Robust Multi-Objective Control toolbox*

Dimitri Peaucelle, Denis Arzelier
LAAS-CNRS,
7, av. du colonel Roche - 31077 Toulouse, FRANCE
peaucelle@laas.fr

February 10, 2006

Abstract

A new "Robust Multi-Objective Control" toolbox is presented. It is freely distributed (`www.laas.fr/OLOCEP/romuloc`) and works in Matlab environment along with the parser YALMIP. The current version tackles stability, pole location, H_∞, H_2 and impulse-to-peak analysis problems. The uncertain models are all in state-space and range from interval matrices to structured rational representations. For each considered problem, the user has the opportunity to choose between parameter dependent or independent Lyapunov functions. This gives more or less conservative conditions associated with different numerical burden. The paper exposes the underlying theoretical results and illustrates the use of the tool on some examples.

1 Introduction

For the past ten years the formalism of Linear Matrix Inequalities has lead to many results for robust analysis and synthesis. One major characteristic of these results is the variety of performances and uncertain models that may be tackled. Although every individual result may be easily coded we felt necessary to gather results in a unique toolbox. This will allow not only a good diffusion of the up-to-date theoretical results and their wide testing on applicative examples, but may also be the starting point of a common platform for all researchers of the field to compare their contributions. These are the reasons why the tool is freely distributed and we are eager to have many contributions.

At this stage of development the focus is made on functionalities for manipulating uncertain LTI models and on implementation of some well established LMI results for robust analysis.

Modeling and manipulation of LTI models is much alike the well known Matlab© Control toolbox. The major difference is that modeling distinguishes explicitly control inputs from disturbance inputs as well as measurement outputs from outputs used

*This work was supported by a "projet LAAS" funding - Projet OLOCEP.

to define performances. Only state-space models (continuous or discrete-time) are considered. Two wide classes of uncertain models are available. One is the class of affine polytopic uncertainties for which many results are provided in the literature, see [6, 8, 16, 22] for example. The second more general class is based on a Linear-Fractional Transformation as in μ-theory. The uncertainty description is more general than the norm-bounded description. At this stage only constant uncertainties (parametric) are considered. Some useful functions are provided to operate models. Note that the objective of the toolbox is not to optimize these model manipulations. For modeling purpose we recommend the use of the LFR toolbox [11] that we plan to interface with RoMulOC.

Robust analysis tackles stability, pole location, H_∞, H_2 and impulse-to-peak analysis problems. For both classes of uncertain models, two categories of LMI-based analysis results are implemented. One relies on "quadratic stability" (i.e. a unique Lyapunov matrix is used to prove robustness for the whole set of uncertainties). The second is based on parameter-dependent Lyapunov functions (PDLF). For the class of polytopic uncertain systems the PDLF results are those of [16]. For the class of LFT uncertain systems, results are derived using the quadratic separation framework [9] (see also the full block S-procedure framework in [20]). The PDLF results for this class of systems are based on results of [10].

The paper is organized as follows. First the different uncertain models are presented. Then section III summarizes the theoretical tools implemented for LMI-based robust analysis. Next, a long section is devoted to the description of the functionalities of the RoMulOC toolbox on a numerical example. Conclusions are finally driven along with a description of future developments.

2 Modeling of uncertain systems

2.1 LFT systems

To cope simultaneously with controller design requirements, input/output performances and uncertainties, the LFT modeling adopted in RoMulOC explicitly distinguishes three pairs of outputs/inputs:

- Measurements outputs/Control inputs. Denoted y/u, these are the actual vector signals accessible for control purpose.
- Control outputs/Perturbations. Denoted z/w, these vector signals are used to define input/output performance specifications such as H_∞ norm, H_2 norm or impulse-to-peak.
- Exogenous outputs/inputs. Denoted z_Δ/w_Δ, these are fictitious vector signals used for LFT modeling of uncertainties. The uncertainties denoted Δ enter the system as $w_\Delta = \Delta z_\Delta$. The uncertain matrix Δ gathers all parametric uncertainties and is defined as unknown inside a set $\mathbf{\Delta}$. The types of sets and structures handled in RoMulOC are detailed in the following.

Both continuous-time and discrete-time LTI models can be defined. For the discrete-time case $\delta[x(t)] = x(t+T)$ where T is the sampling time. For the continuous-time case $\delta[x(t)] = \dot{x}(t)$. The matrices of the state-space model are as follows:

$$\begin{array}{rl} \delta[x(t)] = & Ax(t) + B_\Delta w_\Delta(t) + B_w w(t) + B_u u(t) \\ z_\Delta(t) = & C_\Delta x(t) + D_{\Delta\Delta} w_\Delta(t) + D_{\Delta w} w(t) + D_{\Delta u} u(t) \\ z(t) = & C_z x(t) + D_{z\Delta} w_\Delta(t) + D_{zw} w(t) + D_{zu} u(t) \\ y(t) = & C_y x(t) + D_{y\Delta} w_\Delta(t) + D_{yv} v(t) + D_{yu} u(t) \end{array} \tag{1}$$

where $x \in \mathsf{R}^{\mathsf{n}}$, $w_\Delta \in \mathsf{R}^{\mathsf{m}_\Delta}$, $w \in \mathsf{R}^{\mathsf{m}_\mathsf{w}}$, $u \in \mathsf{R}^{\mathsf{m}_\mathsf{u}}$, $z_\Delta \in \mathsf{R}^{\mathsf{p}_\Delta}$, $z \in \mathsf{R}^{\mathsf{p}_\mathsf{z}}$ and $y \in \mathsf{R}^{\mathsf{p}_\mathsf{y}}$.

