ROBUST $\mathcal{H}_2$ PERFORMANCE ANALYSIS OF UNCERTAIN LTI SYSTEMS VIA POLYNOMIALLY PARAMETER-DEPENDENT LYAPUNOV FUNCTIONS

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Abstract: In this paper, we address the robust $\mathcal{H}_2$ performance analysis problems of linear time-invariant polytopic-type uncertain systems. To obtain numerically verifiable and less conservative analysis conditions, we employ polynomially parameter-dependent Lyapunov functions (PPDLFs) to assess the robust $\mathcal{H}_2$ performance and give a sufficient condition for the existence of such PPDLFs in terms of finitely many linear matrix inequalities (LMIs). The resulting LMI conditions turn out to be a natural extension of those known as extended or dilated LMIs in the literature, where the PDLFs employed were restricted to those depending affinely on the uncertain parameters. It is shown that, by increasing the degree of PPDLFs, we can obtain more accurate (no more conservative) analysis results at the expense of increased computational burden. Exactness of the proposed analysis conditions as well as their computational complexity will be examined through numerical experiments.

Keywords: Robust $\mathcal{H}_2$ performance analysis, polynomially parameter-dependent Lyapunov functions, linear matrix inequalities (LMIs).

1. INTRODUCTION

Robust performance analysis of linear time-invariant (LTI) polytopic-type uncertain systems has been an active research field in the community of control theory (Barmish, 1994). Since the linear matrix inequalities (LMIs) appeared as a powerful tool for such analysis problems, intensive research effort has been made in this direction. Roughly speaking, the robust performance analysis problems of LTI systems affected by real parametric uncertainties can readily be reduced into feasibility/optimization problems of parametrized LMIs (PLMIs). In contrast with a single, or parameter-independent LMI, however, the computation of PLMIs is hardly tractable in general. In these days, intensive studies to overcome this difficulty arouse renewed interest in the fields of LMI-based robust performance analysis. The main attention is directed on how to obtain numerically verifiable and easy to compute LMI-based conditions that enable us to achieve accurate (nonconservative) analysis results.

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To achieve accurate analysis results, the idea to employ parameter-dependent Lyapunov functions (PDLFs) was first introduced in (Gahinet *et al.*, 1996; Feron *et al.*, 1996). Since then, a number of LMI conditions based on PDLFs has been proposed for robust stability/performance analysis; see, for example, (Leite and Peres, 2003; Oliveira *et al.*, 1999; Oliveira and Skelton, 2001; Peaucelle *et al.*, 2000) and references cited therein. In these existing studies, however, the PDLFs employed are almost restricted to those affine in the uncertain parameters. This poses essential limitations on the resulting analysis conditions and does not allow us to achieve exact analysis results in general cases. To get around the conservatism arising from affinely PDLFs, more recently, promising LMI-based conditions using polynomially PDLFs (PPDLFs) have been proposed in (Bliman, 2004; Chesi *et al.*, 2005) and subsequently in (Ebihara *et al.*, 2005).

In this paper, we provide a strategy to assess the robust $\mathcal{H}_2$ performance of polytopic-type uncertain LTI systems by means of PPDLFs. More precisely, we give a sufficient condition for the existence of such PPDLFs in terms of finitely many LMIs evaluated on the vertex of the polytope. This study is motivated by our preceding results in (Ebihara *et al.*, 2005), where we showed that employing Lyapunov functions that can be associated with higher-order time-derivative of the state naturally leads to LMI conditions by means of PPDLFs. The results in the present paper have been obtained by analyzing the form of those preceding LMIs and finding out a specific matrix structure that enables us to obtain finitely many LMIs from infinite-dimensional PLMIs. In the proposed LMIs, it turns out that we can obtain more accurate (no more conservative) analysis results by increasing the degree of PPDLFs, at the expense of increased computational burden.

