Robust analysis and synthesis of linear polytopic discrete-time periodic systems via LMIs

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***Abstract*— A particular class of uncertain linear discrete-time periodic systems is considered. The problem of robust stabilization of real polytopic linear discrete-time periodic systems via a periodic state-feedback law is tackled here. Using additional slack variables and the periodic Lyapunov lemma, an extended sufficient condition of robust stabilization is proposed. Based on periodic parameter-dependent Lyapunov functions, this last condition is shown to be always less conservative than the more classic one based on the quadratic stability framework. This is illustrated on numerical examples from the literature.**

I. INTRODUCTION

One century after the pioneering works on periodic differential equations with periodic coefficients [18], [10], [13], the eighties have witnessed a renewed interest in the modelling, analysis and control of periodic systems (see [8] and the references therein). This interest is mainly due to the variety and the originality of the possible applications of a control theory dedicated to this class of systems. One can first recall the classic examples of control of vibrations in helicopters [3] as well as the attitude control of satellites equipped with magnetorquers [17], [27]

Another interesting and original application of linear periodic systems control theory concerns autonomous orbit control. During the last years, the problem of autonomous orbit control for spacecraft has been largely addressed for various applications ranging from geostationary station keeping to formation flying earth orbiters. For circular orbits, the synthesis problems (compute an adequate control law) are generally tackle via the use of Hill's equations leading to a complete linear time-invariant formulation. On the contrary, for elliptical orbits, the discrete-time approximation of the linearized equations of relative motion of a spacecraft in the orbit plane yields discrete-time linear periodic model. In [21], a time-invariant reformulation [5] is utilized to design a discrete-time optimal periodic controller. Those results are completed in [22] by considering pertubations resulting from atmospheric drag.

When dealing with such intrinsically periodic systems, it is natural to wonder if the well-established analysis and synthesis framework for linear-time invariant systems may be extended to this peculiar class of time-varying models. Indeed, time-invariant reformulations for discrete-time systems [5] paved the way for the development of a broad variety of tools (periodic Lyapunov and Riccati equations) directly extrapolated from the LTI set-up. Structural properties, stability analysis as well as the most popular synthesis techniques (pole placement, Linear Quadratic optimal control) were therefore extended to discrete-time and continuous-time periodic systems [6], [2], [7], [27], [27]. Surprisingly, few results exist that extend the robust control framework to periodic systems. [12], [24], [14].

In this paper, a particular class of uncertain linear discrete-time periodic systems similar to the one presented in [24], is considered. The problem of robust stabilization of real polytopic linear discrete-time periodic systems via a periodic state-feedback law is tackled here. As the usual lifted representations of the periodic systems naturally lead to hard numerical computations due to the sparse and highly structured matrices involved [15], a more straightforward approach based on the periodic Lyapunov lemma [1] is developed here. Using additional slack variables already introduced in the robust LTI context [11], [19], [20], an extended sufficient condition of robust stabilization is proposed. Based on periodic parameter-dependent Lyapunov functions, this last condition is shown to be always less conservative than the more classic one based on the quadratic stability framework and proposed in [24]. Two numerical examples illustrate the relevance of this new condition.

Notations: Notation is standard. The transpose of a matrix A is denoted A'. For symmetric matrices, $\succ$ ($\succeq$) denotes the Löwner partial order, i.e. $A \succ (\succeq) B$ iff $A - B$ is positive (semi) definite. $\mathbf{1}$ stands for the identity matrix and $\mathbf{0}$ for the zero matrix with the appropriate dimensions. $\mathbb{S}_n$ denotes the set of symmetric matrices of $\mathbb{R}^{n\times n}$ and $\mathbb{S}_n^+$, ($\mathbb{S}_n^{+*}$), the cone of positive semi-definite, (definite) matrices in $\mathbb{S}_n$. $\mathbb{R}^+$, ($\mathbb{R}^{+*}$) is the set of positive (strictly positive) real numbers. $\mathbb{C}$ is the set of complex numbers.The symetric part of a square matrix A is denoted $\mathcal{S}\{A\}$, i.e. $\mathcal{S}\{A\} = A + A'$. $\mathrm{co}\{A_1, \cdots, A_N\}$ is the convex hull of the collection of N elements $A_1, \cdots, A_N$.