The uncertain system $\Sigma(\Delta)$ when the exogenous feedback connection $w_\Delta = \Delta z_\Delta$ is applied (the system is assumed to be well-posed, *i.e.* $1 - D_{\Delta\Delta}\Delta$ is non singular for all admissible uncertainties Δ) is denoted:

$$\begin{array}{rl} \delta[x(t)] = & A(\Delta)x(t) + B_w(\Delta)w(t) + B_u(\Delta)u(t) \\ z(t) = & C_z(\Delta)x(t) + D_{zw}(\Delta)w(t) + D_{zu}(\Delta)u(t) \\ y(t) = & C_y(\Delta)x(t) + D_{yw}(\Delta)w(t) + D_{yu}(\Delta)u(t) \end{array} \tag{2}$$

2.2 Polytopic systems

Polytopic uncertain systems are a particular instance of uncertain systems that can be modeled as LFT uncertain systems but for which $D_{\Delta\Delta} = 0$ and Δ belongs to the convex hull of a finite number of vertices. Mathematical properties of such systems have lead to specific analysis results that make it necessary to develop specific functionalities and a specific modeling.

For polytopic modeling there is no need for z_Δ/w_Δ signals (which amounts to $m_\Delta = p_\Delta = 0$). The polytopic modeling is based on definition of all N vertices which are LTI systems of same output/input and state dimensions. Each vertex ($\Sigma^{[i]}$) is defined as:

$$\begin{array}{rl} \delta[x(t)] = & A^{[i]}x(t) + B_w^{[i]}w(t) + B_u^{[i]}u(t) \\ z(t) = & C_z^{[i]}x(t) + D_{zw}^{[i]}w(t) + D_{zu}^{[i]}u(t) \\ y(t) = & C_y^{[i]}x(t) + D_{yw}^{[i]}w(t) + D_{yu}^{[i]}u(t) \end{array}$$

and the uncertain polytopic system with vertices $\Sigma^{[1]}, \Sigma^{[2]}, \dots \Sigma^{[N]}$ is described by the set :

$$\left\{ \Sigma(\Delta) \quad : \quad \Delta = \begin{pmatrix} \zeta_1 \geq 0 \\ \vdots \\ \zeta_N \geq 0 \end{pmatrix} \quad , \quad \sum_{i=1}^{N} \zeta_i = 1 \right\}$$

where matrices of the systems $\Sigma(\Delta)$ are such that:

$$A(\Delta) = \sum_{i=1}^{N} \zeta_i A^{[i]} \quad , \quad B_w(\Delta) = \sum_{i=1}^{N} \zeta_i B_w^{[i]} \quad \dots$$

Note that polytopes have two important sub-cases that are explicitly incorporated in RoMulOC, namely: interval and parallelotopic uncertain systems. Interval systems are defined given two extremal LTI systems $\bar{\Sigma}$ and $\underline{\Sigma}$ with identical output/input and state dimensions. The matrices of the resulting uncertain system $\Sigma(\Delta)$ are such that all components of the system matrices are independent and bounded by those of $\bar{\Sigma}$ and $\underline{\Sigma}$:

$$\underline{A}_{ij} \le A_{ij}(\Delta) \le \bar{A}_{ij} \quad \ldots$$

This modeling is trivially a sub-case of polytopic models with 2^{N_I} vertices where N_I is the number of non equal elements in $\bar{\Sigma}$ and $\underline{\Sigma}$.

Parallelotopic uncertain systems are defined given a central model $\Sigma^{|0|}$ along with a finite number N_P of "axes" $\Sigma^{|i|}$ all of which are LTI systems with identical output/input and state dimensions. The matrices of the resulting uncertain system $\Sigma(\Delta)$ are such that

$$A(\Delta) = A^{|0|} + \sum_{i=1}^{N_P} \delta_i A^{|i|} \quad \ldots$$

where $|\delta_i| \le 1$ are the distances to the center $\Sigma^{|0|}$ along the axis $\Sigma^{|i|}$. The parallelotopic modeling is trivially a sub-case of polytopic models with 2^{N_P} vertices. A function `u2poly` allows to convert interval and parallelotopic models into polytopic models.

2.3 Manipulating models in RoMulOC

In the current version of the toolbox, many functionalities are available for manipulating the upper defined uncertain models. For example: the function `feedback` allows to operate a given control feedback; for LFT models, the function `certain` builds the LTI model for a given fixed value of Δ; the function `shape` allows to add a filter on either the w disturbance input or on the z performance control output... All model definition and manipulation functionalities are precisely described in the User's Guide document [14].

