncertainties.

The paper is organized as follows. First, a section is devoted to modeling control problems as an uncertain feedback on an implicit linear transformation. Artificial augmentation issues are commented. The following section is devoted to technical results, essentially to the central quadratic separation Theorem. Before concluding, sections IV and V formulate the application of the quadratic separation Theorem to stability analysis and robustness issues.

Notations: $\mathsf{R}^{m\times n}$ and $\mathsf{C}^{m\times n}$ are the sets of m-by-n real and complex matrices respectively. A^T is the transpose of the matrix A and A^* is its transpose conjugate. $\mathbf{1}$ and $\mathbf{0}$ are respectively the identity and the zero matrices of appropriate dimensions. For Hermitian matrices, $A > (\geq) B$ means that $A - B$ is positive (semi) definite. $A^{\perp}$ is a full rank matrix whose columns span the null-space of A. In Matlab $A^{\perp}$ is obtained using `null(A)`. Define as well A° as a full rank matrix whose columns span the same space as the columns of A. If $A = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \mathrm{diag}(S, \mathbf{0}) \begin{bmatrix} V_1 & V_2 \end{bmatrix}^*$ is the singular value decomposition with S containing all non zero singular values, then one can choose $A^{\perp} = V_2$ and $A^{\circ} = U_1$. In Matlab A° is obtained using `orth(A)`. In addition, define $A^{\circledast} = A^{*\circ}$ (or $A^{T\circ}$ if real-valued) which is such that the columns of $\begin{bmatrix} A^{\perp} & A^{\circledast} \end{bmatrix}$ span C^m (assuming $A \in \mathsf{C}^{n\times m}$). To shorten some formulas, $\langle A \rangle$ stands for $A + A^*$.

2 Feedback connected implicit models

2.1 Well-posedness condition

Consider two possibly non-square finite dimensional matrices $\mathcal{E}$ and $\mathcal{A}$. Let an uncertain matrix ∇ with appropriate dimensions that belongs to some set $\boldsymbol{\nabla}$. No assumption is made on the uncertainty set $\boldsymbol{\nabla}$ at this stage. This paper investigates stability problems encountered when performing the feedback connection of $\mathcal{E}z = \mathcal{A}w$ with $w = \nabla z$. To formulate these stability problems define two external disturbances $\bar{w}$ and $\bar{z}$ that act on these two equations defining the feedback system of Figure 1.

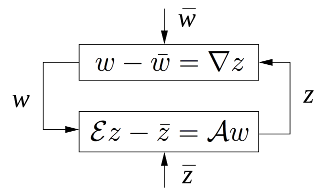

Figure 1: Feedback system

The feedback system of Figure 1 is said to be well-posed if for all uncertainties and all bounded input vectors, the internal vectors characterizing the system are unique and bounded. More specifically, consider the decomposition of z in the $\left[\begin{array}{cc}\mathcal{E}^{\perp} & \mathcal{E}^{\circledast}\end{array}\right]$ basis, *i.e.* $z = \mathcal{E}^{\perp}y_1 + \mathcal{E}^{\circledast}y$. With these notations, the feedback connected system writes

$$\begin{array}{c} w - \bar{w} = \nabla\mathcal{E}^{\perp}y_1 + \nabla\mathcal{E}^{\circledast}y \\ \mathcal{E}\mathcal{E}^{\circledast}y - \bar{z} = \mathcal{A}w \ . \end{array} \tag{1}$$

As ∇ may be rank-deficient, the vector y_1 may be non-unique and unbounded, at least for some values of ∇. The vector y_1 is therefore not an internal variable of the system but rather a perturbation, possibly unbounded. The definition of well-posedness of the feedback connected system is therefore based on proving that for all uncertainties $\nabla \in \boldsymbol{\nabla}$ and all bounded inputs $\bar{w}$ and $\bar{z}$, the internal variables w and y are unique and bounded. Note as well that $\mathcal{E}z$ is bounded if and only if y is bounded. Therefore the y signal is replaced by $\mathcal{E}z$ in the following well-posedness definition. Since only linear transformations enter the feedback system, w and $\mathcal{E}z$ are necessarily unique if we can prove they are bounded. Hence, well-posedness writes as

$$\exists\bar{\gamma} > 0 \ : \ \begin{array}{c}\forall(\bar{z}, \bar{w}) \\ \forall\nabla \in \boldsymbol{\nabla}\end{array} \ , \ \left\| \left(\begin{array}{c} \mathcal{E}z \\ w \end{array} \right) \right\| \leq \bar{\gamma} \left\| \left(\begin{array}{c} \bar{z} \\ \bar{w} \end{array} \right) \right\| \ . \tag{2}$$

Note that (1) implies

$$(\mathcal{E} - \mathcal{A}\nabla)\mathcal{E}^{\circledast}y = \bar{z} + \mathcal{A}\nabla\mathcal{E}^{\perp}y_1 + \mathcal{A}\bar{w} \ .$$

Well-posedness of the system states that for all admissible $\nabla \in \boldsymbol{\nabla}$, the null space of $(\mathcal{E} - \mathcal{A}\nabla)\mathcal{E}^{\circledast}$ is empty (the matrix is non-singular if square) and one gets $((\mathcal{E} - \mathcal{A}\nabla)\mathcal{E}^{\circledast})^{\dagger}\mathcal{A}\nabla\mathcal{E}^{\perp} = \mathbf{0}$ since y is unique for all y_1.