We use the following notations in this paper. For a matrix $A \in \mathbf{R}^{n\times n}$, we define $\mathrm{He}\{A\} := A + A^T$ and $\mathrm{Sq}\{A\} := A^T A$. For a matrix $A \in \mathbf{R}^{m\times n}$ with $\mathrm{rank}(A) = r < n$, $A^\perp \in \mathbf{R}^{n\times(n-r)}$ is a matrix such that $AA^\perp = 0$ and $A^{\perp T}A^\perp > 0$. The symbol $\mathbf{Z}$ denotes the set of nonnegative integers, and $\mathbf{S}_n$ the set of $n \times n$ real symmetric matrices. For given $M \in \mathbf{R}^{n\times n}$ and $r \in \mathbf{Z}$, we define a function $\Gamma(M, r)$ by

$$\Gamma(M, r) := \begin{bmatrix} I_n & M^T & \cdots & (M^r)^T \end{bmatrix}^T \in \mathbf{R}^{(r+1)n\times n}.$$

In particular, we denote $\Gamma(0_n, r)$ by $\Gamma_{n,r}$. We also define a function $\Lambda(M, r)$ by

$$\Lambda(M, r) := \begin{bmatrix} I_r \otimes M & 0_{rn,n} \end{bmatrix} + \begin{bmatrix} 0_{rn,n} & -I_r \otimes I_n \end{bmatrix} \in \mathbf{R}^{rn\times(r+1)n}.$$

Note that $\Gamma(M, r) = \Lambda(M, r)^\perp$ holds for any $M \in \mathbf{R}^{n\times n}$ and $r \in \mathbf{Z}$.

The next lemma is used frequently in the paper.

Lemma 1. (Finsler's Lemma) (Oliveira and Skelton, 2001) Let matrices $Q \in \mathbf{R}^{n\times n}$ and $B \in \mathbf{R}^{m\times n}$ be given such that $\mathrm{rank}(B) < n$. Then, the following conditions are equivalent:

(i) The condition $B^{\perp T} Q B^\perp < 0$ holds.
(ii) There exists $\mu_l \in \mathbf{R}$ such that $Q - \mu B^T B < 0$ holds for all $\mu \geq \mu_l$.
(iii) There exists $F \in \mathbf{R}^{n\times m}$ such that $Q + \mathrm{He}\{FB\} < 0$.

2. PROBLEM STATEMENT

Let us consider the continuous-time uncertain LTI system described by

$$\dot{x} = A(p)x + B(p)w, \quad z = C(p)x \tag{1}$$

where $x \in \mathbf{R}^n$, $w \in \mathbf{R}^m$ and $z \in \mathbf{R}^l$. We assume that the uncertain parameter $p \in \mathbf{R}^q$ is time-invariant and belongs to a polytope $\mathcal{P}$ defined by

$$\mathcal{P} := \left\{ p \in \mathbf{R}^q : \sum_{i=1}^{q} p_i = 1, \quad p_i \geq 0 \right\}. \tag{2}$$

The matrices $A(p) \in \mathbf{R}^{n\times n}$, $B(p) \in \mathbf{R}^{n\times m}$ and $C(p) \in \mathbf{R}^{l\times n}$ are assumed to be linear on p and given by

$$\begin{aligned} A(p) &= \sum_{i=1}^{q} p_i A_i, \quad B(p) = \sum_{i=1}^{q} p_i B_i, \\ C(p) &= \sum_{i=1}^{q} p_i C_i. \end{aligned} \tag{3}$$

Here, A_i, B_i, C_i $(i = 1, \cdots, q)$ are given matrices. We denote the transfer matrix from w to z of (1) by $G(s, p)$. Our goal in this paper is to compute the worst case $\mathcal{H}_2$ norm of the system (1) over $\mathcal{P}$, which is defined by

$$\gamma^\star := \sup_{p\in\mathcal{P}} ||G(s, p)||_2. \tag{4}$$

For the $\gamma^\star$ to be well-defined, in the sequel, we assume $A(p)$ is Hurwitz stable for all $p \in \mathcal{P}$.