The present manuscript is devoted to description of the LMI analysis tools of the first version of RoMulOC. Therefore the signals y/u are removed in the following.

3 LMIs for robust performance analysis

3.1 Performances, an LMI-Lyapunov approach

Before entering into the details of the methodologies used to handle robustness issues, this first sub-section summarizes the types of performances considered in the toolbox and gives the LMI formulations on which are based the results.

Let the following LTI system

$$\begin{array}{rl} \delta[x(t)] = & Ax(t) + B_w w(t) \\ z(t) = & C_z x(t) + D_{zw} w(t) \end{array} \tag{3}$$

where $x \in \mathsf{R}^{\mathsf{n}}$, $w \in \mathsf{R}^{\mathsf{m_w}}$ and $z \in \mathsf{R}^{\mathsf{p_z}}$. Using the Lyapunov theory framework, the five robust performance criteria considered are treated as follows. All formulas are essentially reformulations of results from [4].

• Stability. Define the following matrices for continuous-time and the discrete-time systems respectively:

$$R_{sc} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \ , \ R_{sd} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{4}$$

Classically for the Lyapunov theory, stability is proved by the existence of a positive-definite solution $P > 0$ to the following LMI problem

$$\begin{bmatrix} 1 & A^T \end{bmatrix} R_s \otimes P \begin{bmatrix} 1 \\ A \end{bmatrix} < 0 \ . \tag{5}$$

• Pole location. Any regions such as discs and half-planes are considered. All of these may be summarized by the following expression

$$\begin{pmatrix} 1 & s^* \end{pmatrix} R \begin{pmatrix} 1 \\ s \end{pmatrix} \leq 0 \tag{6}$$

where $R = R^* \in \mathsf{C}^{2\times 2}$ is a Hermitian matrix (real symmetric in case of symmetric regions with respect to the real axis) with non negative bottom right element $R_{22} \geq 0$. Such pole location constraints allow to test the system settling time (half plane $s+s^* \leq -\alpha$), natural frequency ($ss^* \leq \omega_n^2$) or damping (half-plane below the inclined line that makes an angle of ψ with the imaginary axis). As in [16], pole location in such regions is proved by the existence of a solution $P > 0$ to

$$\begin{bmatrix} 1 & A^T \end{bmatrix} R \otimes P \begin{bmatrix} 1 \\ A \end{bmatrix} < 0 \ . \tag{7}$$

• H_∞ performance. Using similar notations as for stability, the H_∞ norm of (2) is below a level γ if and only if there is a solution $P > 0$ to

$$\begin{array}{l} \begin{bmatrix} 1 & A^T \\ 0 & B_w^T \end{bmatrix} R_s \otimes P \begin{bmatrix} 1 & 0 \\ A & B_w \end{bmatrix} \\ + \begin{bmatrix} C_z^T C_z & C_z^T D_{zw} \\ D_{zw}^T C_z & D_{zw}^T D_{zw} - \gamma^2 1 \end{bmatrix} < 0 \ . \end{array} \tag{8}$$

This performance criterion guarantees that the L_2-gain of the channel $w \to z$ is smaller than γ; that asymptotic stability is preserved whatever bounded ($\Omega^*(t)\Omega(t) \leq \gamma^2 1$) perturbations $w(t) = \Omega(t) z(t)$; and that the maximal singular value of the $w \to z$ transfer matrix is smaller than γ for all frequencies.

• H_2 performance. The H_2 norm of (2) is below a level γ if and only if there is a solution $P > 0$ to

$$\begin{array}{c} \begin{bmatrix} 1 & A^T \end{bmatrix} R_s \otimes P \begin{bmatrix} 1 \\ A \end{bmatrix} + C_z^T C_z < 0 \\ \mathrm{Trace}(B_w^T P B_w + D_{zw}^T D_{zw}) \leq \gamma^2 \end{array} \tag{9}$$

plus the constraint $D_{zw} = 0$ in case of continuous-time systems. This performance criterion guarantees that for a white noise input w, the output z variance is bounded by

γ; that for a scalar impulse inputs w the energy of the output z is below γ; and that for scalar outputs the deviation amplitude is bounded by γ given unit energy disturbances w.

• Impulse-to-peak performance. The amplitude of the output $\|z(t)\|$ is bounded by γ for any impulse input w if there is a solution $P > 0$ to

$$\begin{array}{cc} \left[\begin{array}{cc} 1 & A^T \end{array}\right] R_s \otimes P \left[\begin{array}{c} 1 \\ A \end{array}\right] < 0 & B_w^T P B_w \leq \gamma^2 1 \\ C_z^T C_z \leq P & D_{zw}^T D_{zw} \leq \gamma^2 1 \ . \end{array} \tag{10}$$

All these LMI formulations that are suitable for nominal certain models, become intractable for uncertain systems (2) because it is needed to find an infinite number of matrices $P(\Delta)$ for an infinite number of constraints. For each uncertain modeling, RoMulOC gives conservative results for that problem using either the "quadratic stability" framework in which $P(\Delta) = P$ is chosen unique over the uncertainty set, or novel frameworks based on particular instances of parameter-dependent quadratic Lyapunov functions $V(x, \Delta) = x^T P(\Delta)x$. All these methods are described in the following two sub-sections.