Having formulated this very general well-posedness problem for implicit linear transformations with uncertain feedback, the remaining of this section is devoted to illustrating this problem with some important analysis questions.

2.2 Pole location

One most simple problem is pole location analysis of linear descriptor systems. Let the state-space equation $E\dot{x} = Ax$ and define the poles as the values such that $rank(Es -$

A) drops from its normal value. The well known $\mathcal{D}$-stability problem is to demonstrate all poles belong to some region $\mathcal{D}$. In the paper only half-planes and discs of the complex plane are considered. The regions generically write as

$$\mathcal{D} = \{ \ s \in \mathbf{C} \ : \ d_1 + d_2 s + d_2^* s^* + d_3 s s^* < 0 \ \} \,.$$

For many other regions (as well as for unions of regions) results can also be produced by combining the methodology exposed in the present paper with results in [11, 15]. For intersections of regions, the procedure consists in proving pole location in each region independently. The $\mathcal{D}$-stability problem includes Hurwitz stability of continuous time systems when choosing $d_1 = d_3 = 0$ and $d_2 = 1$. Schur stability of discrete-time systems $Ex(t+1) = Ax(t)$ may be treated in the same framework taking $-d_1 = d_3 = 1$ and $d_2 = 0$.

Proposition 1 *$\mathcal{D}$-stability of $E\dot{x} = Ax$ is equivalent to well-posedness of the feedback system of Figure 1 where*

$$\begin{gathered} z = \dot{x} \ , \ \ w = x \ , \ \ \mathcal{E} = E \ , \ \ \mathcal{A} = A \ , \\ \boldsymbol{\nabla} = \left\{ \ s^{-1}\mathbf{1} \ : \ d_1 s^{-1} s^{-*} + d_2 s^{-*} + d_2^* s^{-1} + d_3 \geq 0 \ \right\} . \end{gathered}$$

Proof of this proposition is based on the fact that $\mathcal{D}$-stability is achieved if $Es - A$ remains full rank for all s in the exterior of $\mathcal{D}$.

2.3 Robust $\mathcal{D}$-stability of descriptor systems

Consider the following uncertain descriptor system

$$\begin{aligned} (E_A + (B\Delta - E_B)(E_D - D\Delta)^{-1}E_C)\dot{x} \\ = (A + (B\Delta - E_B)(E_D - D\Delta)^{-1}C)x \end{aligned} \tag{3}$$

where the state-space model matrices are rational functions of the uncertain parameters Δ that are assumed to belong to a set $\boldsymbol{\Delta}$. Note that this very general modeling naturally arises from Linear Fractional Transform (LFT) as attested in [10, 19]. Moreover, it can often give minimal LFT formulations which is of major interest as the numerical complexity of analysis tools grows significantly with the dimensions of the uncertain operator Δ.

Robust $\mathcal{D}$-stability of model (3) matches the framework of Figure 1 if one considers

$$\overbrace{\begin{bmatrix} E_A & E_B \\ E_C & E_D \end{bmatrix}}^{\mathcal{E}} \begin{pmatrix} \dot{x} \\ z_\Delta \end{pmatrix} = \overbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}^{\mathcal{A}} \begin{pmatrix} x \\ w_\Delta \end{pmatrix} \tag{4}$$

along with the set

$$\boldsymbol{\nabla} = \left\{ \begin{bmatrix} s^{-1}\mathbf{1}_n & \mathbf{0} \\ \mathbf{0} & \Delta \end{bmatrix} \ : \ s \in \mathcal{D} \ , \ \Delta \in \boldsymbol{\Delta} \right\} .$$

Most often the uncertainties Δ are composed of several independent elements $\Delta_1, \Delta_2, \dots$ In all cases Δ can be written as a block diagonal matrix of these elements, each of which may be repeated several times, $\mathbf{1}_{r_i} \otimes \Delta_i$. In this paper all uncertainties are assumed to be of the following type.