From (Boyd *et al.*, 1994), we see that the worst case $\mathcal{H}_2$ norm (4) can be characterized via PLMIs as follows:

$$\gamma^\star = \inf_{P(p)\in\mathbf{S}_n, X(p)\in\mathbf{S}_m} \gamma \quad \text{subject to} \tag{5a}$$

$$P(p)A(p) + A(p)^T P(p) + C(p)^T C(p) < 0, \tag{5b}$$

$$-X(p) + B(p)^T P(p) B(p) < 0, \tag{5c}$$

$$\gamma^2 > \mathrm{trace}(X(p)), \quad \forall p \in \mathcal{P}. \tag{5d}$$

Here, $P(p)$ and $X(p)$ are the Lyapunov matrix variable and slack variable to be determined as functions on $p \in \mathcal{P}$. When dealing with the PLMIs as in (5), the following issues would be outstanding in the current state of the art:

1. There is no systematic way to determine the form of the Lyapunov matrix P as the function of the parameter p. In recent studies (Zhang *et al.*,

2003; Henrion *et al.*, 2004), polynomially parameter dependent Lyapunov matrices (PPDLMs) of specific degrees have proved to be enough to achieve exact analysis in the case of robust stability. However, to the best of the author's knowledge, counterpart results have not been obtained for robust performance analysis cases.

2. Solving PLMI problems exactly is a difficult problem in general, even though recently developed LMI relaxation techniques (Scherer, 2003) are expected to be promising to deal with such problem and significant progress has been achieved in these days.

As a remedy to get around these difficulties, in this paper, we confine ourselves to seeking for the PPDLMs of a specific form, and give a sufficient condition for the existence of such PPDLMs in terms of finitely many LMIs.

3. ROBUST $\mathcal{H}_2$ PERFORMANCE ANALYSIS VIA PPDLMS

3.1 Reduction to Quadratic PLMI Problems

To assess the worst case $\mathcal{H}_2$ norm by means of the PLMI (5), let us consider to employ PPDLMs of the form

$$P_r(p) := \Gamma(M(p), r)^T \Pi_r(p) \Gamma(M(p), r). \tag{6}$$

Here, $M(p) : \mathbf{R}^q \to \mathbf{R}^{n\times n}$ is a *given* affine function on $p \in \mathcal{P}$ whereas $\Pi_r(p) : \mathbf{R}^q \to\in \mathbf{S}_{(r+1)n}$ is an affine function to be determined through optimization. The parameter $r \in \mathbf{Z}$ is used to determine the degree of the PPDLMs on p.

On the other hand, we employ an affine function $X_{\mathrm{a}}(p) : \mathbf{R}^q \to \mathbf{S}_m$ for the matrix variable $X(p)$ included in (5c) and (5d), which will be determined also via optimization.

For given $r \in \mathbf{Z}$, let us define

$$\gamma_r^\star = \inf_{P_r(p), X_{\mathrm{a}}(p)} \gamma \quad \text{subject to} \tag{7a}$$

$$P_r(p)A(p) + A(p)^T P_r(p) + C(p)^T C(p) < 0, \tag{7b}$$

$$-X_{\mathrm{a}}(p) + B(p)^T P_r(p) B(p) < 0, \tag{7c}$$

$$\gamma^2 > \mathrm{trace}(X_{\mathrm{a}}(p)), \quad \forall p \in \mathcal{P}. \tag{7d}$$

Then, it is apparent that $\gamma^\star \le \gamma_r^\star$ for all $r \in \mathbf{Z}$. In the sequel, we concentrate our attention on how to compute the upper bounds $\gamma_r^\star$ for the worst case $\mathcal{H}_2$ norm $\gamma^\star$.

As we can see, the PLMI (7b) obtained by simply substituting $P_r(p)$ into (5b) is not easily numerically verifiable mainly because it involves multiple products on the parameter p. However, by revisiting the results in (Ebihara *et al.*, 2005) and applying Finsler's Lemma in a peculiar way, we can simplify the structure of (7b) so that it includes only quadratic terms on the uncertain parameter p. To see this, let us first note that the PLMI (7b) can be rewritten, equivalently, in the following form:

$$\begin{bmatrix} \Gamma(M(p), r) \\ \Gamma(M(p), r)A(p) \end{bmatrix}^T \begin{bmatrix} \Gamma_{n,r} C(p)^T C(p) \Gamma_{n,r}^T & \Pi_r(p) \\ \Pi_r(p) & 0_{(r+1)n} \end{bmatrix} \times \begin{bmatrix} \Gamma(M(p), r) \\ \Gamma(M(p), r)A(p) \end{bmatrix} < 0. \tag{8}$$