3.2 Robustness: Quadratic separation for LFT models

All robust analysis results for LFT models are based on a Lyapunov - Quadratic Separation framework. Parameter-dependent quadratic Lyapunov functions are introduced to deal with system dynamics while quadratic separation play the same role for the uncertainties. Both the Lyapunov matrix and the quadratic separator matrix play the role of additional variables introduced to turn the analysis problem into a tractable one (finite number of convex LMI constraints).

At this stage of implementation, two types of parameter-dependent Lyapunov functions are considered: unique matrix over all uncertainties $P(\Delta) = P$ and quadratic-LFT such that:

$$P(\Delta) = \left[\begin{array}{c} 1 \\ \Delta_c \end{array}\right]^T P \left[\begin{array}{c} 1 \\ \Delta_c \end{array}\right] \ : \ \Delta_c = \Delta(1 - D_{\Delta\Delta}\Delta)^{-1} C_\Delta \ . \tag{11}$$

This last type of parameter-dependent Lyapunov matrix was employed in [10, 15] with similar techniques. In RoMulOC the implementation corresponds to results of [10] that have the advantage compared to [15] to lead to less decision variables and smaller LMIs. The methodology can be summarized to the rewriting of conditions (5-10) in the following form:

$$\left[\begin{array}{c} 1 \\ \Delta_{\mathcal{C}} \end{array}\right]^T \mathcal{P}(P) \left[\begin{array}{c} 1 \\ \Delta_{\mathcal{C}} \end{array}\right] < 0 \ : \ \Delta_{\mathcal{C}} = \Delta_k(1 - \mathcal{D}\Delta_k)^{-1}\mathcal{C} \ . \tag{12}$$

where $\Delta_k = 1_k \otimes \Delta$ and k goes from 1 to 3 depending on the considered constraint and the type of Lyapunov matrix used. For each constraint the corresponding values of k, $\mathcal{P}$, $\mathcal{C}$ and $\mathcal{D}$ can be found in [13]. Quadratic separation results (also known as full-block

S-procedure [20]) are applied to (12) to yield the following LMI formulation:

$$\mathcal{P}(P) \leq \begin{bmatrix} \mathcal{C} & \mathcal{D} \\ 0 & 1 \end{bmatrix}^T \Theta \begin{bmatrix} \mathcal{C} & \mathcal{D} \\ 0 & 1 \end{bmatrix} \tag{13}$$

where Θ should satisfy for all uncertainties $\Delta \in \mathbf{\Delta}$:

$$\begin{bmatrix} 1 \\ 1_k \otimes \Delta \end{bmatrix}^T \Theta \begin{bmatrix} 1 \\ 1_k \otimes \Delta \end{bmatrix} \leq 0 \tag{14}$$

Choosing a separator Θ is not lossless in general and only sufficient conditions are implemented. Three main types of uncertainties are implemented at this time.

• Scalar real repeated $\{x, y, z\}$-dissipative uncertainties $\delta 1_r$ that satisfy a quadratic constraint:

$$x + 2y\delta + z\delta^2 \leq 0 \ \ .$$

This representation allows to consider any uncertain interval, possibly infinite, such as the classical norm-bounded case $|\delta| \leq \bar{\delta}$ and the positive real case $\delta \geq 0$. Based on DG-scaling techniques, the following separators are implemented for these uncertainties:

$$\begin{bmatrix} 1 \\ \delta 1_r \end{bmatrix}^T \begin{bmatrix} xD & yD + G \\ yD - G & zD \end{bmatrix} \begin{bmatrix} 1 \\ \delta 1_r \end{bmatrix} \leq 0 \tag{15}$$

where $D = D^T > 0$ and $G^T = -G$.

• Full-block repeated $\{X, Y, Z\}$-dissipative uncertainties $1_r \otimes \Delta$ that satisfy a quadratic constraint:

$$X + Y\Delta + \Delta^* Y^* + \Delta^* Z \Delta \leq 0 \ \ .$$

Again this modeling includes the classical norm-bounded uncertainties $\Delta^*\Delta \leq \bar{\delta}^2 1$ and passive operators $\Delta + \Delta^* \geq 0$. Based on D-scaling, the following separators are implemented:

$$\begin{bmatrix} 1 \\ 1_r \otimes \Delta \end{bmatrix}^* \begin{bmatrix} D \otimes X & D \otimes Y \\ D \otimes Y^* & D \otimes Z \end{bmatrix} \begin{bmatrix} 1 \\ 1_r \otimes \Delta \end{bmatrix} \leq 0 \tag{16}$$

where $D = D^* > 0$.