II. PROBLEM STATEMENT

A. Linear polytopic discrete-time T-periodic systems

Let us consider the linear polytopic discrete-time T-periodic systems defined by the following state-space realization:

$$x_{k+1} = A_k(\lambda_k)x_k + B_k(\lambda_k)u_k = M_k(\lambda_k)\begin{bmatrix} x_k \\ u_k \end{bmatrix} \quad (1)$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^m$ is the control vector and λ_k is a parametric uncertainty.

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For each k, the parameter-dependent system matrix $M_k(\lambda_k)$ belongs to the convex polytope $\mathcal{M}_k$

$$M_k(\lambda_k) = \begin{bmatrix} A_k(\lambda_k) & B_k(\lambda_k) \end{bmatrix} \in \mathcal{M}_k \tag{2}$$

defined by,

$$\begin{aligned} \mathcal{M}_k &= \text{co}\left\{ M_k^{[1]}(\lambda_k), \cdots, M_k^{[N_k]}(\lambda_k) \right\} \\ &= \left\{ M_k(\lambda_k) = \sum_{i=1}^{N_k} \lambda_k^{[i]} M_k^{[i]} \ : \ \lambda_k^{[i]} \in \Lambda_k \right\} \end{aligned} \tag{3}$$

where

$$\Lambda_k = \left\{ \lambda_k \in \mathbb{R}^{N_k} \ : \ \lambda_k^{[i]} \geq 0 \ , \ \sum_{i=1}^{N_k} \lambda_k^{[i]} = 1 \right\} \tag{4}$$

The sequence of polytopes $\{\mathcal{M}_k\}$ is assumed to be polytopic T-periodic i.e. $\mathcal{M}_{k+T} = \mathcal{M}_k$ for all $k \geq 0$. The resulting system is said to be T-periodic. Note that the number of vertices defining the polytope associated to the uncertain model may be different at each sample-time k.

B. A Periodic Lyapunov framework for robust stabilization

We are interested in existence conditions of a robust stabilizing memoryless T-periodic state-feedback for (1) of the form:

$$u_k = K_k x_k \tag{5}$$

where $K_{k+T} = K_k$ for all $k \geq 0$. Applying the control law (5) to (1), we get the closed-loop polytopic T-periodic system:

$$x_{k+1} = (A_k(\lambda_k) + B_k(\lambda_k)K_k)x_k = A_k^{cl}(\lambda_k)x_k \tag{6}$$

where the closed-loop matrix $A_k^{cl}(\lambda_k)$ belongs to the T-periodic sequence of polytopes $\{\mathcal{M}_k^{cl}\}$ defined by:

$$\mathcal{M}_k^{cl} = \text{co}\left\{ A_k^{[1]} + B_k^{[1]}K_k, \cdots, A_k^{[N_k]} + B_k^{[N_k]}K_k \right\} \tag{7}$$

The robust stability of the uncertain polytopic T-periodic closed-loop system (6) can be proved using the transition matrix (monodromy matrix) defined as:

$$\Psi_{k_0+T,k_0} = A_{k_0+T}^{cl}(\lambda_{k_0+T})A_{k_0+T-1}^{cl}(\lambda_{k_0+T-1}) \cdots A_{k_0}^{cl}(\lambda_{k_0}) \tag{8}$$

If for any $k_0 \geq 0$ and any collection $\{\lambda_k \in \Lambda_k\}_{k \in \{k_0, \cdots, k_0+T\}}$ the eigenvalues of Ψ_{k_0+T,k_0} (referred to as the characteristic multipliers of the system) belong to the open unit disc of $\mathbb{C}$ then the uncertain T-periodic system is robustly stable. In addition, if a state-feedback (5) exists such that (6) is robustly stable then the system (1) is said to be robustly stabilizable via a T-periodic state-feedback.

A useful way to test robust stability of (6) is to use the extended Lyapunov framework proposed in [1].

Definition 1: :

The linear uncertain T-periodic discrete-time system (6) is robustly stable if and only if there exist T parameter-dependent Lyapunov matrices $X_k(\lambda_k) \in \mathbb{S}_n^{+*}$ solutions of the parameter-dependent Lyapunov inequalities $\forall\ k = 1, \cdots, T$ and $\forall\ \lambda_k \in \Lambda_k$:

$$\begin{aligned} & A_k^{cl}(\lambda_k)X_k(\lambda_k)A_k^{cl'}(\lambda_k) - X_{k+1}(\lambda_{k+1}) \prec \mathbf{0} \\ & X_{T+1}(\lambda_{T+1}) = X_1(\lambda_1) \end{aligned} \tag{9}$$

This result is a direct extension of the periodic Lyapunov lemma in the uncertain case.