Definition 1 *An uncertainty Δ_i is said to be $\{\Phi_{1i}, \Phi_{2i}, \Phi_{3i}\}$-structured $\{\Psi_{1i}, \Psi_{2i}, \Psi_{3i}\}$-dissipative if it belongs to the set defined by the following equality and inequality constraints*

$$\begin{bmatrix} \mathbf{1} & \Delta_i^* \end{bmatrix} \begin{bmatrix} \Phi_{1i} & \Phi_{2i} \\ \Phi_{2i}^* & \Phi_{3i} \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \Delta_i \end{bmatrix} = \mathbf{0} \,,$$
$$\begin{bmatrix} \mathbf{1} & \Delta_i^* \end{bmatrix} \begin{bmatrix} \Psi_{1i} & \Psi_{2i} \\ \Psi_{2i}^* & \Psi_{3i} \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \Delta_i \end{bmatrix} \leq \mathbf{0} \,.$$

Special instances of such uncertainties are:

- $\{\mathbf{0}, \mathbf{0}, \mathbf{0}\}$-structured, $\{-\gamma\mathbf{1}, \mathbf{0}, \mathbf{1}\}$-dissipative uncertainties are exactly the well-known norm-bounded uncertainties ($\Delta_i^*\Delta_i \leq \gamma\mathbf{1}$) that equivalently represent H_∞ performances.
- $\{\mathbf{0}, \mathbf{0}, \mathbf{0}\}$-structured, $\{\mathbf{0}, \mathbf{1}, \mathbf{0}\}$-dissipative uncertainties are exactly the well-known positive-real uncertainties involved in passivity framework ($\Delta_i^* + \Delta_i \geq \mathbf{0}$).
- $\{0, 0, 0\}$-structured, $\{-d_3, -d_2^*, -d_1\}$-dissipative uncertainties are exactly scalar complex uncertainties δ constrained in a region $d_1\delta\delta^* + d_2\delta^* + d_2^*\delta + d_3 \geq 0$. Examples of such "uncertainties" are the integral operator s^{-1} but also operators such as e^{-sh}, $(1-e^{-sh})/sh$ and others which allow extensions of the exposed results to time-delay systems (see [21, 9] for details). Following the methodology of [15], the $\{\Phi_{1i}, \Phi_{2i}, \Phi_{3i}\}$-structure may potentially be used to specify properties on frequency ranges ($s = j\omega$, $\omega \in [\underline{\omega} \;\; \overline{\omega}]$).
- $\{0, j, 0\}$-structured, $\{\underline{\delta}\overline{\delta}, -\frac{\underline{\delta}+\overline{\delta}}{2}, 1\}$-dissipative uncertainties are exactly scalar real uncertainties constrained in an interval ($\delta_i \in [\underline{\delta} \;\; \overline{\delta}]$).

As said above, the operator s^{-1} may be seen as among the uncertainties, hence without loosing any generality one may consider the following model

$$\begin{array}{c} \mathcal{E}z = \mathcal{A}w \,,\; w = \nabla z \\ \nabla = \text{diag}(\mathbf{1}_{r_1} \otimes \Delta_1, \dots, \mathbf{1}_{r_N} \otimes \Delta_N) \end{array} \tag{5}$$

composed of N independent uncertainties each being $\{\Phi_{1i}, \Phi_{2i}, \Phi_{3i}\}$-structured, $\{\Psi_{1i}, \Psi_{2i}, \Psi_{3i}\}$-dissipative .

2.4 System augmentation for conservatism reduction

Deriving LMI results for robust analysis of systems as exposed above is known to introduce some conservative in general. This conservatism is usually related to assumptions on the nature of some "slack" variables introduced to make the problem LMI.

One of such variables is the Lyapunov matrix and the assumptions are its dependency with respect to the uncertain parameters. Starting from parameter-independent Lyapunov matrices, result known as the "quadratic stability" framework, many new results have been derived for polynomial or rational parameter-dependent Lyapunov matrices [1, 3, 16, 20, 24, 25]. Moreover, in [21, 22] it is shown that some of these methods have an interpretation in terms of artificially augmented systems. Rather than deriving complex matrix manipulations of parameter-dependent LMIs, this approach suggests to manipulate only the model equations on which a unique simple methodology is applied.

For example, let the uncertain system modeled as

$$\dot{x} = (A + B\Delta_a)x \ , \ \Delta_a = \Delta(\mathbf{1} - D\Delta)^{-1}C$$

with constant uncertain parameters, $\dot{\Delta} = \mathbf{0}$. Proving its stability with the help of a parameter-dependent Lyapunov function of the type

$$V(x, \Delta) = x^* \begin{bmatrix} \mathbf{1} & \Delta_a^* \end{bmatrix} P \begin{bmatrix} \mathbf{1} \\ \Delta_a \end{bmatrix} x$$

is shown in [16] to be related to robust stability of the augmented system

$$\overbrace{\left[\begin{array}{cc|cc} \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & -\mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ -C & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{array}\right]}^{\mathcal{E}} z = \overbrace{\left[\begin{array}{cc|cc} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ A & \mathbf{0} & B & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ C & \mathbf{0} & D & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & D \end{array}\right]}^{\mathcal{A}} w$$

with the feedback operator $\nabla = \begin{bmatrix} s^{-1}\mathbf{1}_{n+m} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_2 \otimes \Delta \end{bmatrix}$ where $z^* = \left(\begin{array}{cc|cc} \dot{x}^* & \dot{z}_\Delta^* & z_\Delta^* & \dot{z}_\Delta^* \end{array} \right)$ and $w^* = \left(\begin{array}{cc|cc} x^* & z_\Delta^* & w_\Delta^* & \dot{w}_\Delta^* \end{array} \right)$.