• Block diagonal polytopic uncertainties $\Delta_{pol} = \text{diag}(\Delta_1, \ldots, \Delta_q)$ that satisfy

$$\Delta_{pol} = \sum_{i=1}^{N} \zeta_i \text{diag}(\Delta_1^{[i]}, \ldots, \Delta_q^{[i]}) = \sum_{i=1}^{N} \zeta_i \Delta_{pol}^{[i]}$$

with $\zeta_i \geq 0$ and $\sum \zeta_i = 1$. Again, as for polytopic systems, interval and parallelotopic uncertainties may be defined and RoMulOC provides specific functions for this purpose. Vertex separators are implemented for this type of uncertainties such that for all vertices

$$\begin{bmatrix} 1 \\ 1_r \otimes \Delta_{pol}^{[i]} \end{bmatrix}^* \begin{bmatrix} \Theta_1 & \Theta_2 \\ \Theta_2^* & \Theta_3 \end{bmatrix} \begin{bmatrix} 1 \\ 1_r \otimes \Delta_{pol}^{[i]} \end{bmatrix} \leq 0$$

and the blocks of the diagonal of Θ_3 corresponding to the $\Delta_1, \ldots, \Delta_q$ subdivision of Δ_{pol} are constrained to be positive semi-definite $\Theta_{3ii} \geq 0$.

• Any block diagonal structure of such 3 types of uncertainties is allowed in RoMulOC as well. The separators are in this case combinations of the upper exposed separators.

Note that scalar $\{x, y, z\}$-dissipative uncertainties bounded in a finite segment may as well be defined as polytopic uncertainties. Doing so is proved to lead to less conservative results at the expense of increasing significantly the numerical burden [10]. This is illustrated in the last section devoted to numerical examples.

3.3 Robustness: Validation on vertices for polytopic models

The case of polytopic uncertain models defined in section 2.2 is now considered. First, assume quadratic stability for which a unique Lyapunov matrix $P(\Delta) = P$ is searched. In this situation, all the LMIs (5-10) prove to be convex with respect to the parameters ζ. Therefore it is necessary and sufficient to test the LMIs for the vertices.

This well known result was extended first in [16] to polytopic type Lyapunov matrices

$$P(\Delta) = \sum_{i=1}^{N} \zeta_i P^{[i]} \quad . \tag{17}$$

The result relies on the inverse use of the elimination lemma [21] that creates "slack variables" [6] leading to "dilated LMIs" [7] based on which are derived sufficient conditions that are proved to be less conservative than quadratic stability. For the case of robust pole location, the result is as follows: find a unique F and N matrices $P^{[i]}$ such that for all N vertices the next LMIs hold

$$R \otimes P^{[i]} + F \begin{bmatrix} A^{[i]} & -1 \end{bmatrix} + \begin{bmatrix} {A^{[i]}}^T \\ -1 \end{bmatrix} F^T < 0 \quad .$$

For all other performances the results are much alike and are therefore not recalled here.

4 Examples

The following example is taken from [8] and [22]. It is used here to illustrate robust H_2 performance analysis. Let us follow step by step the example without getting into details of the functions used.

```
>> sys=ssmodel('Geromel et al. 2002')
name: Geromel et al. 2002
empty SSMODEL
```

ROMULOC works along with a new definition of systems in Matlab environment. Here we have declared a new system, with an identifying tag for nice display purpose. The variable contains no data on the system. Now simply append the model

```
>> sys.A=[0.9 0.1; 0.01, 0.9];
>> sys.Bw=[ 1 0 0 ; 0 1 0];
>> sys.Cy=[1 0;1 1];
>> sys.Dyw=[0 0 1.414;0 0 0];
>> sys.T=1
name: Geromel et al. 2002
               n=2    mw=3
 n=2    dx  =   A*x +  Bw*w
py=2     y  =  Cy*x + Dyw*w
discrete time  ( dx : advance operator ) period T=1
```

At this stage we have declared a discrete-time linear system of order 2, with 3 disturbance inputs (represented by the notation w) and 2 measured outputs (represented by the notation y). In [8] and [22], the system includes two scalar uncertainties on the A matrix such that

$$A(\Delta) = A + \delta_1 \begin{bmatrix} 0 & 0.06 \\ 0 & 0 \end{bmatrix} + \delta_2 \begin{bmatrix} 0 & 0 \\ 0.05 & 0 \end{bmatrix}$$

where the uncertainties are norm-bounded $|\delta_i| \leq 1$. This corresponds to a parallelotopic uncertain model.Variations characterized by the δ's are operated around the 'central' system `sys` along axes that are systems with the same dimensions. Therefore define `axes`: an empty system with identical dimensions as `sys`

```
>> axes=ssmodel(0,sys)
               n=2    mw=3
 n=2    dx  =
py=2     y  =
discrete time  ( dx : advance operator ) period T=1
```

This variable is used as an array of systems with identical dimensions. Each element of the array contains one axis.

```
>> axes(1).A=[0 0.06;0 0];
>> axes(2).A=[0 0;0.05 0]
Array of 2 systems
               n=2    mw=3
 n=2    dx  =   A*x
py=2     y  =
discrete time  ( dx : advance operator ) period T=1
```