If the periodic state-feedback (5) is given, the previous formulation allows to tackle the robust stability analysis problem. Even in the simpler case where (6) is not periodic, this problem is known to be hard to solve exactly. Except for special simple cases, one has to resort to using some relaxations of the previous problem. Mimicking a well-known relaxation for robust stabilization of LTI uncertain systems, the authors of [24] define the quadratic stability and its synthesis counter-part the quadratic stabilizability concept for uncertain periodic linear discrete-time systems.

Definition 2: [24]

The system (1) is quadratically stabilizable via a T-periodic state-feedback law (5) if and only if there exist a periodic state feedback law (5) and T Lyapunov matrices $X_k \in \mathbb{S}_n^{+*}$ solutions of the Lyapunov inequalities $\forall\ k = 1, \cdots, T$ and $\forall\ \lambda_k \in \Lambda_k$:

$$\begin{aligned} & A_k^{cl}(\lambda_k)X_kA_k^{cl'}(\lambda_k) - X_{k+1} \prec \mathbf{0} \\ & X_{T+1} = X_1 \end{aligned} \tag{10}$$

The system (6) is then said to be quadratically stable.

Imposing to the periodic Lyapunov matrices to be independent of the uncertain parameters λ_k clearly shows that the quadratic stability of (6) is a sufficient condition for its robust stability. In [24], necessary and sufficient condition of quadratic stabilizability are given for (1) in terms of solutions to some Linear Matrix Inequalities.

Theorem 1: [24]

The linear discrete-time polytopic T-periodic model (1) is quadratically stabilizable via a T-periodic state-feedback $u_k = K_k x_k$ iff there exist T matrices $X_k \in \mathbb{S}_n^{+*}$ and T matrices $Y_k \in \mathbb{R}^{m \times n}$ solutions of the following LMI constraints:

$$\begin{aligned} & \begin{bmatrix} -X_k & (A_k^{[i]}X_k + B_k^{[i]}Y_k)' \\ A_k^{[i]}X_k + B_k^{[i]}Y_k & -X_{k+1} \end{bmatrix} \prec \mathbf{0} \\ & X_{T+1} = X_1 \\ & \text{for } k = 1, \cdots, T \text{ and } i = 1, \cdots, N_k \end{aligned} \tag{11}$$

The quadratic T-periodic stabilizing gain is reconstructed as:

$$\begin{aligned} & K_k = Y_k X_k^{-1} \quad \text{for } k \in \{1, \cdots, T\} \\ & K_{k+T} = K_k \end{aligned} \tag{12}$$

The closed-loop periodic system (6) obtained by the application of the T-periodic state-feedback gain computed by (12) is said to be quadratically stable.

Based on some recent results [11], [19], [20], an extended robust stability test involving additional slack variables is introduced and it is shown that it always leads to less conservative results than the quadratic one.

III. MAIN RESULTS

A. Extended robust analysis test

The conservatism of the quadratic stability test mainly comes from the use of "single" periodic Lyapunov functions for the whole set of uncertainty. One way to overcome this drawback consists in looking for parameter-dependent periodic Lyapunov functions. Of course, even when the uncertain model is known to have a polytopic structure, it is impossible to know *a priori* the dependency of the Lyapunov function with respect to the unknown parameter. Nevertheless, it seems natural to seek polytopic periodic Lyapunov functions of the form:

$$X_k = \sum_{i=1}^{N_k} \lambda_k^{[i]} X_k^{[i]} \tag{13}$$

An extended robust stability condition based on parameter-dependent periodic Lyapunov functions of the form (13) may now be derived

Theorem 2: If there exist $N_1 + \cdots + N_T$ matrices $X_k^{[i]} \in \mathbb{S}_n^{+*}$ and T matrices $H_k \in \mathbb{R}^{2n\times 2n}$ solutions of the following LMI constraints:

$$\begin{aligned} &\begin{bmatrix} -X_{k+1}^{[i]} & \mathbf{0} \\ \mathbf{0} & X_k^{[i]} \end{bmatrix} + \mathcal{S}\left\{ \begin{bmatrix} A_k^{cl[i]} \\ -\mathbf{1} \end{bmatrix} H_k \right\} \prec \mathbf{0} \\ &X_{T+1}^{[i]} = X_1^{[i]} \\ &\text{for } k = 1, \cdots, T \text{ and } i = 1, \cdots, N_k \end{aligned} \tag{14}$$

then (6) is robustly stable.