The key technical point in the model augmentation is that the inverse of the scalar operator s^{-1} commutes with the matrices C, D and Δ (in case Δ is time varying its derivative is also involved in the augmented model). The more this artificial augmentation is applied, thus introducing further derivatives of z_Δ and w_Δ, the higher degree is the implicit rational parameter-dependent Lyapunov function.

In [1] it is shown that not only the Lyapunov matrix should be parameter-dependent but this also holds for the other "slack" variables introduced to make the problem LMI. This indicates that conservatism may as well be reduced when performing the same artificial augmentation with other uncertainties involved in ∇. At this stage, since the system augmentation needs the "uncertainty" to commute with the model matrices, the technique may only be applied for scalar uncertain elements $\Delta_i = \delta_i$.

Without getting in more details for lack of space, this section indicates the potentiality of well-posedness analysis of models such as in Figure 1. Not only descriptor-type systems justify the choice of studying the situation when $\mathcal{E}$ is different from the identity matrix, but the search for less conservative results through artificial system augmentation does as well, even for usual LTI systems. The next section gives a new quadratic separation Theorem, generalizing that of [14] to implicit linear transformations.

3 Technical results

Theorem 1 *The uncertain feedback system of Figure 1 is well-posed if and only if there exists a Hermitian matrix $\Theta = \Theta^*$ satisfying both conditions*

$$\left[\begin{array}{cc} \mathcal{A}^* & \mathbf{1} \end{array} \right] \Theta \left[\begin{array}{c} \mathcal{A} \\ \mathbf{1} \end{array} \right] > \mathbf{0} \tag{6}$$

$$\left[\begin{array}{cc} \mathcal{E}^* & \nabla^* \end{array} \right] \Theta \left[\begin{array}{c} \mathcal{E} \\ \nabla \end{array} \right] \leq \mathbf{0} \quad , \quad \forall \nabla \in \boldsymbol{\nabla} \,. \tag{7}$$

If $\mathcal{E}$ and $\mathcal{A}$ are real, the equivalence still holds with Θ restricted to be real.

Proof of sufficiency: Assume (6) and (7) hold, then

$$\begin{array}{l} z^* \left[\begin{array}{cc} \mathcal{E}^* & \nabla^* \end{array} \right] \Theta \left[\begin{array}{c} \mathcal{E} \\ \nabla \end{array} \right] z \\ = \left(\begin{array}{c} \mathcal{E} z \\ w - \bar{w} \end{array} \right)^* \Theta \left(\begin{array}{c} \mathcal{E} z \\ w - \bar{w} \end{array} \right) \leq 0 \,. \end{array}$$

and for some sufficiently small positive scalar ϵ:

$$\begin{array}{l} w^* \left[\begin{array}{cc} \mathcal{A}^* & \mathbf{1} \end{array} \right] (\Theta - \epsilon \mathbf{1}) \left[\begin{array}{c} \mathcal{A} \\ \mathbf{1} \end{array} \right] w \\ = \left(\begin{array}{c} \mathcal{E} z - \bar{z} \\ w \end{array} \right)^* (\Theta - \epsilon \mathbf{1}) \left(\begin{array}{c} \mathcal{E} z - \bar{z} \\ w \end{array} \right) \geq 0 \,. \end{array}$$

Subtract the first expression to the second to obtain $X^* \left[\begin{array}{cc} \epsilon \mathbf{1} & T_1 \\ T_1^* & T_2 \end{array} \right] X \leq 0$ where

$$\begin{array}{l} X^* = \left(\begin{array}{cc|cc} z^* \mathcal{E}^* & w^* & \bar{z}^* & \bar{w}^* \end{array} \right) \,, \\ T_1 = (\Theta - \epsilon \mathbf{1}) J_1 - \Theta J_2 \,, \\ T_2 = J_1 (\epsilon \mathbf{1} - \Theta) J_1 + J_2 \Theta J_2 \end{array}$$

and $J_1 = \left[\begin{array}{cc} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{array} \right]$, $J_2 = \left[\begin{array}{cc} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{array} \right]$ with the appropriate partitioning. Take any $\tilde{\epsilon}$ such that $\epsilon > \tilde{\epsilon} > 0$ and take a sufficiently large $\tilde{\gamma} > 0$ such that

$$\left[\begin{array}{cc} \tilde{\epsilon} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\tilde{\gamma} \mathbf{1} \end{array} \right] \leq \left[\begin{array}{cc} \epsilon \mathbf{1} & T_1 \\ T_1^* & T_2 \end{array} \right]$$

to finally obtain $X^* \left[\begin{array}{cc} \tilde{\epsilon} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\tilde{\gamma} \mathbf{1} \end{array} \right] X \leq 0$ which is the well-posedness condition (2) for $\bar{\gamma} = \tilde{\gamma} / \tilde{\epsilon}$. ■