Based on these data, the uncertain parallelotopic system is declared as

```
>> usysP=uparal(sys,axes)
Uncertain model : parallelotope 2 param
-------- WITH --------
               n=2    mw=3
 n=2    dx  =   A*x +  Bw*w
py=2     y  =  Cy*x + Dyw*w
discrete time  ( dx : advance operator ) period T=1
```

In [22] an output filter is designed for signal reconstruction. The performance of the filter with respect to the disturbance input is measured using the H_2 norm of the transfer between w and z where z is the performance output signal of the filter defined as follows:

```
>> filter=ssmodel('filter Xie et al.2004');
>> filter.A=[0.0705 0.0263;1.2779 0.5492];
>> filter.Bu=[0.9114 0;-0.9972 0];
>> filter.Cz=[-1.2885 -0.2382];
>> filter.Dzu=[0 1];
>> filter.T=1
name: filter Xie et al.2004
               n=2    mu=2
 n=2    dx  =   A*x +  Bu*u
pz=1     z  =  Cz*x + Dzu*u
discrete time  ( dx : advance operator ) period T=1
```

The filtered uncertain system is obtained by plugging in the output of the system `sys` into the control input of `filter`. This corresponds to the usual multiplication of systems

```
>> filteredP=filter*usysP;
>> filteredP.name = 'Filtered system'
Uncertain model : parallelotope 2 param
-------- WITH --------
name: Filtered system
               n=4    mw=3
 n=4    dx  =   A*x +  Bw*w
pz=1     z  =  Cz*x
discrete time  ( dx : advance operator ) period T=1
```

The robust analysis methods implemented in ROMULOC deal with polytopic models which are a more general representation than parallelotopes. Polytopes describe uncertain sets as the convex combination of vertices. It is tedious to transform parallelotopes into polytopes but ROMULOC does it for you:

```
>> filteredP=u2poly(filteredP)
Uncertain model : polytope 4 vertices
-------- WITH --------
name: Filtered system
               n=4    mw=3
 n=4    dx  =   A*x +  Bw*w
pz=1     z  =  Cz*x
discrete time  ( dx : advance operator ) period T=1
```

Now that the uncertain system is declared, the ROMULOC functions allow to test two methods for computing its robust H_2 cost. This is done in three steps. First declare the type of problem you want to solve and the methodology to be used.

```
>> quiz=ctrpb
 CHOOSE A CONTROL PROBLEM
```

```
(a) Analysis
(b) State-Feedback design
(c) Full-Order Dynamic Output-Feedback
choice > a
 CHOOSE A TYPE OF LYAPUNOV FUNCTION
(a) Unique over all uncertainties
(b) Quadratic w.r.t. del*(I-Ddd*del)^-1*Cd
(c) Polytopic
choice > a

control problem: ANALYSIS
Lyapunov function: UNIQUE (quadratic stability)
No specified performance
```

Second, append the `quiz` variable to indicate that H_2 performance of `filteredP` is required

```
>> quiz=h2(quiz,filteredP)
control problem: ANALYSIS
Lyapunov function: UNIQUE (quadratic stability)
Specified performances / systems:
# minimize H2 / Filtered system
```

At this stage an LMI problem has been declared and stored in YALMIP format in the variable `quiz`. The analysis problem can be solved as a third and last step, using whatever `sdpsettings` of YALMIP.

```
>> solvesdp(quiz ,sdpsettings('solver','sdpt3'))
...
SDPT3 display
...
H2 norm = 4.70402 guaranteed for all uncertainties
ans =
    4.7040
```

Note that the optimization has been done here using SDPT3 and the results are satisfying. This may not be the case for all examples and for all solvers. A significant easy-to-use feature of YALMIP (and therefore of ROMULOC) is to propose different solvers. For example, for the same problem perform the optimization using SeDuMi gives:

```
>> solvesdp( quiz , sdpsettings('solver','sedumi') )
...
SeDuMi display
...
Feasibility is not determined
Worst constraint residual is -4.14989e-11 < 0
4.70402 may be a guaranteed H2 norm
ans =
   NaN
```

The two results are close except that SeDuMi converged to a non strictly feasible point. The distance to feasibility is evaluated as equal to `4.14989e-11`.

At this stage we have obtained that the H_2 norm of the uncertain system is less than 4.70402 whatever the uncertainties. Now test the second method that involves parameter-dependent Lyapunov functions.

```
>> quiz=ctrpb;
 CHOOSE A CONTROL PROBLEM
(a) Analysis
(b) State-Feedback design
(c) Full-Order Dynamic Output-Feedback
choice > a
 CHOOSE A TYPE OF LYAPUNOV FUNCTION
(a) Unique over all uncertainties
(b) Quadratic w.r.t. del*(I-Ddd*del)^-1*Cd
(c) Polytopic
choice > c

>> quiz=h2(quiz,filteredP)

control problem: ANALYSIS
Lyapunov function: POLYTOPIC
Specified performances / systems:
# minimize H2 / Filtered system

>> solvesdp(quiz,sdpsettings('solver','sdpt3'))
...
H2 norm = 3.92362 guaranteed for all uncertainties
```

For analysis purpose, parameter-dependent Lyapunov function based methods are all proved to be less conservative than quadratic stability. The conservatism is reduced at the expense of larger LMI problems to solve. Here the number of constraints grows from 11 to 73 and the number of decision variables from 20 to 36. Meanwhile, the computation time grows (on the SunBlade 150 computer I used) from 1.8 to 2.4. In terms of conservatism, the computed upper bound on the worst-case H_2 norm is reduced from 4.70402 to 3.92362.