Proof:

If there exist $N_1 + \cdots + N_T$ matrices $X_k^{[i]} \in \mathbb{S}_n^{+*}$ and T matrices $H_k \in \mathbb{R}^{2n\times 2n}$ solutions of (14) then the computation the k convex combinations over the N_k vertices allows to write

$$\begin{aligned} &\begin{bmatrix} -X_{k+1}(\lambda_k) & \mathbf{0} \\ \mathbf{0} & X_k(\lambda_k) \end{bmatrix} + \mathcal{S}\left\{ \begin{bmatrix} A_k^{cl}(\lambda_k) \\ -\mathbf{1} \end{bmatrix} H_k \right\} \prec \mathbf{0} \\ &X_{T+1}(\lambda_{T+1}) = X_1(\lambda_1) \end{aligned} \tag{15}$$

for $k = 1, \cdots, T$ and $\lambda_k \in \Lambda_k$. Pre multiplying inequality k by $\begin{bmatrix} \mathbf{1} & A_k^{cl}(\lambda_k) \end{bmatrix}$ and post multiplying it by its transpose leads to

$$-X_{k+1}(\lambda_k) + A_k^{cl}(\lambda_k) X_k(\lambda_k) (A_k^{cl}(\lambda_k))' \prec \mathbf{0} \tag{16}$$

for $k = 1, \cdots, T$ and $\lambda_k \in \Lambda_k$. These last inequalities are equivalent to the T-periodic Lyapunov lemma [6] for all $\lambda_k \in \Lambda_k$. ■

The main reason to introduce slack variables consists in the decoupling between the periodic Lyapunov functions $X_k^{[i]}$ and the matrix $A_k^{cl[i]}$ thus allowing the use of parameter-dependent periodic Lyapunov functions of the form (13) where $X_k^{[i]}$ is defined at each vertex of the polytope $\mathcal{M}_k^{cl}$. The matrices H_k play the role of Lagrangian relaxation variables as the single Lyapunov function does in the quadratic stability set-up [9]. It is then possible to show that this new robust stability test is always less conservative than the one elaborated in the quadratic context.

Lemma 1: If (6) is quadratically stable then there exist $N_1 + \cdots + N_T$ matrices $X_k^{[i]} \in \mathbb{S}_n^{+*}$ and T matrices $H_k \in \mathbb{R}^{2n\times 2n}$ solutions of (14)

Proof:

If (6) is quadratically stable then the set of matrices

$$\begin{aligned} X_k^{[i]} &= X_k & k = 1, \cdots, T \quad i = 1, \cdots, N_k \\ H_k &= \begin{bmatrix} \mathbf{0} & X_k \end{bmatrix} & k = 1, \cdots, T \end{aligned} \tag{17}$$

are solutions of the set of LMIs (14). ■

This lemma shows that quadratic stability results may always be recovered by the test defined by (14) when giving a special structure to the Lagrangian multipliers H_k.

According to the dimensions, the matrix H_k may be partitioned as $H_k = \begin{bmatrix} F_k & G_k \end{bmatrix}$ where $F_k,\ G_k \in \mathbb{R}^{n\times n}$. In the LMIs (14), the bloc $(2, 2)$ impose to the matrices G_k to be invertible. It is therefore possible to get the following reformulation of the previous robust stability test.

Lemma 2: If there exist $N_1 + \cdots + N_T$ matrices $X_k^{[i]} \in \mathbb{S}_n^{+*}$ and T matrices $G_k \in \mathbb{R}^{n\times n}$, $A_k^0 \in \mathbb{R}^{n\times n}$ solutions of the following LMI constraints:

$$\begin{aligned} &\begin{bmatrix} -X_{k+1}^{[i]} & \mathbf{0} \\ \mathbf{0} & X_k^{[i]} \end{bmatrix} + \mathcal{S}\left\{ \begin{bmatrix} A_k^{cl[i]} \\ -\mathbf{1} \end{bmatrix} (-G_k) \begin{bmatrix} A_k^0 & -\mathbf{1} \end{bmatrix} \right\} \prec \mathbf{0} \\ &X_{T+1}^{[i]} = X_1^{[i]} \\ &\text{for } k = 1, \cdots, T \text{ and } i = 1, \cdots, N_k \end{aligned} \tag{18}$$

then (6) is robustly stable.