Proof of necessity: Assume the system in Figure 1 is well-posed, *i.e.* (2) holds for some $\bar{\gamma}$. Note that (2) also holds for all $\gamma \geq \bar{\gamma}$. Define $Y = \left(\begin{array}{cc|cc} z^* & w^* & \bar{z}^* & \bar{w}^* \end{array} \right)^*$,

$$\mathcal{M} = \left[\begin{array}{cc|cc} -\nabla & \mathbf{1} & \mathbf{0} & -\mathbf{1} \\ \mathcal{E} & -\mathcal{A} & -\mathbf{1} & \mathbf{0} \end{array} \right]$$

and

$$\Xi = \left[\begin{array}{cc|c} \mathcal{E}^*\mathcal{E} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & -\gamma\mathbf{1} \end{array} \right].$$

Well-posedness of (1) implies that, for all $\gamma \geq \bar{\gamma}$ and for all $\nabla \in \boldsymbol{\nabla}$, if the equality constraint $\mathcal{M}Y = 0$ holds, then the quadratic constraint $Y^*\Xi Y \leq \mathbf{0}$ also holds. Due to Finsler's lemma [26], it is equivalent to

$$\mathcal{M}^{\perp^*}\Xi\mathcal{M}^{\perp} \leq \mathbf{0}\,. \tag{8}$$

Take $\mathcal{M}^{\perp *} = \left[\begin{array}{cc|cc} \mathbf{1} & \mathbf{0} & \mathcal{E}^* & -\nabla^* \\ \mathbf{0} & \mathbf{1} & -\mathcal{A}^* & \mathbf{1} \end{array} \right]$, the inequality (8) writes as

$$\left[\begin{array}{cc} (1-\gamma)\mathcal{E}^*\mathcal{E} - \gamma\nabla^*\nabla & \gamma(\mathcal{E}^*\mathcal{A} + \nabla^*) \\ \gamma(\mathcal{A}^*\mathcal{E} + \nabla) & \mathbf{1} - \gamma(\mathbf{1} + \mathcal{A}^*\mathcal{A}) \end{array} \right] \leq \mathbf{0}$$

Define $Q = \mathbf{1} + \mathcal{A}^*\mathcal{A}$ and choose γ sufficiently large such that $\mathbf{1} - \gamma Q < \mathbf{0}$. Applying a Schur complement argument on the block $\mathbf{1} - \gamma Q$, one gets inequality (7) where $\Theta =$

$$\left[\begin{array}{cc} \mathbf{1} - \gamma\mathbf{1} - \gamma^2\mathcal{A}(\mathbf{1} - \gamma Q)^{-1}\mathcal{A}^* & -\gamma^2\mathcal{A}(\mathbf{1} - \gamma Q)^{-1} \\ -\gamma^2(\mathbf{1} - \gamma Q)^{-1}\mathcal{A}^* & -\gamma\mathbf{1} - \gamma^2(\mathbf{1} - \gamma Q)^{-1} \end{array} \right].$$

This matrix is real if $\mathcal{E}$ and $\mathcal{A}$ are real. Let us prove now that (6) also holds.

$$\left[\begin{array}{cc} \mathcal{A}^* & \mathbf{1} \end{array} \right] \Theta \left[\begin{array}{c} \mathcal{A} \\ \mathbf{1} \end{array} \right] = \mathcal{A}^*\mathcal{A} - \gamma Q - \gamma^2 Q(\mathbf{1} - \gamma Q)^{-1}Q$$

Recall the matrix inversion lemma $(a + bcd)^{-1} = a^{-1} - a^{-1}b(c^{-1} + da^{-1}b)^{-1}da^{-1}$ to conclude

$$\begin{array}{l} \left[\begin{array}{cc} \mathcal{A}^* & \mathbf{1} \end{array} \right] \Theta \left[\begin{array}{c} \mathcal{A} \\ \mathbf{1} \end{array} \right] = \mathcal{A}^*\mathcal{A} - ((\gamma Q)^{-1} - \mathbf{1})^{-1} \\ = \mathcal{A}^*\mathcal{A} - (\gamma Q)^{-1/2}(\mathbf{1} - \gamma Q)^{-1}(\gamma Q)^{-1/2} > \mathbf{0}\,. \end{array}$$

Both inequalities (6) and (7) hold for any γ sufficiently large to ensure $\mathbf{1} - \gamma Q < \mathbf{0}$. ■

Remark that the heart of the proof relies on the use of Finsler's lemma. As in [6, 7] this is the key tool that enables to deal with implicit linear transformation constraints.