Furthermore, we see that the last term in the above inequality satisfies

$$\begin{bmatrix} \Gamma(M(p), r) \\ \Gamma(M(p), r)A(p) \end{bmatrix} = \begin{bmatrix} \Gamma_{n,r}^T A(p) & -\Gamma_{n,r}^T \\ \Lambda(M(p), r) & 0 \\ 0 & \Lambda(M(p), r) \end{bmatrix}^\perp . \tag{9}$$

It is also straightforward to verify that the PLMI (7c) can be rewritten as

$$\begin{bmatrix} I_m \\ \Gamma(M(p), r)B(p) \end{bmatrix}^T \begin{bmatrix} -X_{\mathrm{a}}(p) & 0_{m,(r+1)n} \\ 0_{(r+1)n,m} & \Pi_r(p) \end{bmatrix} \times \begin{bmatrix} I_m \\ \Gamma(M(p), r)B(p) \end{bmatrix} < 0. \tag{10}$$

Similarly to (9), we see that the last term in the above inequality satisfies

$$\begin{bmatrix} I_m \\ \Gamma(M(p), r)B(p) \end{bmatrix} = \begin{bmatrix} B(p) & -\Gamma_{n,r}^T \\ 0_{rn,m} & \Lambda(M(p), r) \end{bmatrix}^\perp . \tag{11}$$

The results in (9) and (11) clearly indicate that the PLMIs (7b) and (7c) are given in the form of the condition (i) in Lemma 1. Noting this fact and applying Lemma 1, we can readily obtain the next results.

Theorem 1. For given $r \in \mathbf{Z}$, the upper bound $\gamma_r^\star$ defined by (7) can be characterized as

$$\gamma_r^\star = \inf_{\Pi_r(p), X_{\mathrm{a}}(p), \mu \in \mathbf{R}} \gamma \quad \text{subject to}$$

$$\begin{bmatrix} \Gamma_{n,r} C(p)^T C(p) \Gamma_{n,r}^T & \Pi_r(p) \\ \Pi_r(p) & 0_{(r+1)n} \end{bmatrix} - \mu \mathrm{Sq}\left\{ \begin{bmatrix} \Gamma_{n,r}^T A(p) & -\Gamma_{n,r}^T \\ \Lambda(M(p), r) & 0 \\ 0 & \Lambda(M(p), r) \end{bmatrix} \right\} < 0, \tag{12a}$$

$$\begin{bmatrix} -X_{\mathrm{a}}(p) & 0_{m,(r+1)n} \\ 0_{(r+1)n,m} & \Pi_r(p) \end{bmatrix} - \mu \mathrm{Sq}\left\{ \begin{bmatrix} B(p) & -\Gamma_{n,r}^T \\ 0_{rn,m} & \Lambda(M(p), r) \end{bmatrix} \right\} < 0, \tag{12b}$$

$$\gamma^2 > \mathrm{trace}(X_{\mathrm{a}}(p)), \quad \forall p \in \mathcal{P}. \tag{12c}$$

The upper bound $\gamma_r^\star$ can also be characterized as

$$\gamma_r^\star = \inf_{\Pi_r(p), X_{\mathrm{a}}(p), \mathcal{F}(p), \mathcal{G}(p)} \gamma \quad \text{subject to}$$

$$\begin{bmatrix} \Gamma_{n,r} C(p)^T C(p) \Gamma_{n,r}^T & \Pi_r(p) \\ \Pi_r(p) & 0_{(r+1)n} \end{bmatrix} + \mathrm{He}\left\{ \mathcal{F}(p) \begin{bmatrix} \Gamma_{n,r}^T A(p) & -\Gamma_{n,r}^T \\ \Lambda(M(p), r) & 0 \\ 0 & \Lambda(M(p), r) \end{bmatrix} \right\} < 0, \tag{13a}$$