This reformulation is interesting because of the interpretation of the extra variables A_k^0. Applying elimination lemma [23] to (18) shows that these matrices must verify

$$\begin{bmatrix} \mathbf{1} & A_k^0 \end{bmatrix} \begin{bmatrix} -X_{k+1}^{[i]} & \mathbf{0} \\ \mathbf{0} & X_k^{[i]} \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ A_k^{0\prime} \end{bmatrix} \prec \mathbf{0} \tag{19}$$

for $X_k^{[i]} \in \mathbb{S}_n^{+*}$, $k \geq 0$, $i = 1, \cdots, N_k$ such that $X_{T+1}^{[i]} = X_1^{[i]}$. This means that the sequence of matrices $\{A_k^0\}$ solutions of (18) must be p-stable. In particular, a possible choice is $\{A_k^0\} = \{\mathbf{0}\}$ for all $k \leq 0$.

Lemma 3: If there exist $N_1 + \cdots + N_T$ matrices $X_k^{[i]} \in \mathbb{S}_n^{+*}$ and T matrices $G_k \in \mathbb{R}^{n\times n}$ solutions of the following

LMI constraints:

$$\begin{aligned}
&\begin{bmatrix} -X_{k+1}^{[i]} & \mathbf{0} \\ \mathbf{0} & X_k^{[i]} \end{bmatrix} + \mathcal{S}\left\{ \begin{bmatrix} A_k^{cl[i]} \\ -\mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{0} & G_k \end{bmatrix} \right\} \prec \mathbf{0} \\
&X_{T+1}^{[i]} = X_1^{[i]} \\
&\text{for } k = 1, \cdots, T \text{ and } i = 1, \cdots, N_k
\end{aligned} \tag{20}$$

then (6) is robustly stable.

Proof:

The proof of theorem 1 and noticing that the robust stability test (20) is nothing but the robust stability test (14) with the following structure on the matrix H_k

$$H_k = \begin{bmatrix} \mathbf{0} & G_k \end{bmatrix} \tag{21}$$

leads to the result ■

The proof of lemma 1 shows in addition that imposing the above structure on H_k surely introduces additional conservatism for the test (20) with respect to (14) but does not prevent (20) to be always better than the quadratic stability one. The hierarchy of robust stability test is therefore the following.

$$\text{Quadratic stability } \Rightarrow (20) \Rightarrow (14) \tag{22}$$

In the next section, this structure is defined to take advantage of the extended robust stability test for robust periodic state-feedback stabilization.

B. Application to the robust stabilizability problem

Considering that $A_k^{cl} = A_k + B_k K_k$ and that $H_k \in \mathbb{R}^{2n \times 2n}$ may be partitioned as

$$H_k = \begin{bmatrix} F_k & G_k \end{bmatrix} \quad F_k \in \mathbb{R}^{n \times n} \quad G_k \in \mathbb{R}^{n \times n} \tag{23}$$

it is clear that the usual linearizing change of variables used in the quadratic stabilizability context cannot be directly extended to the robust stability conditions (14). Similarly to the idea proposed in the reference [19], for the special case of discrete-time systems, it is possible to use the result of lemma 3 imposing a particular structure to the multiplier H_k while keeping some of the advantages related to the role of the slack variables H_k. A sufficient condition of robust stabilization via T-periodic state-feedback based on the robust stability test (20) may be derived.

Theorem 3: If there exist $N_1 + \cdots + N_T$ matrices $X_k^{[i]} \in \mathbb{S}_n^{+*}$ and T matrices $Y_k \in \mathbb{R}^{m \times n}$, $G_k \in \mathbb{R}^{n \times n}$ solutions of the following LMI constraints:

$$\begin{aligned}
&\begin{bmatrix} -X_{k+1}^{[i]} & (A_k^{[i]} G_k + B_k^{[i]} Y_k)' \\ A_k^{[i]} G_k + B_k^{[i]} Y_k & X_k^{[i]} - G_k - G_k' \end{bmatrix} \prec \mathbf{0} \\
&X_{T+1}^{[i]} = X_1^{[i]} \\
&\text{for } k = 1, \cdots, T \text{ and } i = 1, \cdots, N_k
\end{aligned} \tag{24}$$

then the linear discrete-time polytopic T-periodic model (1) is robustly stabilizable via a T-periodic state-feedback $u_k = K_k x_k$ defined as:

$$\begin{aligned}
&K_k = Y_k G_k^{-1} \quad k \in \{1, \cdots, T\} \\
&K_{k+T} = K_k
\end{aligned} \tag{25}$$