Lemma 1 *Consider a Linear Matrix Inequality (LMI) problem of the type*

$$\mathcal{L}MI(X, Y, Z) < \mathbf{0} \;\; , \;\; E^*XE \geq \mathbf{0} \tag{9}$$

where $\mathcal{L}MI$ gathers possibly several LMI definite positive or definite negative constraints and where X and Y are possibly composed of several variables. This LMI problem is feasible if and only if there is a solution to the second LMI problem

$$\mathcal{L}MI(X,Y) < \mathbf{0} \ \ , \ \ E^{\circledast *}E^{*}XEE^{\circledast} > \mathbf{0} \tag{10}$$

Proof : Assume $(\hat{X},\hat{Y})$ solution to problem (9). Consider the decomposition of z in the $\left[\begin{array}{cc} E^{\perp} & E^{\circledast}\end{array}\right]$ basis, *i.e.* $z = E^{\perp}y_1 + E^{\circledast}y$. Then

$$z^{*}E^{*}\hat{X}Ez = y^{*}E^{\circledast *}E^{*}\hat{X}EE^{\circledast}y \geq 0$$

which makes $E^{\circledast *}E^{*}\hat{X}EE^{\circledast} \geq \mathbf{0}$ and $E^{*}\hat{X}E \geq \mathbf{0}$ equivalent. The constraint

$$\mathcal{L}MI(\hat{X},\hat{Y}) < \mathbf{0}$$

defines an open set, therefore there exist a possibly small scalar $\epsilon > 0$ such that $\mathcal{L}MI(X,Y) < \mathbf{0}$ with $X = \hat{X} - \epsilon\mathbf{1}$ and $Y = \hat{Y}$. Moreover, one gets that

$$E^{\circledast *}E^{*}XEE^{\circledast} \geq \epsilon E^{\circledast *}E^{*}EE^{\circledast} > \mathbf{0}$$

due to the fact that $EE^{\circledast}$ is full rank. ■

This lemma gives an effective way to code LMIs such as (9) with only strict LMI constraints. It is of interest from a numerical point of view and should be preferred when implementing the LMI constraints exposed further in the paper.

4 Stability of descriptor systems

Theorem 1 is now applied to $\mathcal{D}$-stability of linear descriptor systems.

Corollary 1 *The descriptor system $E\dot{x} = Ax$ is $\mathcal{D}$-stable, if (6) holds for*

$$\Theta = \left[\begin{array}{cc} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & E^{*}\end{array}\right]\Theta_{\mathcal{D}}\left[\begin{array}{cc} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & E\end{array}\right] + \left\langle \left[\begin{array}{c} E^{*\perp} \\ \mathbf{0}\end{array}\right] Y \right\rangle \tag{11}$$

constrained by

$$\Theta_{\mathcal{D}} = \left[\begin{array}{cc} d_3 X & d_2^{*}X \\ d_2 X & d_1 X\end{array}\right] , \ E^{*}XE \geq \mathbf{0} \ . \tag{12}$$

This corollary includes results of [28] (in which only discs are considered), when taking $Y = \mathbf{0}$. Moreover, the corollary can be proved to be a necessary and sufficient condition. Rather than providing the proof, two special instances (Hurwitz and Schur stability) are considered illustrating that there is no conservatism.

Corollary 2 *The continuous-time descriptor system $E\dot{x} = Ax$ is admissible (i.e. regular, stable and impulse free [18]) if and only if (6) holds for Θ constrained by either one of the constraints (13) or (14)*

$$\Theta = \begin{bmatrix} \mathbf{0} & -P^* \\ -P & \mathbf{0} \end{bmatrix}, \; E^*P^* = PE \geq \mathbf{0} \tag{13}$$

$$\Theta = \begin{bmatrix} \mathbf{0} & E^{*\perp}Y_2 - XE \\ (E^{*\perp}Y_2 - XE)^* & \mathbf{0} \end{bmatrix}, \; E^*XE \geq \mathbf{0}. \tag{14}$$

Proof : Sufficiency is proved applying Theorem 1. With any of these two choices of separator Θ conditions (7) holds.