$$\begin{bmatrix} -X_{\mathrm{a}}(p) & 0_{m,(r+1)n} \\ 0_{(r+1)n,m} & \Pi_r(p) \end{bmatrix} + \mathrm{He}\left\{ \mathcal{G}(p) \begin{bmatrix} B(p) & -\Gamma_{n,r}^T \\ 0_{rn,m} & \Lambda(M(p), r) \end{bmatrix} \right\} < 0, \tag{13b}$$

$$\gamma^2 > \mathrm{trace}(X_{\mathrm{a}}(p)), \quad \forall p \in \mathcal{P}, \tag{13c}$$

where $\mathcal{F}(p) : \mathbf{R}^q \to \mathbf{R}^{2(r+1)n\times(2r+1)n}$ and $\mathcal{G}(p) : \mathbf{R}^q \to \mathbf{R}^{((r+1)n+m)\times(r+1)n}$ are the multipliers that can be chosen as affine on p.

Proof: See the appendix section. □

It should be noted that the PLMIs (12) and (13) in Theorem 1 only involve quadratic terms on p even if the original one (7) may involve multiple products of p in higher degree. This interesting feature motivates us to develop a particular technique to solve quadratic PLMI problems for the exact computation of $\gamma_r^\star$, by exploiting its special quadratic structure. Unfortunately, however, we have not obtained definite results in this direction and this topic is currently under investigation.

On the other hand, even though the PLMI (13) is still hard to solve exactly as in (7), the reformulation from (7) into (13) should be meaningful. This is because (13) enables us to derive numerically verifiable PLMI conditions for the computation of *the upper bound* of $\gamma_r^\star$, as discussed in the next subsection.

3.2 Numerically Verifiable PLMI Conditions via Parameter Independent Multipliers

In the preceding subsection, we have derived PLMI conditions (12) and (13) by means of the Finsler's Lemma. This strategy based on the particular application of the Finsler's Lemma was first introduced in (Oliveira and Skelton, 2001; Peaucelle *et al.*, 2000) to obtain numerically verifiable PLMI condition for the robust stability/performance analysis of uncertain LTI systems. By following similar lines to (Oliveira and Skelton, 2001; Peaucelle *et al.*, 2000) and employing parameter-independent multipliers in (13), we readily obtain the next results.

Theorem 2. For given $r \in \mathbf{Z}$, $\gamma_r^\star \le \bar{\gamma}_r^\star$ holds where

$$\bar{\gamma}_r^\star = \inf_{\Pi_r(p), X_{\mathrm{a}}(p), \mathcal{F}, \mathcal{G}} \gamma \quad \text{subject to}$$

$$\begin{bmatrix} \Gamma_{n,r} C(p)^T C(p) \Gamma_{n,r}^T & \Pi_r(p) \\ \Pi_r(p) & 0_{(r+1)n} \end{bmatrix} + \mathrm{He}\left\{ \mathcal{F} \begin{bmatrix} \Gamma_{n,r}^T A(p) & -\Gamma_{n,r}^T \\ \Lambda(M(p), r) & 0 \\ 0 & \Lambda(M(p), r) \end{bmatrix} \right\} < 0, \tag{14a}$$

$$\begin{bmatrix} -X_{\mathrm{a}}(p) & 0_{m,(r+1)n} \\ 0_{(r+1)n,m} & \Pi_r(p) \end{bmatrix} + \mathrm{He}\left\{ \mathcal{G} \begin{bmatrix} B(p) & -\Gamma_{n,r}^T \\ 0_{rn,m} & \Lambda(M(p), r) \end{bmatrix} \right\} < 0, \tag{14b}$$

$$\gamma^2 > \mathrm{trace}(X_{\mathrm{a}}(p)), \quad \forall p \in \mathcal{P}_{\mathrm{v}}. \tag{14c}$$

Here, $\mathcal{P}_{\mathrm{v}}$ denotes the set of the vertices of $\mathcal{P}$.

In Theorem 2, we have derived numerically verifiable finitely many LMIs (14) from intractable PLMI (13) by restricting the multipliers $\mathcal{F}(p)$ and $\mathcal{G}(p)$ to be parameter-independent. This is the source of the conservatism of the proposed analysis condition (14) and there might be a certain gap between $\gamma_r^\star$ and $\bar{\gamma}_r^\star$. To obtain more accurate and verifiable LMI conditions, it is possible to employ parameter-dependent multipliers by following the methodologies, for example, in (Leite and Peres, 2003). However, we do not pursue this direction in the paper.