Proof:

(24) and (25) may be rewritten as:

$$\begin{aligned}
&\begin{bmatrix} -X_{k+1}^{[i]} & \mathbf{0} \\ \mathbf{0} & X_k^{[i]} \end{bmatrix} \\
&+\mathcal{S}\left\{ \begin{bmatrix} A_k^{[i]} + B_k^{[i]} K_k \\ -\mathbf{1} \end{bmatrix} G_k \begin{bmatrix} \mathbf{0} & \mathbf{1} \end{bmatrix} \right\} \prec \mathbf{0}
\end{aligned} \tag{26}$$

for $k = 1, \cdots, T$ and $i = 1, \cdots, N_k$. This last inequality is nothing but (20) for the closed-loop ■

An identical hierarchy to the one defined for robust analysis conditions may be proposed for robust stabilizability conditions.

Lemma 4: If the linear discrete-time polytopic T-periodic model (1) is quadratically stabilizable via a T-periodic state-feedback $u_k = K_k x_k$ then there always exist $N_1 + \cdots + N_T$ matrices $X_k^{[i]} \in \mathbb{S}_n^{+*}$ and T matrices $Y_k \in \mathbb{R}^{m \times n}$, $G_k \in \mathbb{R}^{n \times n}$ solutions of the system of LMIs (24).

Proof:

It is clear that if there exist $X_k \in \mathbb{S}^{+*}$ and $Y_k \in \mathbb{R}^{m \times n}$ solutions of the set of LMIs (11) then the set of matrices

$$\begin{array}{lcll}
G_k & = & X_k & k = 1, \cdots, T \\
Y_k & = & Y_k & k = 1, \cdots, T \\
X_k^{[i]} & = & X_k & k = 1, \cdots, T \quad i = 1, \cdots, N_k
\end{array} \tag{27}$$

are solutions of the set of LMIs (24). ■

The sufficient conditions robust stabilizability (24) are defined for $\{A_k^0\} = \{\mathbf{0}\}$ for all $k \geq 0$. For different choices of p-stable matrices $\{A_k^0\}$, different sufficient conditions of robust stabilizability via periodic state-feedback may be given.

Theorem 4: For a sequence of p-stable matrices $\{A_k^0\}$, if there exist $N_1 + \cdots + N_T$ matrices $X_k^{[i]} \in \mathbb{S}_n^{+*}$ and T matrices $Y_k \in \mathbb{R}^{m \times n}$, $G_k \in \mathbb{R}^{n \times n}$ solutions of the following LMI constraints:

$$\begin{aligned}
&\begin{bmatrix} -X_{k+1}^{[i]} & \mathbf{0} \\ \mathbf{0} & X_k^{[i]} \end{bmatrix} + \\
&\begin{bmatrix} -A_k^{[i]} G_k - B_k^{[i]} Y_k \\ G_k \end{bmatrix} \begin{bmatrix} A_k^0 & -\mathbf{1} \end{bmatrix} \prec \mathbf{0} \\
&X_{T+1}^{[i]} = X_1^{[i]} \\
&\text{for } k = 1, \cdots, T \text{ and } i = 1, \cdots, N_k
\end{aligned} \tag{28}$$

then the linear discrete-time polytopic T-periodic model (1) is robustly stabilizable via a T-periodic state-feedback $u_k = K_k x_k$ defined as:

$$\begin{aligned}
&K_k = Y_k G_k^{-1} \quad k \in \{1, \cdots, T\} \\
&K_{k+T} = K_k
\end{aligned} \tag{29}$$

Proof:

The proof is obvious from the previous analysis results and is omitted for space reasons ■

The main drawback of those last conditions comes from the difficulty to choose adequately the sequence of p-stable $\{A_k^0\}$. One simple and systematic way to do so is to apply any nominal periodic stabilization procedure at one vertex of the polytope for each sample-time k. Note also that the results given by the conditions (24) may always be recovered by the choice $\{A_k^0\} = \{\mathbf{0}\}$. However, as will be shown by the numerical example 2, for a very simple choice of the sequence $\{A_k^0\}$, better results may also be obtained.