Necessity is obtained since the resulting LMI conditions are respectively exactly those of [18]

$$E^*P^* = PE \geq \mathbf{0} \;\; , \;\; A^*P^* + PA < \mathbf{0}\,. \tag{15}$$

or those of [27] and [2]. In [2] the LMIs are more precisely

$$\begin{array}{c} E^*XE + E^{*\perp}\hat{X} + (E^{*\perp}\hat{X})^* > \mathbf{0} \\ A^*(XE - E^{*\perp}Y_2) + (XE - E^{*\perp}Y_2)^*A < \mathbf{0}\,. \end{array} \tag{16}$$

They prove to be equivalent to that obtained when applying Lemma 1 to $E^*XE \geq \mathbf{0}$ and elimination lemma with $E^{\circledast *}E^{*\perp} = \mathbf{0}$ to the first inequality in (16). ■

Note that corollary 2 indicates that not all elements of the Y matrix in (11) may be needed to have necessary and sufficient conditions. Note as well that the second type of separator (14) is preferable from numerical point of view because the LMI conditions can be coded with only strict matrix inequalities (see Lemma 1) and avoids the coding of equality constraints.

Corollary 3 *The discrete-time descriptor system $Ex(t+1) = Ax(t)$ is stable if and only if (6) holds for*

$$\Theta = \begin{bmatrix} -X & \mathbf{0} \\ \mathbf{0} & E^*XE \end{bmatrix}, \; E^*XE \geq \mathbf{0}. \tag{17}$$

The result corresponds the generalized discrete Lyapunov inequality of [12] [29, lemma 1] that writes $E^*XE \geq \mathbf{0}$, $E^*XE > A^*XA$. Again it indicates that the variable Y may not be needed in all cases thus indicating possible reduction of the dimensions of the LMI problems.

5 LMI formulas for robust stability analysis

At this stage it is assumed that the model has been manipulated following the lines exposed in Section 2 and has lead to a formulation of the type (5). Well-posedness of this feedback connected system corresponds to a robust $\mathcal{D}$-stability problem.

To formulate the results, let the matrices with the following structure

$$\begin{matrix} J_i = \left[\begin{array}{ccc} \mathbf{0} & \mathbf{1} & \mathbf{0} \end{array}\right] \\ K_i = \left[\begin{array}{ccc} \mathbf{0} & \mathbf{1} & \mathbf{0} \end{array}\right] \end{matrix}, \; L_i = \left[\begin{array}{cc} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{array}\right] = K_i^{\perp}$$

with appropriate dimensions to satisfy for all $i = 1 \ldots N$ the formulas $J_i \nabla = (\mathbf{1}_{r_i} \otimes \Delta_i) K_i$. Let also $F_i = K_i \mathcal{E}^*$ and $G_i = (L_i^T \mathcal{E}^*)^{\perp} F_i^{\circledast}$.

Theorem 2 *(5) is well-posed if there exist a matrix Y, $2N$ Hermitian matrices Θ_i, P_i and N Hermitian positive semi-definite matrices $Q_i \geq \mathbf{0}$ such that (6) holds where the separator is structured as*

$$\Theta = \sum_{i=1}^{N} \left[\begin{array}{cc} G_i & \mathbf{0} \\ \mathbf{0} & J_i^T \end{array}\right] \Theta_i \left[\begin{array}{cc} G_i & \mathbf{0} \\ \mathbf{0} & J_i^T \end{array}\right]^* + \left\langle \left[\begin{array}{c} \mathcal{E}^{*\perp} \\ \mathbf{0} \end{array}\right] Y \right\rangle \tag{18}$$

and constrained by the inequalities

$$\left[\begin{array}{cc} F_i F_i^{\circledast} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{array}\right] \Theta_i \left[\begin{array}{cc} F_i F_i^{\circledast} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{array}\right]^* \leq \Lambda_i(P_i, Q_i) \tag{19}$$

where $\Lambda_i(P_i, Q_i) =$

$$\left[\begin{array}{cc} P_i \otimes \Phi_{1i} & P_i \otimes \Phi_{2i} \\ P_i \otimes \Phi_{2i}^* & P_i \otimes \Phi_{3i} \end{array}\right] + \left[\begin{array}{cc} Q_i \otimes \Psi_{1i} & Q_i \otimes \Psi_{2i} \\ Q_i \otimes \Psi_{2i}^* & Q_i \otimes \Psi_{3i} \end{array}\right].$$

If the data $\mathcal{E}$ and $\mathcal{A}$ are real, then the Θ_i are real.

Proof: Note that

$$\mathcal{E}^*(L_i^T \mathcal{E}^*)^{\perp} = \left[\begin{array}{cc} L_i & K_i^T \end{array}\right] \left[\begin{array}{c} L_i^T \\ K_i \end{array}\right] \mathcal{E}^*(L_i^T \mathcal{E}^*)^{\perp} = K_i^T F_i$$

which implies that

$$\left[\begin{array}{cc} \mathcal{E}^* & \nabla^* \end{array}\right] \left[\begin{array}{cc} G_i & \mathbf{0} \\ \mathbf{0} & J_i^T \end{array}\right] = K_i^T \left[\begin{array}{cc} F_i F_i^{\circledast} & \mathbf{1}_{r_i} \otimes \Delta_i^* \end{array}\right].$$