By solving (14), we can compute $\bar{\gamma}_r^\star$ that satisfy $\gamma^\star \le \gamma_r^\star \le \bar{\gamma}_r^\star$. It is expected that, by increasing $r \in \mathbf{Z}$ that determines the degree of the PPDLMs to be employed, we can obtain $\bar{\gamma}_r^\star$ that gives no more conservative upper bounds for $\gamma^\star$. This is indeed true and we can prove that the following results hold.

Theorem 3. For each fixed affine function $M(p)$, the following condition holds in (14):

$$\gamma^\star \le \bar{\gamma}_{r_2}^\star \le \bar{\gamma}_{r_1}^\star, \quad r_1 \le r_2. \tag{15}$$

Proof: Omitted due to limited space. □

From Theorem 3, we can construct monotonically non-increasing sequence of $\bar{\gamma}_r^\star$ that may approach to $\gamma^\star$. However, the computational complexity for solving LMIs (14) grows rapidly by increasing r since the size of the matrix variables in (14) depend on r as in $\Pi_r(p) \in \mathbf{S}_{(r+1)n}$, $\mathcal{F} \in \mathbf{R}^{2(r+1)n\times(2r+1)n}$ and $\mathcal{G} \in \mathbf{R}^{((r+1)n+m)\times(r+1)n}$. It should be noted also that the value $\bar{\gamma}_r^\star$ in (14) depends on the choice of the affine function $M(p)$, whose reasonable choice is an outstanding issue to us. In the next section, we will compare the exactness of the proposed analysis condition under several choice of the matrix function $M(p)$ via numerical examples.

Before proceeding to numerical experiments, we will briefly discuss the connections among the proposed condition and several existing results. We first note that, by letting $r = 0$ in Theorem 2, the following results can be derived.

Corollary 1. $\gamma_0^\star \le \bar{\gamma}_0^\star$ holds where

$$\bar{\gamma}_0^\star = \inf_{\Pi_0(p), X_{\mathrm{a}}(p), \mathcal{F}, \mathcal{G}} \gamma \quad \text{subject to}$$

$$\begin{bmatrix} C(p)^T C(p) & \Pi_0(p) \\ \Pi_0(p) & 0_n \end{bmatrix} + \mathrm{He}\left\{ \mathcal{F} \begin{bmatrix} A(p) & -I_n \end{bmatrix} \right\} < 0, \tag{16a}$$

$$\begin{bmatrix} -X_{\mathrm{a}}(p) & 0_{m,n} \\ 0_{n,m} & \Pi_0(p) \end{bmatrix} + \mathrm{He}\left\{ \mathcal{G} \begin{bmatrix} B(p) & -I_n \end{bmatrix} \right\} < 0, \tag{16b}$$

$$\gamma^2 > \mathrm{trace}(X_{\mathrm{a}}(p)), \quad \forall p \in \mathcal{P}_{\mathrm{v}}. \tag{16c}$$

Note that the condition (16) is exactly the same as the robust $\mathcal{H}_2$ performance analysis condition known as extended or dilated LMI in the literature (Oliveira and Skelton, 2001; Peaucelle *et al.*, 2000; Ebihara and Hagiwara, 2005). In these preceding studies, the condition (16) has proved to be no more conservative than the quadratic-stability based robust $\mathcal{H}_2$ performance analysis condition (Boyd *et al.*, 1994). Hence, it follows from Theorem 3 that the proposed condition (14) with $r \geq 1$ is no more conservative than the quadratic-stability based analysis condition and dilated LMI based condition (16), regardless of the choice of the affine function $M(p)$. However, it can be also proved that the proposed condition (14) with $r \geq 1$ is essentially equivalent to (16) if we take $M(p)$ as a constant over $\mathcal{P}$. Thus we cannot expect improvement of the analysis results by increasing $r \in \mathbf{Z}$ unless we employ $M(p)$ that varies over the uncertainty domain $\mathcal{P}$.