IV. NUMERICAL EXAMPLES

Some numerical examples borrowed from the literature are now presented to illustrate the relevance of this new approach.

A. Example 1

Let us consider the first example presented in [24] defined by its state-space realizations:

$$A_1(\alpha) = \begin{bmatrix} -3-\alpha & 2 \\ -3 & 3 \end{bmatrix} \quad A_2(\alpha) = \begin{bmatrix} -1 & -\alpha 2 \\ 0.5 & 0 \end{bmatrix}$$
$$A_3(\alpha) = \begin{bmatrix} 1-\alpha & 2 \\ 2.5 & 3 \end{bmatrix} \quad |\alpha| \leq \overline{\alpha}$$
$$B_1(\beta) = \begin{bmatrix} 1 \\ \beta \end{bmatrix} \quad B_2(\beta) = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$$
$$B_3(\beta) = \begin{bmatrix} 0.5(\beta+1) \\ 1 \end{bmatrix} \quad 0 \leq \beta \leq 1 \tag{30}$$

where (α, β) are two uncertain parameters defining a polytope of system matrices with 4 vertices. As in the first example, quadratic 3-periodic state-feedback stabilizing gain obtained from theorem 1 and 3-periodic state-feedback stabilizing gain computed thanks to theorem 3 are compared.

$$\overline{\alpha}_Q^{max} = 0.152 \qquad \overline{\alpha}_G^{max} = 0.49$$
$$K_Q^1 = \begin{bmatrix} 3.217 & -2.938 \end{bmatrix} \quad K_G^1 = \begin{bmatrix} 3.02 & -2.26 \end{bmatrix}$$
$$K_Q^2 = \begin{bmatrix} 1.114 & -1.467 \end{bmatrix} \quad K_G^2 = \begin{bmatrix} 1.167 & -2.037 \end{bmatrix}$$
$$K_Q^3 = \begin{bmatrix} -1.602 & -2.292 \end{bmatrix} \quad K_G^3 = \begin{bmatrix} -2.212 & -2.313 \end{bmatrix} \tag{31}$$

In this case, the new bound $\overline{\alpha}$ is improved by 222 %. For the same example, some preliminary experiments have been performed with the conditions (28). It appears with the simple choice:

$$A_1^0 = A_2^0 = A_3^0 = -0.05 * \mathbf{1}_2 \tag{32}$$

we get an intermediate result between the quadratic stabilization procedure and the one based on (24):

$$\overline{\alpha}_{A_0}^{max} = 0.45 \tag{33}$$

More interesting, when choosing

$$A_1^0 = A_2^0 = A_3^0 = 0.35 * \mathbf{1}_2 \tag{34}$$

we get a better result (improvement of 14.3 %) than for the procedure based on (24):

$$\overline{\alpha}_{A_0}^{max} = 0.56 \tag{35}$$

Considering now the same example but with a modified period $T = 2$. This means that only $M_1 = \begin{bmatrix} A_1 & B_1 \end{bmatrix}$ and $M_2 = \begin{bmatrix} A_2 & B_2 \end{bmatrix}$ are taken into account. The previous results are confirmed since we get:

$$\overline{\alpha}_Q^{max} = 0.218 \qquad \overline{\alpha}_G^{max} = 0.8$$
$$K_Q^1 = \begin{bmatrix} 0.419 & -0.209 \end{bmatrix} \quad K_G^1 = \begin{bmatrix} 2.791 & -1.953 \end{bmatrix}$$
$$K_Q^2 = \begin{bmatrix} 1.123 & -1.425 \end{bmatrix} \quad K_G^2 = \begin{bmatrix} 1.275 & -1.689 \end{bmatrix} \tag{36}$$

leading to an improvement of 266 %. The next figure shows the map of the closed-loop poles of the monodromy matrix $\Psi_A^T = (A_2(\alpha) + B_2(\beta)K_2)(A_1(\alpha) + B_1(\beta)K_1)$ where the uncertain parameters (α, β) vary in a grid of 9000 points.