To criticize these results, compute the H_2 norm of some specific systems within the uncertain set. First, test the central 'nominal' system (calculated as the center of the polytope):

```
>> norm(filteredP,2)
Nominal
ans =
    2.9636
```

Second test any of the four vertices of the polytope, for example the fourth:

```
>> norm(filteredP(4),2)
System is stable
ans =
    3.9236
```

The H_2 norm of this vertex is equal to the upper bound found by the polytopic parameter-dependent Lyapunov based method. Therefore, we can conclude that it is the worst-case H_2 norm for the filtered system. This is also assessed by gridding in [22].

Now, concerning LFT modeling based methods. As mentioned previously, any polytopic model may as well be converted to an LFT uncertain model. For the current example it is done by writing

$$A(\Delta) = A + \underbrace{\begin{bmatrix} 0.06 & 0 \\ 0 & 0.05 \end{bmatrix}}_{B_\Delta} \underbrace{\begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}}_{\Delta} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{C_\Delta}$$

```
>> sys.Bd=[0.06 0; 0, 0.05];
>> sys.Cd=[ 0 1 ; 1 0]
name: Geromel et al. 2002
              n=2    md=2     mw=3
 n=2    dx   =   A*x +  Bd*wd +  Bw*w
pd=2    zd   =  Cd*x
py=2     y   =  Cy*x          + Dyw*w
discrete time  ( dx : advance operator ) period T=1
```

and uncertainties can be defined either as intervals (and hence polytopic)

```
>> u11 = uinter( -1 , 1 , 'delta1' );
>> u12 = uinter( -1 , 1 , 'delta2' );
>> u1 = diag( u11 , u12 )
diagonal structured uncertainty
size: 2x2 | nb blocks: 2 | independent blocks: 2
 wd = diag( #1 #2 ) * zd
index   size    constraint              name
#1      1x1     interval 1 param        delta1
#2      1x1     interval 1 param        delta2
>> u1 = u2poly( u1 )
diagonal structured uncertainty
size: 2x2 | nb blocks: 2 | independent blocks: 2
 wd = diag( #3 #4 ) * zd
index   size    constraint              name
#3      1x1     polytope 2 vertices     delta1
#4      1x1     polytope 2 vertices     delta2
```

or as norm-bounded (hence dissipative)

```
>> u21 = unb( 1 , 1 , 1 , 'delta1' );
>> u22 = unb( 1 , 1 , 1 , 'delta2' );
>> u2 = diag( u21 , u22 );
```

Based on these two definitions of the uncertainties, two uncertain models are derived

```
>> usys1 = ussmodel( sys, u1 );
>> usys2 = ussmodel( sys, u2 );
```

Table 1: Results for the LFT modeling case

	$P(\Delta) = P$	$P(\Delta)$ as (11)
Polytopic Δ	4.70402	4.33218
Norm bounded Δ	4.70561	4.35878

and two new filtered uncertain models are obtained

```
>> filtered1 = filter*usys1;
>> filtered2 = filter*usys2
Uncertain model : LFT
-------- WITH --------
               n=4    md=2     mw=3
 n=4    dx  =   A*x +  Bd*wd +  Bw*w
pd=2    zd  =  Cd*x
pz=1     z  =  Cz*x
discrete time  ( dx : advance operator ) period T=1
-------- AND  --------
diagonal structured uncertainty
size: 2x2 | nb blocks: 2 | independent blocks: 2
 wd = diag( #5 #6 ) * zd
index   size    constraint             name
#5      1x1     norm-bounded by 1      delta1
#6      1x1     norm-bounded by 1      delta2
```

For each of these, define the analysis problems as previously. Parameter-dependent Lyapunov methods are selected by choosing `b` rather than `c` since LFT modeling is considered. Table 1 gives the results returned by RoMulOC for the four problems. For this particular example, LFT modeling based methods perform worse than polytopic based methods. For the LFT modeling based methods, DG-scalings are more conservative than vertex separator. Improvement due to parameter-dependent Lyapunov functions is confirmed as well.

5 Conclusions and future works

The presented first version of the RoMulOC toolbox is a basis for future developments. At this step it illustrates the potentialities of a tool including contemporary research results combined with a versatile tool for solving LMIs. More developments are to come and we hope them to be available by the time this paper is presented at the CACSD conference in Munich.

The planned future new functionalities are as follows. For modeling: implement an interface with the LFR toolbox [11]; include complex and time-varying uncertainties; extend modeling to descriptor systems. For analysis: include results of [12, 5, 17] for polytopic systems and results of [3, 18] for LFT uncertain models. For controller design: implement state-feedback and full-order output-feedback multi-objective synthesis based on the Lyapunov Shaping Paradigm [19] and extensions [1, 2].