4. ILLUSTRATIVE EXAMPLES

4.1 The Case where $q = 2$

Let us consider the system (1) with $q = 2$ whose vertex matrices are given by

$$A_1 = \begin{bmatrix} -0.6659 & 0.1141 & 0.4734 \\ 0.6114 & -0.8433 & 0.6832 \\ 0.8473 & 0.1148 & -1.3593 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -0.7599 & 0.5660 & 0.0160 \\ 0.0713 & -0.9687 & 0.3847 \\ 0.5812 & 0.2385 & -0.4667 \end{bmatrix}, \tag{17}$$
$$B_i = \begin{bmatrix} 0.5752 \\ 0.4081 \\ 0.1957 \end{bmatrix}, \quad C_i = \begin{bmatrix} 0.5122 \\ 0.7133 \\ 0.8674 \end{bmatrix}^T \quad (i = 1, 2).$$

To compute the upper bounds of the worst case $\mathcal{H}_2$ norm $\gamma^\star$, we solved the proposed LMI condition (14) by letting $M(p) = \sum_{i=1}^{2} p_i M_i$ where $M_i = I$ $(i = 1, 2)$ in the first case whereas $M_i = A_i$ $(i = 1, 2)$ in the second case.

Table 1. Computation Results
($M_1 = M_2 = I$).

Quadratic Stability	2.5203	$(N = 9)$
Existing Condition (16)	$\bar{\gamma}_0^\star = 2.4237$	$(N = 45)$
Proposed Condition (14)	$\bar{\gamma}_1^\star = 2.4237$	$(N = 195)$
Proposed Condition (14)	$\bar{\gamma}_2^\star = 2.4237$	$(N = 453)$

(N: The number of scalar variables in each LMI.)

Table 2. Computation Results
($M_1 = A_1$, $M_2 = A_2$).

Proposed Condition (14)	$\bar{\gamma}_1^\star = 2.4192$	$(N = 195)$
Proposed Condition (14)	$\bar{\gamma}_2^\star = 2.4192$	$(N = 453)$

(N: The number of scalar variables in each LMI.)

From Tables 1 and 2, we can confirm that the upper bounds can be improved from $\bar{\gamma}_0^\star$ to $\bar{\gamma}_1^\star$ by selecting $M(p)$ that is not constant over $\mathcal{P}$. In this example, we also computed a lower bound of $\gamma^\star$ via fine griding over $\mathcal{P}$, which turns out to be 2.4192. Hence, in the second case, we can conclude that the proposed condition achieves an exact analysis result with $r = 1$.

4.2 The Case where $q = 3$

As the second example, let us consider the system (1) with $q = 3$ whose vertex matrices are given by

$$A_1 = \begin{bmatrix} 0.0061 & -0.2630 & 0.2748 \\ 0.1266 & 0.1242 & -0.3029 \\ -0.5100 & 0.4678 & -0.9712 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0.1330 & 0.2009 & 0.1672 \\ 0.1224 & -0.5987 & 0.3100 \\ -0.5235 & 0.0297 & -0.4784 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} -0.2733 & -0.1868 & -0.0077 \\ -0.0253 & -0.2828 & 0.6112 \\ -0.2412 & -0.0844 & -0.8024 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} -0.2215 \\ 0.1195 \\ 0.2996 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -0.6798 \\ 0.2351 \\ -0.5082 \end{bmatrix},$$
$$B_3 = \begin{bmatrix} 0.5425 \\ -0.0847 \\ -0.0655 \end{bmatrix}, \quad C_1 = \begin{bmatrix} -0.6078 \\ 0.5412 \\ 0.2940 \end{bmatrix}^T,$$
$$C_2 = \begin{bmatrix} -0.1872 \\ -0.6046 \\ 0.5114 \end{bmatrix}^T, \quad C_3 = \begin{bmatrix} -0.5776 \\ 0.2493 \\ 0.2031 \end{bmatrix}^T.$$

By letting $M(p) = \sum_{i=1}^{3} p_i M_i$, we solved the PLMI condition (14) under different set of the matrices M_i $(i = 1, 2, 3)$. Namely, in the first case we take $M_1 = I$ and $M_2 = M_3 = 0$ whereas in the second case we take $M_i = A_i$ $(i = 1, 2, 3)$. It should be noted that PPDLMs of the type (6) have been introduced previously in (Ebihara *et al.*, 2005) with the particular choice of the matrices $M(p) = A(p)$. This result was based on the fact that redundant system description naturally leads to this particular choice and its effectiveness was proved on examples for stability analysis cases. The same choice is therefore tested here and except for $\bar{\gamma}_2^\star$ in Table 4, it does give better results than other tested choices.