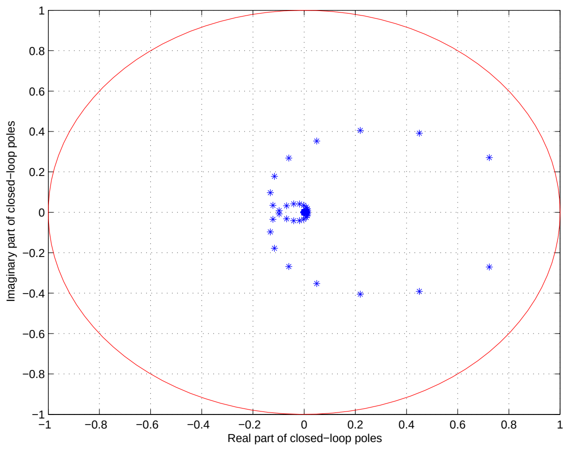

FIG. 1 –. *Samples of the closed-loop poles of the monodromy matrix*

B. Example 2

Consider now the uncertain discrete-time system defined by its state-space realization.

$$A(\alpha) = \begin{bmatrix} 0 & 1 \\ 1 & -1+\alpha \end{bmatrix} \quad B(\beta) = \begin{bmatrix} 1-\beta \\ \beta \end{bmatrix} \tag{37}$$

The uncertain parameters (α, β) are defined as $|\alpha| \leq \overline{\alpha}$ and $0 \leq \beta \leq 1$. This defines a polytope of matrices (A, B) with 4 vertices. This example is borrowed from [24] where a quadratic static output-feedback periodic gain is computed. Note that in the previous reference, no uncertainty was considered in the input matrix B. After a standard analysis of the pair $(A(\alpha), B(\beta))$, it is possible to show that the uncertain system has non stabilizable mode given by:

$$\lambda = \frac{-1-\sqrt{5}}{2}$$

for

$$\alpha = 0 \quad \beta = \frac{3-\sqrt{5}}{2} \tag{38}$$

Consequently, the uncertain system (37) is not robustly stabilizable nor quadratically stabilizable.

As in [24], a 2-periodic transformation V_k is defined by:

$$V_0 = \begin{bmatrix} -0.95 & -1 \\ 1.95 & 1 \end{bmatrix} \quad V_1 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \tag{39}$$

We look for the maximum value of $\overline{\alpha}$ for which a robust 2-periodic state-feedback gain may be found using the quadratic stabilizability-based condition ($\overline{\alpha}_Q^{max}$) and the extended condition based on parameter-dependent 2-periodic Lyapunov functions ($\overline{\alpha}_G^{max}$). Here parser YALMIP [16] and LMI solver SDPT3 [25] are used to get the solution of the semi-definite problem associated with those two LMI conditions and allow to find:

$$\overline{\alpha}_Q^{max} = 0.6 \quad \overline{\alpha}_G^{max} = 0.827 \tag{40}$$

This result clearly shows that the new extended robust synthesis condition improves the one based on quadratic stabilizability by almost 38 %.

The corresponding robust stabilizing 2-periodic gains obtained with the quadratic condition and with the extended one are respectively given by:

$$K_Q^1 = \begin{bmatrix} -0.543 & -0.043 \end{bmatrix} \quad K_Q^2 = \begin{bmatrix} -0.458 & 0.043 \end{bmatrix}$$
$$K_G^1 = \begin{bmatrix} -0.822 & -0.139 \end{bmatrix} \quad K_G^2 = \begin{bmatrix} -0.538 & -0.232 \end{bmatrix} \tag{41}$$

V. CONCLUSIONS AND FUTURE WORKS

A new robust stability test based on periodic parameter-dependent Lyapunov functions is developed in this paper and is applied to the robust analysis of linear polytopic periodic discrete-time systems. The use of additional slack variables allows to decouple the state matrices from the Lyapunov matrices leading to less conservative results than previous tests based on quadratic stability. An extended sufficient condition for robust stabilization via periodic state-feedback is then derived. It mainly relies on a convex relaxation of the previous robust stability test. Numerical examples have shown the possible gain obtained by this extended robust design. Some points concerning robust stabilizability conditions (28) and particularly the adequate choice of the sequence $\{A_k^0\}$ have to be carefully investigated.

These results are preliminary and could be extended to consider similar problems where the whole state is not available for control. Static as well as dynamic output-feedback are currently under investigation for linear periodic discrete-time systems.

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