References

[1] D. Arzelier and D. Peaucelle. Robust multi-objective state-feedback control for real parametric uncertainties via parameter-dependent Lyapunov functions. In *IFAC Symposium on Robust Control Design*, volume 1, pages 213–218, Prague, June 2000.

[2] D. Arzelier and D. Peaucelle. Multiobjective H_2/H_∞/impulse-to-peak synthesis: Application to the control of an aerospace launcher. In *IFAC Symposium on Automatic Control in Aerospace*, St Petersburg, Russia, June 2004.

[3] P.-A. Bliman. A convex approach to robust stability for linear systems with uncertain scalar parameters. *SIAM J. Control and Optimization*, 42:2016–2042, 2004.

[4] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. SIAM Studies in Applied Mathematics, Philadelphia, 1994.

[5] G. Chesi, A. Garulli, A. Tesi, and A. Vicino. Robust stability of polytopic systems via polynomially parameter-dependent Lyapunov functions. In *IEEE Conference on Decision and Control*, Maui, Hawaii, USA, December 2003.

[6] M.C. de Oliveira, J. Bernussou, and J.C. Geromel. A new discrete-time stability condition. *Systems & Control Letters*, 37(4):261–265, July 1999.

[7] Y. Ebihara and T. Hagiwara. Robust controller synthesis with parameter-dependent Lyapunov variables: A dilated LMI approach. In *IEEE Conference on Decision and Control*, pages 4179–4184, Las Vegas, Nevada, December 2002.

[8] J.C. Geromel, M.C. de Oliveira, and J. Bernussou. Robust filtering of discrete-time linear systems with parameter dependent Lyapunov functions. *SIAM J. Control and Optimization*, 41:700–711, 2002.

[9] T. Iwasaki and S. Hara. Well-posedness of feedback systems: Insights into exact robustness analysis and approximate computations. *IEEE Trans. on Automat. Control*, 43(5):619–630, 1998.

[10] T. Iwasaki and G. Shibata. LPV system analysis via quadratic separator for uncertain implicit systems. *IEEE Trans. on Automat. Control*, 46(8):1195–1207, August 2001.

[11] J.F. Magni. User manual of the linear fractional representation toolbox version 2.0. Technical report, ONERA - Systems Control and Flight Dynamics Depertment, October 2005. URL: `www.cert.fr/dcsd/idco/perso/Biannic%` verb+/toolboxes.html+.

[12] R.C.L.F. Oliveira and P.L.D Peres. Stability of polytopes of matrices via affine parameter-dependent Lyapunov functions: Asymptotically exact LMI conditions. *Linear Algebra and Its Applications*, 405:209–228, 2005.

[13] D. Peaucelle. *Formulation Générique de Problèmes en Analyse et Commande Robuste par les Fonctions de Lyapunov Dépendant des Paramètres*. PhD thesis, Université Toulouse III - Paul Sabatier, France, July 2000.

[14] D. Peaucelle. RoMulOC: a YALMIP-Matlab based robust multi-objective control toolbox. Technical Report 05377, LAAS-CNRS, Toulouse, June 2005.

[15] D. Peaucelle and D. Arzelier. Robust performance analysis with LMI-based methods for real parametric uncertainty via parameter-dependent Lyapunov functions. *IEEE Trans. on Automat. Control*, 46(4):624–630, April 2001.

[16] D. Peaucelle, D. Arzelier, O. Bachelier, and J. Bernussou. A new robust D-stability condition for real convex polytopic uncertainty. *Systems & Control Letters*, 40(1):21–30, May 2000.

[17] D. Peaucelle, Y. Ebihara, D. Arzelier, and H. Tomomichi. General polynomial parameter-dependent lyapunov functions for polytopic uncertain systems. In *International Symposium on Mathematical Theory of Networks and Systems*, Kyoto, July 2006. Submitted.

[18] D. Peaucelle, D. Henrion, and D. Arzelier. Quadratic separation for feedback connection of an uncertain matrix and an implicit linear transformation. In *16th IFAC World Congress*, Prague, Czech Republic, July 2005.

[19] C. Scherer, P. Gahinet, and M. Chilali. Multiobjective output-feedback control via LMI optimization. *IEEE Trans. on Automat. Control*, 42(7):896–911, July 1997.

[20] C.W. Scherer. *Advances in Linear Matrix Inequality Methods in Control*, chapter 10 Robust Mixed Control and Linear Parameter-Varying Control with Full Block Scallings, pages 187–207. Advances in Design and Control. SIAM, 2000. edited by L. El Ghaoui and S.-I. Niculescu.

[21] R.E. Skelton, T. Iwazaki, and K. Grigoriadis. *A unified Approach to Linear Control Design*. Taylor and Francis series in Systems and Control, 1998.

[22] L. Xie, L. Lilei, D. Zhang, and H. Zhang. Improved H_2 and H_∞ filtering for uncertain discrete-time systems. *Automatica*, 40:873–880, 2004.