Table 3. Computation Results
($M_1 = I$, $M_2 = M_3 = 0$).

Quadratic Stability	18.1490	$(N = 10)$
Existing Condition (16)	$\bar{\gamma}_0^\star = 8.3072$	$(N = 52)$
Proposed Condition (14)	$\bar{\gamma}_1^\star = 4.8268$	$(N = 217)$
Proposed Condition (14)	$\bar{\gamma}_2^\star = 4.1726$	$(N = 499)$
Proposed Condition (14)	$\bar{\gamma}_3^\star = 3.9783$	$(N = 898)$

(N: The number of scalar variables in each LMI.)

Table 4. Computation Results
($M_1 = A_1$, $M_2 = A_2$, $M_3 = A_3$).

Proposed Condition (14)	$\bar{\gamma}_1^\star = 4.7339$	$(N = 217)$
Proposed Condition (14)	$\bar{\gamma}_2^\star = 4.2177$	$(N = 499)$
Proposed Condition (14)	$\bar{\gamma}_3^\star = 3.8307$	$(N = 898)$

(N: The number of scalar variables in each LMI.)

In this example, we applied fine griding search over $\mathcal{P}$ and computed a lower bound of the worst case $\mathcal{H}_2$ norm $\gamma^\star$ as 1.3208. Even though the upper bounds resulting from the proposed analysis condition could be improved by further increasing the degree r, it turned out that to solve resulting LMI conditions becomes computationally very demanding if $r \geq 4$. Hence, in this example, the proposed condition fails to achieve exact analysis within the acceptable computation time.

5. CONCLUSION

In this paper, we dealt with the robust $\mathcal{H}_2$ performance analysis problems of polytopic-type uncertain LTI systems. We proposed LMI-based conditions to assess the robust $\mathcal{H}_2$ performance by means of polynomially parameter-dependent Lyapunov functions. We showed that, by increasing the degree of the Lyapunov functions, we can obtain more accurate (no more conservative) analysis results but the computational burden grows rapidly as well. Thus the reduction of the computational complexity is an important topic for our future research.

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APPENDIX

Proof of Theorem 1: From (9) and (11), it is apparent that if (13) holds then (7) holds for the same variable $\Pi_r(p)$ and $X_{\mathrm{a}}(p)$. It is also obvious that if (12) holds, then (13) holds for the same $\Pi_r(p)$ and $X_{\mathrm{a}}(p)$ by letting

$$\begin{aligned} \mathcal{F}(p) &= \mu \begin{bmatrix} \Gamma_{n,r}^T A(p) & -\Gamma_{n,r}^T \\ \Lambda(M(p),r) & 0 \\ 0 & \Lambda(M(p),r) \end{bmatrix}^T, \\ \mathcal{G}(p) &= \mu \begin{bmatrix} B(p) & -\Gamma_{n,r}^T \\ 0_{n,m} & \Lambda(M(p),r) \end{bmatrix}^T, \end{aligned} \tag{18}$$

which are surely affine on p. Hence, it suffices to prove that if (7) holds, then (12) holds for the same $\Pi_r(p)$ and $X_{\mathrm{a}}(p)$, which is almost trivial since the existence of a fixed μ_1 that satisfies (12a) with μ replaced by μ_1 over $\mathcal{P}$ can be ensured from the assertions in Lemma 1 and the compactness of the set $\mathcal{P}$. Similarly, the existence of a fixed μ_2 that satisfies (12b) with μ replaced by μ_2 over $\mathcal{P}$ can also be ensured. Hence, by letting $\mu := \max\{\mu_1, \mu_2\}$, we can confirm that the conditions in (12) are satisified. This completes the proof. □

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