Therefore the formula (18) gives

$$\left[\begin{array}{c} \mathcal{E} \\ \nabla \end{array}\right]^* \Theta \left[\begin{array}{c} \mathcal{E} \\ \nabla \end{array}\right] \leq \sum_{i=1}^{N} \left[\begin{array}{c} (F_i F_i^{\circledast})^* \\ \mathbf{1}_{r_i} \otimes \Delta_i \end{array}\right]^* \Theta_i \left[\begin{array}{c} (F_i F_i^{\circledast})^* \\ \mathbf{1}_{r_i} \otimes \Delta_i \end{array}\right].$$

Pre and post multiply inequalities (19) by $\left[\begin{array}{cc} \mathbf{1} & \mathbf{1}_{r_i} \otimes \Delta_i \end{array}\right]$ and its conjugate-transpose respectively to get

$$\begin{aligned} &\left[\begin{array}{c} (F_i F_i^{\circledast})^* \\ \mathbf{1}_{r_i} \otimes \Delta_i \end{array}\right]^* \Theta_i \left[\begin{array}{c} (F_i F_i^{\circledast})^* \\ \mathbf{1}_{r_i} \otimes \Delta_i \end{array}\right] \\ &\qquad \leq P_i \otimes \left(\left[\begin{array}{cc} \mathbf{1} & \Delta_i^* \end{array}\right] \left[\begin{array}{cc} \Phi_{1i} & \Phi_{2i} \\ \Phi_{2i}^* & \Phi_{3i} \end{array}\right] \left[\begin{array}{c} \mathbf{1} \\ \Delta_i \end{array}\right] \right) \\ &\qquad + Q_i \otimes \left(\left[\begin{array}{cc} \mathbf{1} & \Delta_i^* \end{array}\right] \left[\begin{array}{cc} \Psi_{1i} & \Psi_{2i} \\ \Psi_{2i}^* & \Psi_{3i} \end{array}\right] \left[\begin{array}{c} \mathbf{1} \\ \Delta_i \end{array}\right] \right) \end{aligned}$$

which is negative semi-definite due to the $\{\Phi_{1i}, \Phi_{2i}, \Phi_{3i}\}$-structured $\{\Psi_{1i}, \Psi_{2i}, \Psi_{3i}\}$-dissipative properties of Δ_i. Hence both conditions (6) and (7) of Theorem 1 hold. ■

Remark that (19) is not quite the same as the separators proposed in Section 4. Actually, some cases can be distinguished for which the number of decision variables may be reduced.

In case the uncertainty element is a scalar $\Delta_i = \delta_i$, then one can restrict

$$\Theta_i = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & F_i F_i^{\circledast} \end{bmatrix} \Lambda_i(P_i, Q_i) \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & F_i F_i^{\circledast} \end{bmatrix}^*$$

with the constraint $F_i F_i^{\circledast} Q_i (F_i F_i^{\circledast})^* \geq \mathbf{0}$ in replacement of $Q_i \geq \mathbf{0}$. The restriction corresponds exactly to the situation considered in Section 4.

In case $F_i F_i^{\circledast}$ is square and non-singular, which happens for many cases among which the case of usual non-descriptor systems, then one can take

$$\Theta_i = \begin{bmatrix} (F_i F_i^{\circledast})^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \Lambda_i(P_i, Q_i) \begin{bmatrix} (F_i F_i^{\circledast})^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}^* .$$

Before concluding, note that the $\Lambda(P, Q)$ formula is directly related to the well-known D and DG-scalings [8, 14].

- "D-scaling": A complex norm-bounded scalar uncertainty $|\delta| \leq 1$ is a $\{0, 0, 0\}$-structured, $\{-1, 0, 1\}$-dissipative uncertainty and one gets

$$\Lambda(P, Q) = \begin{bmatrix} -Q & \mathbf{0} \\ \mathbf{0} & Q \end{bmatrix}, \; Q \geq \mathbf{0} .$$

- "DG-scaling": A real norm-bounded scalar uncertainty $|\delta| \leq 1$ is a $\{0, j, 0\}$-structured, $\{-1, 0, 1\}$-dissipative uncertainty and one gets

$$\Lambda(P, Q) = \begin{bmatrix} -Q & jP \\ -jP & Q \end{bmatrix}, \; P = P^* \; , \; Q \geq \mathbf{0} .$$

  As the separator is known to be real in case of real-valued systems, $P = jH$ with H real, skew-symmetric.

6 Conclusion

Motivated by conservatism reduction issues, a general modeling of uncertain descriptor systems is proposed for which new quadratic separation results are derived. The results extend those of [21] in two ways. First, a more general uncertain modeling framework is exposed in which there is no more need to distinguish scalar uncertainties from full-block uncertainties, nor real uncertainties from complex uncertainties. Second, the LMI formulas have the advantage to involve no equality constraints and mostly strict inequalities. This can have good impact for numerical issues and will be illustrated on examples in a journal version of this paper.

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